Problem 1:

For single-access user i, the transmitted signal is

$$s_i(t) = \sum_{k=-\infty}^{\infty} m_{i,k} d_{i,l} p_T(t-kT_b)\psi(t-lT_c) = \sum_{l=-\infty}^{\infty} m_{i,k} d_{i,l}\psi(t-lT_c)$$

$$= \sum_{l=-\infty}^{\infty} m_{i,\left\lfloor \frac{l}{N} \right\rfloor} d_{i,l}\psi(t-lT_c)$$

$$R_{S_iS_i}(t,t+\tau) = \sum_{n=-\infty}^{\infty} R_{I_iI_i}(n) \sum_{l=-\infty}^{\infty} \psi(t-lT_c)\psi(t+\tau-lT_c-nT_c),$$

where $R_{I_iI_i}(n) = E\{I_{i,l}I_{i,l+n}\}$, $I_{i,l} = m_{i,\left\lfloor \frac{l}{N} \right\rfloor} d_{i,l}$.

$$\mu_{I_i} = E\{I_{i,l}\} = \sum_{\alpha,\beta\in\{-1,1\}}\sum \Pr\left\{ m_{i,\left\lfloor \frac{l}{N} \right\rfloor} = \alpha;\, d_{i,l} = \beta \right\}\alpha\beta = \frac{1}{4}[1-1-1+1] = 0$$

$$R_{I_iI_i}(n) = E\{I_{i,l}I_{i,l+n}\}$$

$$= \begin{cases} \sum\sum\sum\sum_{\alpha,\beta,\chi,\delta\in\{-1,1\}} \Pr\left\{ m_{i,\left\lfloor \frac{l}{N} \right\rfloor} = \alpha;\, d_{i,l} = \beta;\, m_{i,\left\lfloor \frac{l+n}{N} \right\rfloor} = \chi;\, d_{i,l+n} = \delta \right\}\alpha\beta\chi\delta, & n \neq 0 \\ \sum\sum_{\alpha,\beta\in\{-1,1\}} \Pr\left\{ m_{i,\left\lfloor \frac{l}{N} \right\rfloor} = \alpha;\, d_{i,l} = \beta \right\}\alpha^2\beta^2, & n = 0 \end{cases}$$

$$= \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} = \delta(n)$$

The power spectral density function of transmitted signal is

$S_{S_iS_i}(\omega) = \frac{1}{T_c}|\Psi(\omega)|^2 S_{I_iI_i}(\omega)$ and $S_{I_iI_i}(\omega) = 1$.

$$\Psi(\omega) = F[\psi(t)] = \int_0^{T_c} e^{-j\omega t}dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_0^{T_c} = \frac{-2je^{-j\frac{\omega T_c}{2}}\sin\left(\frac{\omega T_c}{2}\right)}{-j\omega}$$

$$= T_c e^{-j\frac{\omega T_c}{2}} \sin c\left(\frac{\omega T_c}{2}\right), \text{ where } \sin c(x) = \frac{\sin(x)}{x}.$$

Therefore, $S_{S_iS_i}(\omega) = T_c \sin c^2\left(\frac{\omega T_c}{2}\right)$.

$\frac{\omega_B T_c}{2} = \pi \Rightarrow \pi f_B T_c = \pi \Rightarrow f_B = \frac{1}{T_c}$. Effective bandwidth is $\frac{1}{T_c}$ Hz.

Problem 2:

Similar to Problem 1,

$$s_i(t) = \sum_{k=-\infty}^{\infty} m_{i,k} g_i(t - kT_b), \text{ where } g_i(t) = \begin{cases} 1, & (i-1)T_c \le t < iT_c \\ 0, & elsewhere \end{cases}.$$

$$R_{S_iS_i}(t, t+\tau) = \sum_{n=-\infty}^{\infty} R_{M_iM_i}(n) \sum_{k=-\infty}^{\infty} g_i(t - kT_c) g_i(t + \tau - kT_c - nT_c)$$

where $R_{M_iM_i}(n) = E\{m_{i,k} m_{i,k+n}\}$.

$$\mu_{M_i} = E\{m_{i,k}\} = \sum_{\alpha \in \{-1,1\}} \Pr\{m_{i,k} = \alpha\}\alpha = \frac{1}{2}[-1+1] = 0$$

$$R_{M_iM_i}(n) = E\{m_{i,k} m_{i,k+n}\}$$

$$= \begin{cases} \sum_{\alpha,\beta \in \{-1,1\}} \Pr\{m_{i,k} = \alpha;\ m_{i,k+n} = \beta\}\alpha\beta = \frac{1}{4}[1-1-1+1] = 0, & n \ne 0 \\ \sum_{\alpha \in \{-1,1\}} \Pr\{m_{i,k} = \alpha\}\alpha^2 = \frac{1}{2}(1+1) = 1, & n = 0 \end{cases} = \delta(n),$$

$$G_i(\omega) = F\{g_i(t)\} = \int_{(i-1)T_c}^{iT_c} e^{-j\omega t} dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{(i-1)T_c}^{iT_c} = \frac{-2je^{-j\frac{(2i-1)\omega T_c}{2}} \sin\left(\frac{\omega T_c}{2}\right)}{-j\omega}$$

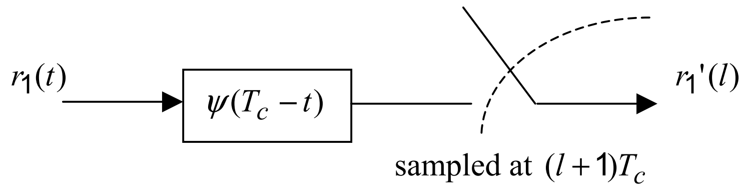
$$= T_c e^{-j\frac{(2i-1)\omega T_c}{2}} \sin c\left(\frac{\omega T_c}{2}\right).$$

$$S_{S_iS_i}(\omega) = \frac{1}{T_c}|G_i(\omega)|^2 S_{M_iM_i}(\omega) \text{ and } S_{M_iM_i}(\omega) = 1.$$

Therefore, $S_{S_iS_i}(\omega) = \frac{T_c^2}{T_b} \sin c^2\left(\frac{\omega T_c}{2}\right)$.

$\frac{\omega_B T_c}{2} = \pi \Rightarrow \pi f_B T_c = \pi \Rightarrow f_B = \frac{1}{T_c}$. Effective bandwidth is $\frac{1}{T_c}$ Hz.

Problem 3:



For single-access user 1, the received signal is $r_1(t) = s_1(t) + cs_2(t-\tau) + n(t)$.
The demodulated signal is

$$r_1'(l) = \int_{lT_c}^{(l+1)T_c} r_1(\varsigma)\psi(T_c - t + \varsigma)\Big|_{t=(l+1)T_c} d\varsigma = s'_{1,l} + cs'_{2,l} + n_l ,$$

where $s'_{1,l} \equiv s_1(t) \otimes \psi(T_c - t)\big|_{t=(l+1)T_c}$, $s'_{2,l} \equiv s_2(t-\tau) \otimes \psi(T_c - t)\big|_{t=(l+1)T_c}$

and $n_l \equiv n(t) \otimes \psi(T_c - t)\big|_{t=(l+1)T_c}$.

$$s'_{1,l} = \sum_{l_1=-\infty}^{\infty} m_{1,\left\lfloor \frac{l_1}{N} \right\rfloor} d_{1,l_1} \int_{lT_c}^{(l+1)T_c} \psi(\varsigma - l_1 T_c)\psi(T_c - t + \varsigma)\Big|_{t=(l+1)T_c} d\varsigma$$

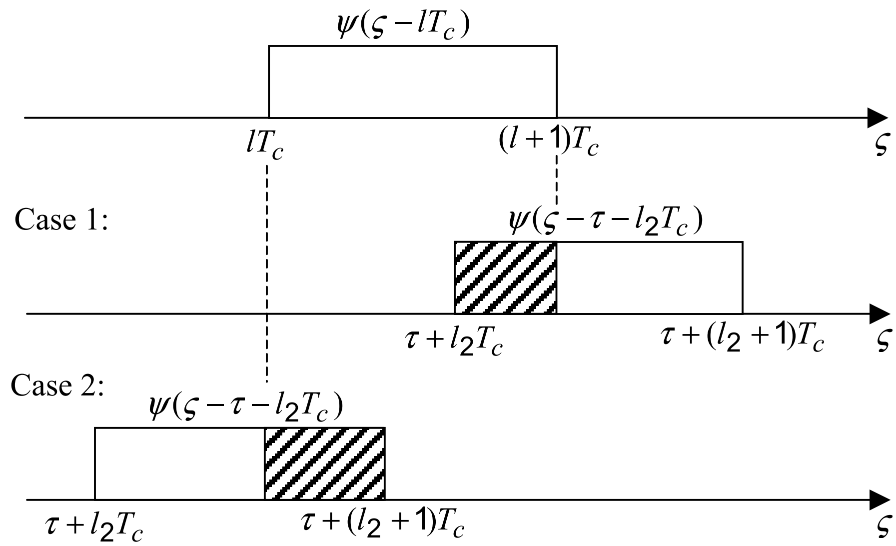
$$= \sum_{l_1=-\infty}^{\infty} m_{1,\left\lfloor \frac{l_1}{N} \right\rfloor} d_{1,l_1} \delta(l - l_1) = m_{1,\left\lfloor \frac{l}{N} \right\rfloor} d_{1,l}$$

since $\int_{lT_c}^{(l+1)T_c} \psi(\varsigma - l_1 T_c)\psi(T_c - t + \varsigma)\Big|_{t=(l+1)T_c} d\varsigma = \int_{lT_c}^{(l+1)T_c} \psi(\varsigma - l_1 T_c)\psi(\varsigma - lT_c) d\varsigma$

$$= \begin{cases} 1, & l = l_1 \\ 0, & l \neq l_1 \end{cases} = \delta(l - l_1)$$

$$s'_{2,l} = \sum_{l_2=-\infty}^{\infty} m_{2,\left\lfloor \frac{l_2}{N} \right\rfloor} d_{2,l_2} \int_{lT_c}^{(l+1)T_c} \psi(\varsigma - \tau - l_2 T_c)\psi(T_c - t + \varsigma)\Big|_{t=(l+1)T_c} d\varsigma$$

$$= \sum_{l_2=-\infty}^{\infty} m_{2,\left\lfloor \frac{l_2}{N} \right\rfloor} d_{2,l_2} \int_{lT_c}^{(l+1)T_c} \psi(\varsigma - \tau - l_2 T_c)\psi(\varsigma - lT_c) d\varsigma$$



Case 1: $lT_c < \tau + l_2T_c \le (l+1)T_c \Rightarrow lT_c - \tau < l_2T_c \cap l_2T_c \le (l+1)T_c - \tau$

$$\Rightarrow l - \frac{\tau}{T_c} < l_2 \le l + 1 - \frac{\tau}{T_c} \Rightarrow l_2 = l - \left\lfloor \frac{\tau}{T_c} \right\rfloor.$$

Case 2: $\tau + l_2T_c < lT_c \le \tau + (l_2+1)T_c \Rightarrow l_2T_c \le lT_c - \tau \cap (l-1)T_c - \tau < l_2T_c$

$$\Rightarrow l - 1 - \frac{\tau}{T_c} < l_2 \le l - \frac{\tau}{T_c} \Rightarrow l_2 = l - \left\lfloor \frac{\tau}{T_c} \right\rfloor - 1.$$

Normalized inner product is

$$\frac{\int_{lT_c}^{(l+1)T_c} \psi(\varsigma - \tau - l_2T_c)\psi(\varsigma - lT_c)d\varsigma}{\int_{lT_c}^{(l+1)T_c} \psi^2(\varsigma - lT_c)d\varsigma} = \begin{cases} \int_{\tau + l_2T_c}^{(l+1)T_c} \frac{1}{T_c} d\varsigma, & l_2 = l - \left\lfloor \frac{\tau}{T_c} \right\rfloor \\ \int_{lT_c}^{\tau + (l_2+1)T_c} \frac{1}{T_c} d\varsigma, & l_2 = l - \left\lfloor \frac{\tau}{T_c} \right\rfloor - 1 \\ 0, & \text{for all other } l_2 \end{cases}$$

$$= \begin{cases} \frac{(l - l_2 + 1)T_c - \tau}{T_c}, & l_2 = l - \left\lfloor \frac{\tau}{T_c} \right\rfloor \\ \frac{(l_2 - l + 1)T_c + \tau}{T_c}, & l_2 = l - \left\lfloor \frac{\tau}{T_c} \right\rfloor - 1 \\ 0, & \text{for all other } l_2 \end{cases}$$

The sampled noise is

$$n_l = \int_{lT_c}^{(l+1)T_c} n(\varsigma)\psi(T_c - t + \varsigma)\Big|_{t=(l+1)T_c} d\varsigma = \int_{lT_c}^{(l+1)T_c} n(\varsigma)\psi(\varsigma - lT_c)d\varsigma .$$

$$E\{n_l\} = \int_{lT_c}^{(l+1)T_c} E\{n(\varsigma)\}\psi(\varsigma - lT_c)d\varsigma = 0$$

$$E\{n_l n_{l'}\} = \int_{lT_c}^{(l+1)T_c}\int_{l'T_c}^{(l'+1)T_c} E\{n(\varsigma)n(\xi)\}\psi(\varsigma - lT_c)\psi(\xi - l'T_c)d\varsigma d\xi = \frac{N_0}{2}\delta(l-l')$$

Define $\kappa(l,l_2,\tau,T_c) \equiv \begin{cases} \dfrac{(l-l_2+1)T_c - \tau}{T_c}, & l_2 = l - \left\lfloor \dfrac{\tau}{T_c} \right\rfloor \\ \dfrac{(l_2-l+1)T_c + \tau}{T_c}, & l_2 = l - \left\lfloor \dfrac{\tau}{T_c} \right\rfloor - 1 \\ 0, & \text{for all other } l_2 \end{cases}$

$$r_1'(l) = s'_{1,l} + cs'_{2,l} + n_l = m_{1,\left\lfloor \frac{l}{N} \right\rfloor} d_{1,l} + c\sum_{l_2=\tilde{l}_2-1}^{\tilde{l}_2} m_{2,\left\lfloor \frac{l_2}{N} \right\rfloor} d_{2,l_2}\kappa(l,l_2,\tau,T_c) + n_l$$

The unspreaded signal is

$$\begin{aligned} r_{1,p} &= \sum_{l'=0}^{N-1} r_1'(l'+pN)d_{1,l'+pN}, \quad l = l'+pN \\ &= \sum_{l'=0}^{N-1} s'_{1,l'+pN}\, d_{1,l'+pN} + c\sum_{l'=0}^{N-1} s'_{2,l'+pN}\, d_{1,l'+pN} + \sum_{l'=0}^{N-1} n_{l'+pN} d_{1,l'+pN} \\ &= \sum_{l'=0}^{N-1} d^2_{1,l'+pN}\, m_{1,\left\lfloor \frac{l'+pN}{N} \right\rfloor} + c\sum_{l'=0}^{N-1}\sum_{l_2=\tilde{l}_2(l')-1}^{\tilde{l}_2(l')} m_{2,\left\lfloor \frac{l_2}{N} \right\rfloor} d_{2,l_2} d_{1,l'+pN}\kappa(l,l_2,\tau,T_c) \\ &\quad + \sum_{l'=0}^{N-1} n_{l'+pN} d_{1,l'+pN}, \end{aligned}$$

where $\tilde{l}_2(l') \equiv l - \left\lfloor \frac{\tau}{T_c} \right\rfloor = l'+pN - \left\lfloor \frac{\tau}{T_c} \right\rfloor$ and $\sum_{l'=0}^{N-1} d^2_{1,l'+pN} = N$.

The unspreaded noise is

$$w_p \equiv \sum_{l'=0}^{N-1} n_{l'+pN} d_{1,l'+pN} .$$

Hence $E\{w_p\} = \sum_{l'=0}^{N-1} E\{n_{l'+pN}\} d_{1,l'+pN} = 0$ and

$$E\{w_p^2\} = \sum_{l'=0}^{N-1}\sum_{q=0}^{N-1} E\{n_{l'+pN} n_{q+pN}\} d_{1,l'+pN} d_{1,q+pN}$$

$$= \sum_{l'=0}^{N-1}\sum_{q=0}^{N-1} \frac{N_0}{2}\delta(l'-q) d_{1,l'+pN} d_{1,q+pN} = \sum_{l'=0}^{N-1} d^2_{1,l'+pN} \frac{N_0}{2} = \frac{NN_0}{2}$$

$$\therefore r_{1,p} = Nm_{1,p}$$

$$+ c\sum_{l'=0}^{N-1}\left\{ m_{2,\left\lfloor \frac{\tilde{l}_2(l')-1}{N} \right\rfloor} d_{2,\tilde{l}_2(l')-1} d_{1,l'+pN} \kappa(l, \tilde{l}_2(l')-1, \tau, T_c) \right\}$$

$$+ c\sum_{l'=0}^{N-1}\left\{ m_{2,\left\lfloor \frac{\tilde{l}_2(l')}{N} \right\rfloor} d_{2,\tilde{l}_2(l')} d_{1,l'+pN} \kappa(l, \tilde{l}_2(l'), \tau, T_c) \right\} + w_p$$

$$f_{R_{1,p}|H_0}\left(r_{1,p} \middle| H_0\right)$$

$$= \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(0)-1}{N} \right\rfloor}, } \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(1)-1}{N} \right\rfloor}, } \cdots \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(N-1)-1}{N} \right\rfloor}, } \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(0)}{N} \right\rfloor}, } \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(1)}{N} \right\rfloor}, } \cdots \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(N-1)}{N} \right\rfloor} \in \{-1,1\}} \frac{1}{\sqrt{\pi N N_0}}$$

$$\times \Pr\left\{ m_{2,\left\lfloor \frac{\tilde{l}_2(0)-1}{N} \right\rfloor}; m_{2,\left\lfloor \frac{\tilde{l}_2(1)-1}{N} \right\rfloor}; \cdots; m_{2,\left\lfloor \frac{\tilde{l}_2(N-1)-1}{N} \right\rfloor}; m_{2,\left\lfloor \frac{\tilde{l}_2(0)}{N} \right\rfloor}; m_{2,\left\lfloor \frac{\tilde{l}_2(1)}{N} \right\rfloor}; \cdots; m_{2,\left\lfloor \frac{\tilde{l}_2(N-1)}{N} \right\rfloor} \right\}$$

$$\times \exp\left\{ -\frac{1}{NN_0}\left[r_{1,p} + N + c\sum_{l'=0}^{N-1} m_{2,\left\lfloor \frac{\tilde{l}_2(l')-1}{N} \right\rfloor} d_{2,\tilde{l}_2(l')-1} d_{1,l'+pN} \kappa(l, \tilde{l}_2(l')-1, \tau, T_c) \right.\right.$$

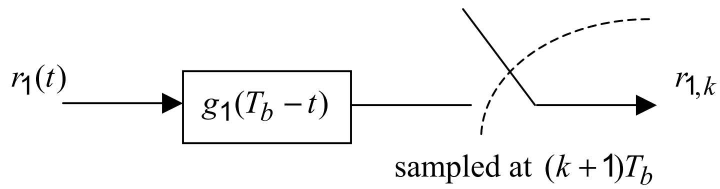
$$\left.\left. + c\sum_{l'=0}^{N-1} m_{2,\left\lfloor \frac{\tilde{l}_2(l')}{N} \right\rfloor} d_{2,\tilde{l}_2(l')} d_{1,l'+pN} \kappa(l, \tilde{l}_2(l'), \tau, T_c) \right]^2 \right\}$$

$$
\begin{aligned}
&f_{R_{1,p}|H_1}\left(r_{1,p}\middle|H_1\right) \\
&= \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(0)-1}{N} \right\rfloor},} \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(1)-1}{N} \right\rfloor},} \cdots \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(N-1)-1}{N} \right\rfloor},} \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(0)}{N} \right\rfloor},} \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(1)}{N} \right\rfloor},} \cdots \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(N-1)}{N} \right\rfloor} \in\{-1,1\}} \Pr\Bigg\{ m_{2,\left\lfloor \frac{\tilde{l}_2(0)-1}{N} \right\rfloor}; \\
&m_{2,\left\lfloor \frac{\tilde{l}_2(1)-1}{N} \right\rfloor}; \cdots ; m_{2,\left\lfloor \frac{\tilde{l}_2(N-1)-1}{N} \right\rfloor}; m_{2,\left\lfloor \frac{\tilde{l}_2(0)}{N} \right\rfloor}; m_{2,\left\lfloor \frac{\tilde{l}_2(1)}{N} \right\rfloor}; \cdots ; m_{2,\left\lfloor \frac{\tilde{l}_2(N-1)}{N} \right\rfloor} \Bigg\} \times \frac{1}{\sqrt{\pi N N_0}} \\
&\times \exp\Bigg\{ -\frac{1}{NN_0} \Bigg[r_{1,p} - N + c \sum_{l'=0}^{N-1} m_{2,\left\lfloor \frac{\tilde{l}_2(l')-1}{N} \right\rfloor} d_{2,\tilde{l}_2(l')-1} d_{1,l'+pN}\, \kappa(l,\tilde{l}_2(l')-1,\tau,T_c) \\
&+ c \sum_{l'=0}^{N-1} m_{2,\left\lfloor \frac{\tilde{l}_2(l')}{N} \right\rfloor} d_{2,\tilde{l}_2(l')} d_{1,l'+pN}\, \kappa(l,\tilde{l}_2(l'),\tau,T_c) \Bigg]^2 \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
P_e &= \frac{1}{2}\int_0^{\infty} f_{R_{1,p}|H_0}\left(r_{1,p}\middle|H_0\right) dr_{1,p} + \frac{1}{2}\int_{-\infty}^{0} f_{R_{1,p}|H_1}\left(r_{1,p}\middle|H_1\right) dr_{1,p} \\
&= \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(0)-1}{N} \right\rfloor},} \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(1)-1}{N} \right\rfloor},} \cdots \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(N-1)-1}{N} \right\rfloor},} \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(0)}{N} \right\rfloor},} \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(1)}{N} \right\rfloor},} \cdots \sum_{m_{2,\left\lfloor \frac{\tilde{l}_2(N-1)}{N} \right\rfloor} \in\{-1,1\}} \Pr\Bigg\{ m_{2,\left\lfloor \frac{\tilde{l}_2(0)-1}{N} \right\rfloor}; \\
&m_{2,\left\lfloor \frac{\tilde{l}_2(1)-1}{N} \right\rfloor}; \cdots ; m_{2,\left\lfloor \frac{\tilde{l}_2(N-1)-1}{N} \right\rfloor}; m_{2,\left\lfloor \frac{\tilde{l}_2(0)}{N} \right\rfloor}; m_{2,\left\lfloor \frac{\tilde{l}_2(1)}{N} \right\rfloor}; \cdots ; m_{2,\left\lfloor \frac{\tilde{l}_2(N-1)}{N} \right\rfloor} \Bigg\} \\
&\times \phi\Bigg\{ \frac{\sqrt{2}}{\sqrt{NN_0}} \Bigg[-N + c \sum_{l'=0}^{N-1} m_{2,\left\lfloor \frac{\tilde{l}_2(l')-1}{N} \right\rfloor} d_{2,\tilde{l}_2(l')-1} d_{1,l'+pN}\, \kappa(l,\tilde{l}_2(l')-1,\tau,T_c) \\
&\quad + c \sum_{l'=0}^{N-1} m_{2,\left\lfloor \frac{\tilde{l}_2(l')}{N} \right\rfloor} d_{2,\tilde{l}_2(l')} d_{1,l'+pN}\, \kappa(l,\tilde{l}_2(l'),\tau,T_c) \Bigg] \Bigg\}
\end{aligned}
$$

Problem 4:

$s_1(t) = \sum_{k'=-\infty}^{\infty} m_{1,k'} g_1(t - k'T_b)$. $s_2(t) = \sum_{k_2=-\infty}^{\infty} m_{2,k_2} g_2(t - k_2 T_b)$



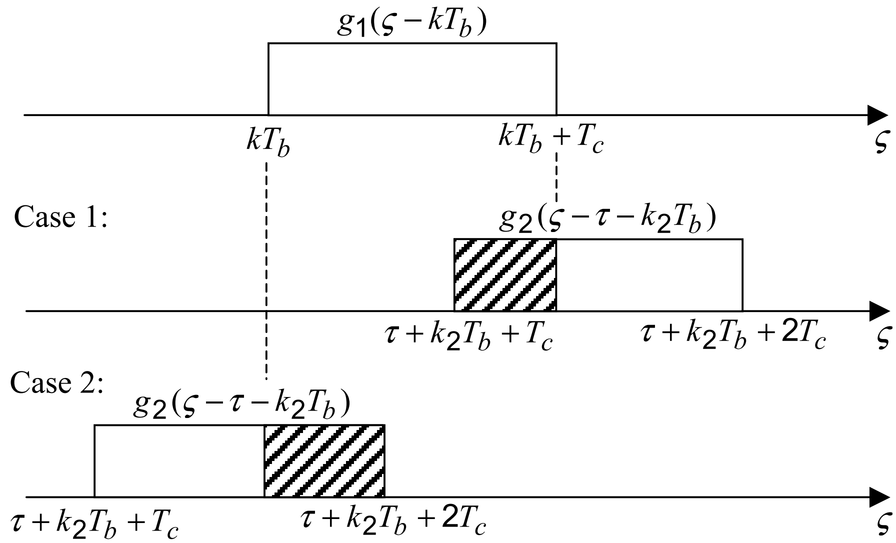
$$r_{1,k} = \int_{kT_b}^{(k+1)T_b} r_1(\varsigma) g_1(T_b - t + \varsigma)\Big|_{t=(k+1)T_b} d\varsigma = s'_{1,k} + cs'_{2,k} + n_k$$

$$s'_{1,k} = \sum_{k'=-\infty}^{\infty} m_{1,k'} \int_{kT_b}^{(k+1)T_b} g_1(\varsigma - k'T_b) g_1(T_b - t + \varsigma)\Big|_{t=(k+1)T_b} d\varsigma$$

$$= \sum_{k'=-\infty}^{\infty} m_{1,k'} \int_{kT_b}^{kT_b + T_c} g_1(\varsigma - k'T_b) d\varsigma = \sum_{k'=-\infty}^{\infty} m_{1,k'} \delta(k - k') = m_{1,k}$$

$$s'_{2,k} = \sum_{k_2=-\infty}^{\infty} m_{2,k_2} \int_{kT_b}^{(k+1)T_b} g_2(\varsigma - \tau - k_2 T_b) g_1(T_b - t + \varsigma)\Big|_{t=(k+1)T_b} d\varsigma$$

$$= \sum_{k_2=-\infty}^{\infty} m_{2,k_2} \int_{kT_b}^{kT_b + T_c} g_2(\varsigma - \tau - k_2 T_b) d\varsigma$$



Case 1: $kT_b < \tau + k_2T_b + T_c \le kT_b + T_c \Rightarrow kT_b - T_c - \tau < k_2T_b \cap k_2T_b \le kT_b - \tau$

$$\Rightarrow k - \frac{T_c}{T_b} - \frac{\tau}{T_b} < k_2 \le k - \frac{\tau}{Tb}.$$

Case 2: $kT_b < \tau + k_2T_b + 2T_c \le kT_b + T_c \Rightarrow kT_b - 2T_c - \tau < k_2T_b \cap k_2T_b \le kT_b - T_c - \tau$

$$\Rightarrow k - \frac{2T_c}{T_b} - \frac{\tau}{T_b} < k_2 \le k - \frac{T_c}{T_b} - \frac{\tau}{T_b}.$$

Normalized inner product is

$$\frac{\int_{kT_b}^{kT_b+T_c} g_2(\varsigma - \tau - k_2T_b)d\varsigma}{\int_{kT_b}^{kT_b+T_c} 1 d\varsigma} = \begin{cases} \int_{\tau+k_2T_b+T_c}^{kT_b+T_c} \frac{1}{T_c} d\varsigma, & k - \frac{T_c}{T_b} - \frac{\tau}{T_b} < k_2 \le k - \frac{\tau}{Tb} \\ \int_{kT_b}^{\tau+k_2T_b+2T_c} \frac{1}{T_c} d\varsigma, & k - \frac{2T_c}{T_b} - \frac{\tau}{T_b} < k_2 \le k - \frac{T_c}{T_b} - \frac{\tau}{T_b} \\ 0, & \textit{for all other } k_2 \end{cases}$$

$$= \begin{cases} \frac{(k - k_2)T_b - \tau}{T_c}, & k - \frac{T_c}{T_b} - \frac{\tau}{T_b} < k_2 \le k - \frac{\tau}{Tb} \\ \frac{(k_2 - k)T_b + 2T_c + \tau}{T_c}, & k - \frac{2T_c}{T_b} - \frac{\tau}{T_b} < k_2 \le k - \frac{T_c}{T_b} - \frac{\tau}{T_b} \\ 0, & \textit{for all other } k_2 \end{cases}$$

The sampled noise is

$$n_k = \int_{kT_b}^{kT_b+T_c} n(\varsigma) g_1(T_b - t + \varsigma)\Big|_{t=(k+1)T_b} d\varsigma = \int_{kT_b}^{kT_b+T_c} n(\varsigma) g_1(\varsigma - kT_b) d\varsigma .$$

$$E\{n_k\} = \int_{kT_b}^{kT_b+T_c} E\{n(\varsigma)\} g_1(\varsigma - kT_b) d\varsigma = 0$$

$$E\{n_k n_q\} = \int_{kT_b}^{kT_b+T_c} \int_{qT_b}^{qT_b+T_c} E\{n(\varsigma) n(\xi)\} g_1(\varsigma - kT_b) g_1(\xi - qT_b) d\varsigma d\xi = \frac{N_0}{2} \delta(k-q)$$

Define

$$\lambda(k, k_2, T_b, T_c, \tau) \equiv \begin{cases} \dfrac{(k-k_2)T_b - \tau}{T_c}, & k - \dfrac{T_c}{T_b} - \dfrac{\tau}{T_b} < k_2 \le k - \dfrac{\tau}{Tb} \\ \dfrac{(k_2 - k)T_b + 2T_c + \tau}{T_c}, & k - \dfrac{2T_c}{T_b} - \dfrac{\tau}{T_b} < k_2 \le k - \dfrac{T_c}{T_b} - \dfrac{\tau}{T_b} \\ 0, & \textit{for all other } k_2 \end{cases}$$

and $\tilde{k}_{2,1}(k) \equiv k - \left\lceil \dfrac{\tau}{T_b} \right\rceil$, $\tilde{k}_{2,2}(k) \equiv k - \left\lceil \dfrac{T_c}{T_b} + \dfrac{\tau}{T_b} \right\rceil$.

The demodulated signal is

$$r_{1,k} = m_{1,k} + c \sum_{j=1}^{2} m_{2,\tilde{k}_{2,j}(k)} \lambda(k, \tilde{k}_{2,j}(k), T_b, T_c, \tau) + n_k$$

$$f_{R_{1,k}|H_0}\left(r_{1,k} \middle| H_0\right) = \sum_{m_{2,\tilde{k}_{2,1}(k)}, } \sum_{m_{2,\tilde{k}_{2,2}(k)} \in \{-1,1\}} \Pr\left[m_{2,\tilde{k}_{2,1}(k)} ; m_{2,\tilde{k}_{2,2}(k)} \right] \frac{1}{\sqrt{\pi N_0}} \times \exp\left\{ - \frac{\left[r_{1,k} + 1 + c \sum_{j=1}^{2} m_{2,\tilde{k}_{2,j}(k)} \lambda(k, \tilde{k}_{2,j}(k), T_b, T_c, \tau) \right]^2}{N_0} \right\}$$

$$f_{R_{1,k}|H_1}\left(r_{1,k} \middle| H_1\right) = \sum_{m_{2,\tilde{k}_{2,1}(k)}, } \sum_{m_{2,\tilde{k}_{2,2}(k)} \in \{-1,1\}} \Pr\left[m_{2,\tilde{k}_{2,1}(k)} ; m_{2,\tilde{k}_{2,2}(k)} \right] \frac{1}{\sqrt{\pi N_0}}$$

$$\times \exp\left\{-\frac{\left[r_{1,k}-1+c\sum_{j=1}^{2} m_{2,\tilde{k}_{2,j}(k)}\lambda(k,\tilde{k}_{2,j}(k),T_b,T_c,\tau)\right]^2}{N_0}\right\}$$

$$P_e=\frac{1}{2}\int_0^{\infty} f_{R_{1,k}|H_0}\left(r_{1,k}\middle|H_0\right)dr_{1,k}+\frac{1}{2}\int_{-\infty}^{0} f_{R_{1,k}|H_1}\left(r_{1,k}\middle|H_1\right)dr_{1,k}$$

$$=\sum_{m_{2,\tilde{k}_{2,1}(k)}}\sum_{,m_{2,\tilde{k}_{2,2}(k)}\in\{-1,1\}}\Pr\left[m_{2,\tilde{k}_{2,1}(k)};m_{2,\tilde{k}_{2,2}(k)}\right]$$

$$\times\phi\left\{\frac{\sqrt{2}}{\sqrt{N_0}}\left[-1+c\sum_{j=1}^{2} m_{2,\tilde{k}_{2,j}(k)}\lambda(k,\tilde{k}_{2,j}(k),T_b,T_c,\tau)\right]\right\}$$