

**EE7000 Advanced Digital Signal Processing for Wireless
Communications**

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Midterm Examination, Spring of 2003

Time: 11:40 a.m. ~ 12:30 p.m., Wednesday, March 12 of 2003

Please ALWAYS work on the FRONT SIDE of each page. No answer on the BACK SIDE of any page will be GRADED! Ask for additional blank papers if necessary!

You may check any textbook, classnote or other references alone during the test.

However, NO CHEATING or COLLABORATION is allowed and such violation of the university regulations will be reported.

Please write down you name and social security number here:

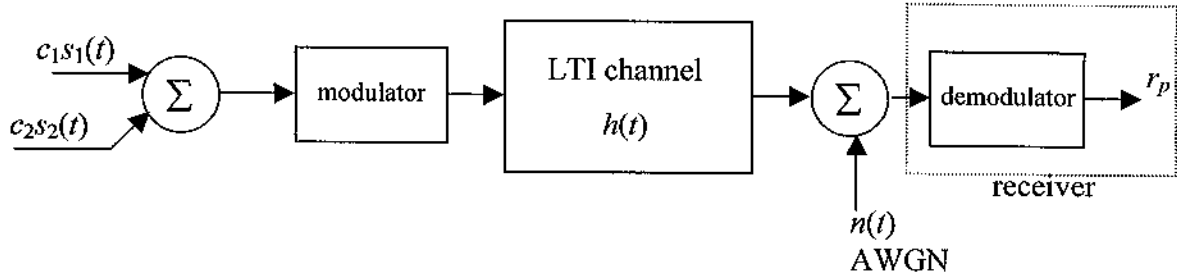
Full Name: Solution

Social Security Number: _____

Partition	Score
Question 1	
Question 2	
Question 3	
Total	

Question 1: (40%)

A Multi-access BPSK communication system is depicted as below. $n(t)$ is an additive



white Gaussian channel noise (zero mean and variance σ^2) and the transmitted signal $s_k(t)$ is a BPSK rectangular pulse train such that

$$s_k(t) = \sum_{i=-\infty}^{\infty} m_{k,i} p(t - iT_b), \text{ where } m_{k,i} = \pm 1 \text{ and } p(t) = \begin{cases} 1, & 0 \leq t \leq T_b \\ 0, & \text{elsewhere} \end{cases}$$

$k=1, 2$, is the user index. The channel is assumed to be distortionless, i.e., $h(t) = \delta(t)$.

We define two hypotheses for each user k here:

$H_{-c_k k}$: Hypothesis for the negative pulse to be sent by user k .

$H_{c_k k}$: Hypothesis for the positive pulse to be sent by user k .

(a) What are the four *a priori* conditional probability density functions

$$f_{R|H_{-c_k k}}(r | H_{-c_k k}) \text{ and } f_{R|H_{c_k k}}(r | H_{c_k k}), k=1, 2? \text{ (20\%)}$$

$$\hat{m} = 1$$

(b) According to a single decision rule $r > 0$, if the error probability for user k is $P_{e,k}$

$$\hat{m} = -1$$

and the average user error probability $P_e = \frac{1}{2} \sum_{k=1}^2 P_{e,k}$, what is P_e in terms of c_k 's and

ϕ function (or Q functions)? (20%)

Answer to Question 1:

$$(a) \quad r_p = C_1 m_{1,p} + C_2 m_{2,p} + n_p$$

$$f_{IR|H_{C_1,1}}(r|H_{C_1,1})$$

$$= \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \otimes \delta(r+C_1)$$

$$\otimes \left[\frac{1}{2} \delta(r+C_2) + \frac{1}{2} \delta(r-C_2) \right]$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}\sigma^2} \left[e^{-\frac{(r+C_1+C_2)^2}{2\sigma^2}} + e^{-\frac{(r+C_1-C_2)^2}{2\sigma^2}} \right]$$

$$f_{IR|H_{C_1,1}}(r|H_{C_1,1})$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}\sigma^2} \left[e^{-\frac{(r-C_1+C_2)^2}{2\sigma^2}} + e^{-\frac{(r-C_1-C_2)^2}{2\sigma^2}} \right]$$

$$f_{IR|H_{C_2,2}}(r|H_{C_2,2})$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}\sigma^2} \left[e^{-\frac{(r+C_1+C_2)^2}{2\sigma^2}} + e^{-\frac{(r-C_1+C_2)^2}{2\sigma^2}} \right]$$

$$f_{IR|H_{C_2,2}}(r|H_{C_2,2})$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}\sigma^2} \left[e^{-\frac{(r+C_1-C_2)^2}{2\sigma^2}} + e^{-\frac{(r-C_1-C_2)^2}{2\sigma^2}} \right]$$

(b)

$$\begin{aligned} P_{e,k} &= \frac{1}{2} \int_0^{\infty} f_{IR} | H_{-C_k, k} (r | H_{-C_k, k}) \, dr \\ &+ \frac{1}{2} \int_{-\infty}^0 f_{IR} | H_{C_k, k} (r | H_{C_k, k}) \, dr \\ &= \frac{1}{4\sqrt{2\pi}\sigma^2} \int_0^{\infty} \left[e^{-\frac{(r+C_k-C_m)^2}{2\sigma^2}} + e^{-\frac{(r+C_k+C_m)^2}{2\sigma^2}} \right] dr \\ &+ \frac{1}{4\sqrt{2\pi}\sigma^2} \int_{-\infty}^0 \left[e^{-\frac{(r-C_k-C_m)^2}{2\sigma^2}} + e^{-\frac{(r-C_k+C_m)^2}{2\sigma^2}} \right] dr \\ &= \frac{1}{4} \left[\phi\left(\frac{C_k-C_m}{\sigma}\right) + \phi\left(\frac{C_k+C_m}{\sigma}\right) \right. \\ &\quad \left. + \phi\left(\frac{C_k+C_m}{\sigma}\right) + \phi\left(\frac{C_k-C_m}{\sigma}\right) \right] \\ &= \frac{1}{2} \left[\phi\left(\frac{C_k+C_m}{\sigma}\right) + \phi\left(\frac{C_k-C_m}{\sigma}\right) \right], \text{ where } k \neq m \end{aligned}$$

$$\begin{aligned} \therefore P_e &= \frac{1}{2} \sum_{k=1}^2 P_{e,k} = \frac{1}{4} \left[\phi\left(\frac{C_1+C_2}{\sigma}\right) \right. \\ &\quad \left. + \phi\left(\frac{C_1-C_2}{\sigma}\right) + \phi\left(\frac{C_2+C_1}{\sigma}\right) + \phi\left(\frac{C_2-C_1}{\sigma}\right) \right] \\ &= \frac{1}{4} \left[2\phi\left(\frac{C_1+C_2}{\sigma}\right) + \phi\left(\frac{C_1-C_2}{\sigma}\right) + 1 - \phi\left(\frac{C_1-C_2}{\sigma}\right) \right] \\ &= \frac{1}{4} \left[1 + 2\phi\left(\frac{C_1+C_2}{\sigma}\right) \right] \end{aligned}$$

$$\text{where } \phi(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

Question 2: (40%)

The digital channel for a BPSK communication system can be described as

$$h_k \begin{cases} \neq 0, & 0 \leq k \leq 1 \\ = 0, & \text{elsewhere} \end{cases}$$

The discrete-time received sequence is $r_k = \begin{cases} -\frac{1}{2}, & k = 0 \\ \frac{1}{2}, & k = 1 \\ 0, & \text{otherwise} \end{cases}$ and the provided training

sequence is $m_k = \begin{cases} -1, & k = -1 \\ 1, & k = 0 \\ 1, & k = 1 \end{cases}$.

- (a) What is the minimum-mean-square-error channel estimate \hat{h}_{MMSE} ? (20%)
- (b) What is the probability of error based on the channel distortion estimated in (a), if a sign detector is applied for the received sequence $r_k = m_k \otimes h_k + n_k$, where n_k is the zero-mean AWGN with variance σ^2 ? (Hint: no equalization is applied here.) (20%)

Answer to Question 2:

$$(a) \quad \vec{m}_i = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \vec{r}_i = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\hat{h}_{MMSE} = (\vec{m}_i^T \vec{m}_i)^{-1} \vec{m}_i^T \vec{r}_i$$

$$= \left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \hat{h}_0 \\ \hat{h}_1 \end{bmatrix}$$

$$\Rightarrow h_k = \delta(k-1)$$

$$(b) \quad r_k = \frac{1}{2} m_{k-1} + n_k$$

$$\Rightarrow P_e = \int_0^{\infty} \frac{1}{2\sqrt{2\pi}\sigma^2} e^{-\frac{(r+\frac{1}{2})^2}{2\sigma^2}} dr + \int_{-\infty}^0 \frac{1}{2\sqrt{2\pi}\sigma^2} e^{-\frac{(r-\frac{1}{2})^2}{2\sigma^2}} dr$$

$$= \frac{1}{2} \Phi\left(\frac{1}{2\sigma}\right) + \frac{1}{2} \Phi\left(\frac{1}{2\sigma}\right)$$

$$= \Phi\left(\frac{1}{2\sigma}\right)$$

Question 3: (20%)

A training sequence needs to be designed for the minimum-mean-square-error channel estimate \hat{h}_{MMSE} in a BPSK communication system. \hat{h}_{MMSE} is a 2×1 vector and the training sequence contains three consecutive bits $[m_{i-1}, m_i, m_{i+1}]$.

- (a) Suggest one appropriate training sequence, which can provide a minimum-mean-square-error channel estimate. (5%)
- (b) How many training sequences out of all eight three-bit sequences are appropriate for minimum-mean-square-error channel estimation? (15%)

Answer to Question 3:

$$(a) \quad \vec{m}_i = \begin{bmatrix} m_i & m_{i-1} \\ m_{i+1} & m_i \end{bmatrix}$$

$$\text{Choose } \begin{bmatrix} m_{i-1} & m_i & m_{i+1} \\ 1 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow \vec{m}_i = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{ is full rank}$$

(b)

$$\text{Since } \vec{m}_i = \begin{bmatrix} m_i & m_{i-1} \\ m_{i+1} & m_i \end{bmatrix}$$

$[m_{i-1}, m_i]^T$ has to be linearly independent of $[m_i, m_{i+1}]^T$ to make a full-rank \vec{m}_i .

Hence, if $m_{i-1} = m_i$, m_{i+1} has to satisfy that $m_{i+1} = -m_i$ to make a full-rank \vec{m}_i . If $m_{i-1} = -m_i$, m_{i+1} has to satisfy that $m_{i+1} = m_i$ to make a full-rank \vec{m}_i . There are $2^2 \times 1 = 4$ appropriate sequences to make a full rank \vec{m}_i .