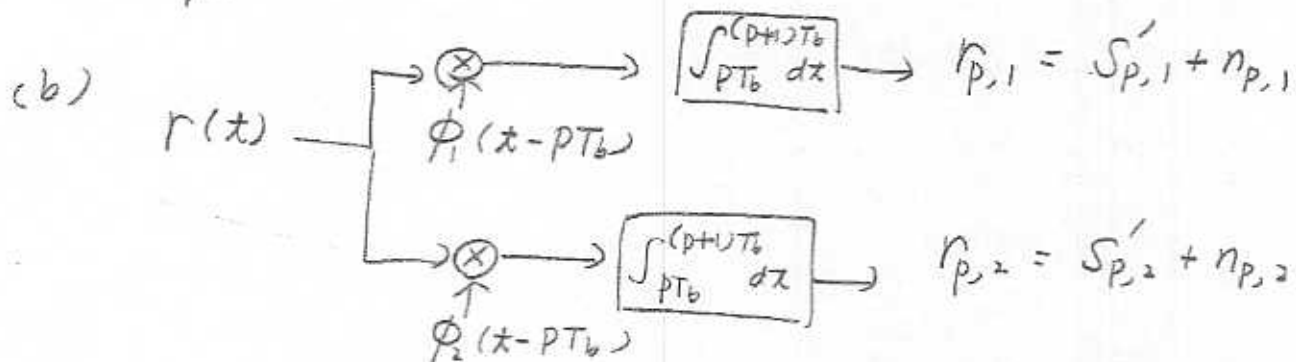


# Homework 3 Solution

Spring of 2003

1. (a)  $\phi_1(x) = \cos(\omega_c x)$ ,  $0 \leq x < T_b$

$\phi_2(x) = \sin(\omega_c x)$ ,  $0 \leq x < T_b$



(c)  $r(x) = s(x) \otimes h(x) + n(x)$

$$= \sum_{k=-\infty}^{\infty} \{ a_k \cos[\omega_c(x - kT_b)]$$

$$+ b_k \sin[\omega_c(x - kT_b)] \} \times \text{rect}\left(\frac{x - kT_b}{T_b}\right)$$

$$\otimes h(x) + n(x)$$

$$= s'(x) + n(x),$$

where  $s'(x) = \sum_{k=-\infty}^{\infty} \{ a_k \cos[\omega_c(x - kT_b)]$

$$+ b_k \sin[\omega_c(x - kT_b)] \} \text{rect}\left[\frac{x - kT_b}{T_b}\right]$$

$$\otimes h(x), \quad \text{rect}(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$S'(x) = \sum_{k=-\infty}^{\infty} a_k \int_0^{B_f T_b} \cos[\omega_c(x - kT_b - \tau)]$$

$$\times \text{rect}\left[\frac{x - kT_b - \tau}{T_b}\right] h(\tau) d\tau$$

$$+ \sum_{k=-\infty}^{\infty} b_k \int_0^{B_f T_b} \sin[\omega_c(x - kT_b - \tau)]$$

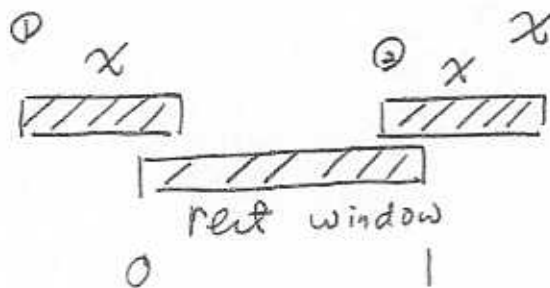
$$\times \text{rect}\left[\frac{x - kT_b - \tau}{T_b}\right] h(\tau) d\tau$$

Since  $0 \leq \tau \leq B_f T_b$  and  $pT_b \leq x < (p+1)T_b$

$$(p - B_f)T_b \leq x - \tau < (p+1)T_b$$

$$\Rightarrow p - B_f \leq \frac{x - \tau}{T_b} < p + 1$$

$$\Rightarrow p - k - B_f \leq \underbrace{\frac{x - kT_b - \tau}{T_b}} < p + 1 - k$$



$$\therefore \textcircled{1} \quad 0 < p + 1 - k \quad \text{and} \quad \textcircled{2} \quad p - k - B_f < 1, \quad p \in \mathbb{Z}$$

$$\Rightarrow k < p + 1 \quad \cap \quad k > p - B_f - 1, \quad k \in \mathbb{Z}$$

$$\Rightarrow p - B_f \leq k \leq p \quad (\text{the transmitted symbol indices interfering the current})$$

symbol  $p$  at the receiver)

$$\begin{aligned}
 S_{p,1}' &= \int_{pT_b}^{(p+1)T_b} s'(x) \cos [\omega_c (x - pT_b)] dx \\
 &= \sum_{k=p-B_f}^p a_k \int_{pT_b}^{(p+1)T_b} \int_{t-(k+1)T_b}^{t-kT_b} \cos [\omega_c (x - kT_b - \tau)] \\
 &\quad \times \cos [\omega_c (x - pT_b)] h(\tau) d\tau dt \\
 &\quad + \sum_{k=p-B_f}^p b_k \int_{pT_b}^{(p+1)T_b} \int_{t-(k+1)T_b}^{t-kT_b} \sin [\omega_c (x - kT_b - \tau)] \\
 &\quad \times \cos [\omega_c (x - pT_b)] h'(\tau) d\tau dt
 \end{aligned}$$

$$\begin{aligned}
 S_{p,2}' &= \sum_{k=p-B_f}^p a_k \int_{pT_b}^{(p+1)T_b} \int_{t-(k+1)T_b}^{t-kT_b} \cos [\omega_c (x - kT_b - \tau)] \\
 &\quad \times \sin [\omega_c (x - pT_b)] h(\tau) d\tau dt \\
 &\quad + \sum_{k=p-B_f}^p b_k \int_{pT_b}^{(p+1)T_b} \int_{t-(k+1)T_b}^{t-kT_b} \sin [\omega_c (x - kT_b - \tau)] \\
 &\quad \times \sin [\omega_c (x - pT_b)] h'(\tau) d\tau dt
 \end{aligned}$$

$$n_{p,1} = \int_{pT_b}^{(p+1)T_b} n(x) \cos [\omega_c (x - pT_b)] dx$$

$$n_{p,2} = \int_{pT_b}^{(p+1)T_b} n(x) \sin [\omega_c (x - pT_b)] dx$$

$$r_{p,1} = s'_{p,1} + n_{p,1}$$

$$r_{p,2} = s'_{p,2} + n_{p,2}, \quad \forall p \in \mathbb{Z}$$

$$(d) \quad E[n_{p,1}] = \int_{pT_b}^{(p+1)T_b} E[n(x)] \cos[\omega_c(x-pT_b)] dx$$

$$= 0$$

$$E[n_{p,2}] = \int_{pT_b}^{(p+1)T_b} E[n(x)] \sin[\omega_c(x-pT_b)] dx$$

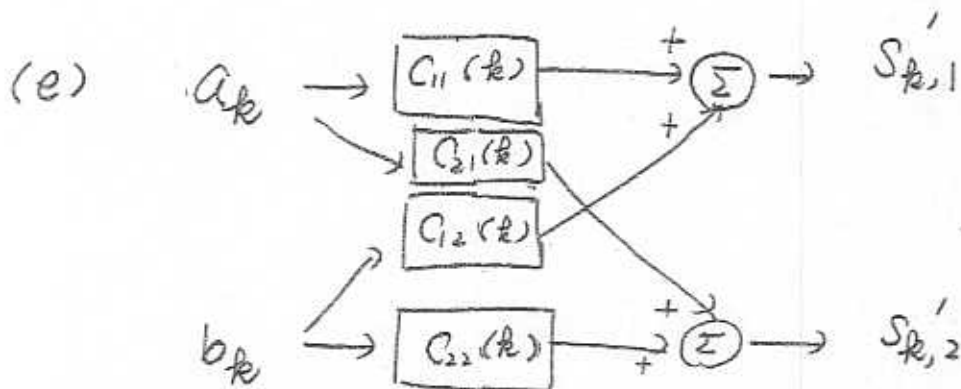
$$= 0$$

$$E\{n_{p,i} n_{q,j}\}$$

$$= \int_{pT_b}^{(p+1)T_b} \int_{qT_b}^{(q+1)T_b} E[n(x) n(x')] \phi_i(x-pT_b) \phi_j(x'-qT_b) dx dx'$$

$$= \delta(p-q) \delta(i-j) \int_{pT_b}^{(p+1)T_b} \frac{N_0}{2} \phi_i^2(x-pT_b) dx$$

$$= \frac{N_0 T_b}{4} \delta(p-q) \delta(i-j), \quad \text{where } \omega_c \gg 1$$



$$C_{11}(k) = \int_{pT_b}^{(p+1)T_b} \int_0^{B_f T_b} \cos[\omega_c(x - kT_b - \tau)]$$

$$\text{rect}\left[\frac{x - kT_b - \tau}{T_b}\right] h(\tau) d\tau \cos[\omega_c(x - pT_b)]$$

$$\text{rect}\left[\frac{t - pT_b}{T_b}\right] dt$$

$$C_{12}(k) = \int_{pT_b}^{(p+1)T_b} \int_0^{B_f T_b} \sin[\omega_c(x - kT_b - \tau)]$$

$$\text{rect}\left[\frac{(x - kT_b - \tau)}{T_b}\right] h(\tau) \cos[\omega_c(x - pT_b)]$$

$$\text{rect}\left[\frac{x - pT_b}{T_b}\right] dt$$

$$C_{21}(k) = \int_{pT_b}^{(p+1)T_b} \int_0^{B_f T_b} \cos[\omega_c(x - kT_b - \tau)]$$

$$\text{rect}\left[\frac{(x - kT_b - \tau)}{T_b}\right] h(\tau) \sin[\omega_c(x - pT_b)]$$

$$\text{rect}\left[\frac{x - pT_b}{T_b}\right] dt$$

$$C_{22}(k) = \int_{pT_b}^{(p+1)T_b} \int_0^{B_f T_b} \sin[\omega_c(x - kT_b - \tau)]$$

$$\text{rect}\left[\frac{(x - kT_b - \tau)}{T_b}\right] h(\tau) \sin[\omega_c(x - pT_b)]$$

$$\text{rect}\left[\frac{x - pT_b}{T_b}\right] dt$$

(f)

$$\text{Let } x' = x - pT_b, \quad l = p - k,$$

$$C_{11}(l) = \int_0^{B_f T_b} h(\tau) \int_0^{T_b} \cos[\omega_c(x' + lT_b - \tau)]$$

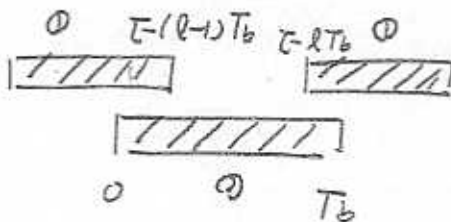
$$\text{rect}\left[\frac{x' + lT_b - \tau}{T_b}\right] \cos[\omega_c x'] \text{rect}\left[\frac{x'}{T_b}\right]$$

$$dx' d\tau$$

$$\text{Window } \textcircled{1} : 0 \leq x' + lT_b - \tau < T_b$$

$$\Rightarrow \tau - lT_b \leq x' < \tau - (l-1)T_b$$

$$\text{Window } \textcircled{2} : 0 \leq x' < T_b$$



$\therefore$  overlap of  $\textcircled{1}$  &  $\textcircled{2}$

$$\Rightarrow 0 < \tau - (l-1)T_b \leq T_b$$

$$\text{or } 0 < \tau - lT_b \leq T_b$$

$$\Rightarrow (l-1)T_b < \tau \leq lT_b$$

$$\text{or } lT_b < \tau \leq (l+1)T_b$$

$$\begin{aligned}
 C_{11}(l) &= \int_{(l-1)T_b}^{lT_b} h(\tau) \int_0^{\tau-(l-1)T_b} \cos[\omega_c(x'+lT_b-\tau)] \\
 &\quad \times \cos[\omega_c x'] dx' d\tau \\
 &+ \int_{lT_b}^{(l+1)T_b} h(\tau) \int_{\tau-lT_b}^{T_b} \cos[\omega_c(x'+lT_b-\tau)] \\
 &\quad \times \cos[\omega_c x'] dx' d\tau
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 C_{12}(l) &= \int_{(l-1)T_b}^{lT_b} h(\tau) \int_0^{\tau-(l-1)T_b} \sin[\omega_c(x'+lT_b-\tau)] \\
 &\quad \times \cos[\omega_c x'] dx' d\tau
 \end{aligned}$$

$$\begin{aligned}
 &+ \int_{lT_b}^{(l+1)T_b} h(\tau) \int_{\tau-lT_b}^{T_b} \sin[\omega_c(x'+lT_b-\tau)] \\
 &\quad \times \cos[\omega_c x'] dx' d\tau
 \end{aligned}$$

$$\begin{aligned}
 C_{21}(l) &= \int_{(l-1)T_b}^{lT_b} h(\tau) \int_0^{\tau-(l-1)T_b} \cos[\omega_c(x'+lT_b-\tau)] \\
 &\quad \times \sin[\omega_c x'] dx' d\tau
 \end{aligned}$$

$$\begin{aligned}
 &+ \int_{lT_b}^{(l+1)T_b} h(\tau) \int_{\tau-lT_b}^{T_b} \cos[\omega_c(x'+lT_b-\tau)] \\
 &\quad \times \sin[\omega_c x'] dx' d\tau
 \end{aligned}$$

$$C_{22}(l) = \int_{(l-1)T_b}^{lT_b} h(\tau) \int_0^{\tau-(l-1)T_b} \sin[\omega_c(x'+lT_b-\tau)] \times \sin[\omega_c x'] dx' d\tau$$

$$+ \int_{lT_b}^{(l+1)T_b} h(\tau) \int_{\tau-lT_b}^{T_b} \sin[\omega_c(x'+lT_b-\tau)] \times \sin[\omega_c x'] dx' d\tau$$

$$\int_x^y \cos[\omega_c(x'+lT_b-\tau)] \cos(\omega_c x') dx'$$

$$= \int_x^y \frac{1}{2} \left\{ \cos[\omega_c(lT_b-\tau)] + \cos[2\omega_c x' + \omega_c(lT_b-\tau)] \right\} dx'$$

$$= \frac{(y-x)}{2} \cos[\omega_c(lT_b-\tau)] + \frac{\sin[2\omega_c x' + \omega_c(lT_b-\tau)]}{4\omega_c} \Big|_x^y$$

$\omega_c \gg 1$

$$\approx \frac{(y-x)}{2} \cos[\omega_c(lT_b-\tau)]$$

$$\int_x^y \sin[\omega_c(x'+lT_b-\tau)] \cos[\omega_c x'] dx'$$

$$= \int_x^y \frac{1}{2} \left\{ \sin[\omega_c(lT_b-\tau)] + \sin[2\omega_c x' + \omega_c(lT_b-\tau)] \right\} dx'$$

$$= \frac{(y-x)}{2} \sin[\omega_c(lT_b-\tau)] - \frac{\cos[2\omega_c x' + \omega_c(lT_b-\tau)]}{4\omega_c} \Big|_x^y$$

$\omega_c \gg 1$



$$\approx \frac{(y-x)}{2} \sin [\omega_c (lT_b - \tau)]$$

$$\int_x^y \sin [\omega_c (t' + lT_b - \tau)] \sin (\omega_c t') dt'$$

$$= \int_x^y \frac{1}{2} \left\{ \cos [\omega_c (lT_b - \tau)] - \cos [2\omega_c t' + \omega_c (lT_b - \tau)] \right\} dt'$$

$$= \frac{(y-x)}{2} \cos [\omega_c (lT_b - \tau)] - \frac{\sin [2\omega_c t' + \omega_c (lT_b - \tau)]}{4\omega_c} \Big|_x^y$$

$\omega_c \gg 1$

$$\approx \frac{(y-x)}{2} \cos [\omega_c (lT_b - \tau)]$$

$$\int_x^y \cos [\omega_c (t' + lT_b - \tau)] \sin (\omega_c t') dt'$$

$$= \int_x^y \frac{1}{2} \left\{ -\sin [\omega_c (lT_b - \tau)] + \sin [2\omega_c t' + \omega_c (lT_b - \tau)] \right\} dt'$$

$$= -\frac{(y-x)}{2} \sin [\omega_c (lT_b - \tau)] + \frac{\cos [2\omega_c t' + \omega_c (lT_b - \tau)]}{4\omega_c} \Big|_x^y$$

$\omega_c \gg 1$

$$\approx \frac{(x-y)}{2} \sin [\omega_c (lT_b - \tau)]$$

$$\text{Hence } C_{11}(l) = C_{22}(l)$$

$$= \int_{(l-1)T_b}^{lT_b} h(\tau) \left[ \frac{\tau - (l-1)T_b}{2} \right] \cos[\omega_c(lT_b - \tau)] d\tau$$

$$+ \int_{lT_b}^{(l+1)T_b} h(\tau) \left[ \frac{T_b - \tau + lT_b}{2} \right] \cos[\omega_c(lT_b - \tau)] d\tau$$

$$C_{12}(l) = -C_{21}(l)$$

$$= \int_{(l-1)T_b}^{lT_b} h(\tau) \left[ \frac{\tau - (l-1)T_b}{2} \right] \sin[\omega_c(lT_b - \tau)] d\tau$$

$$+ \int_{lT_b}^{(l+1)T_b} h(\tau) \left[ \frac{T_b - \tau + lT_b}{2} \right] \sin[\omega_c(lT_b - \tau)] d\tau$$

$$p - B_f \leq k \leq p, \quad l = p - k \Rightarrow 0 \leq l \leq B_f$$

$$\therefore S_{p,1}' = \sum_{l=0}^{B_f} a_{p-l} C_{11}(l) + \sum_{l=0}^{B_f} b_{p-l} C_{12}(l)$$

$$S_{p,2}' = \sum_{l=0}^{B_f} a_{p-l} C_{21}(l) + \sum_{l=0}^{B_f} b_{p-l} C_{22}(l)$$

$$= - \sum_{l=0}^{B_f} a_{p-l} C_{12}(l) + \sum_{l=0}^{B_f} b_{p-l} C_{11}(l)$$

Refine  $\tilde{S}_p' = S_{p,1}' + \sqrt{-1} S_{p,2}'$

$$\tilde{S}_p = a_p + \sqrt{-1} b_p$$

$$\tilde{C}(l) = C_{11}(l) - \sqrt{-1} C_{12}(l)$$

Therefore,

$$S_{p,1}' = \operatorname{Re} \{ \tilde{S}_p' \}$$

$$= \operatorname{Re} \left\{ \sum_{\ell=0}^{B_f} \tilde{S}_{p-\ell} \tilde{Q}(\ell) \right\}$$

$$S_{p,2}' = \operatorname{Im} \{ \tilde{S}_p' \}$$

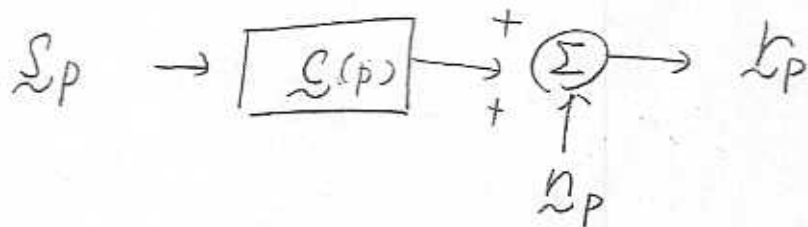
$$= \operatorname{Im} \left\{ \sum_{\ell=0}^{B_f} \tilde{S}_{p-\ell} \tilde{Q}(\ell) \right\}$$

$$\therefore \tilde{S}_p' = \sum_{\ell=0}^{B_f} \tilde{S}_{p-\ell} \tilde{Q}(\ell), \quad p \in \mathbb{Z}$$

Define  $\tilde{N}_p = n_{p,1} + \sqrt{-1} n_{p,2}$

$$\tilde{L}_p = r_{p,1} + \sqrt{-1} r_{p,2}$$

Then the system can be modeled as



$$\tilde{L}_p = \tilde{S}_p \otimes \tilde{Q}(p) + \tilde{N}_p$$

2.

(a)

$$l = -1, \Rightarrow$$

$$C_{11}(-1) = \int_{0^+}^0 \delta(\tau) \left[ \frac{-\tau}{2} \right] \cos [\omega_c (-T_b - \tau)] d\tau = 0$$

$$\cos [\omega_c (-T_b - \tau)] d\tau = 0$$

$$C_{12}(-1) = \int_{0^+}^0 \delta(\tau) \left[ \frac{-\tau}{2} \right] \sin [\omega_c (-T_b - \tau)] d\tau = 0$$

$$l = 0, \Rightarrow$$

$$C_{11}(0) = \int_{-T_b^+}^0 \delta(\tau) \left[ \frac{\tau + T_b}{2} \right] \cos [\omega_c \tau] d\tau = \frac{T_b}{2}$$

$$C_{12}(0) = \int_{-T_b^+}^0 \delta(\tau) \left[ \frac{\tau + T_b}{2} \right] \sin [-\omega_c \tau] d\tau = 0$$

$$l = 40, \Rightarrow$$

$$C_{11}(40) = \int_{-39T_b^+}^{40T_b} 0.631 \delta(\tau - 40T_b) \left[ \frac{\tau - 39T_b}{2} \right] \times \cos [\omega_c (40T_b - \tau)] d\tau = \frac{T_b}{2} \times 0.6310$$

$$C_{12}(40) = \int_{39T_b}^{40T_b} 0.631 \delta(\tau - 40T_b) \left[ \frac{\tau - 39T_b}{2} \right] \\ \times \sin[\omega_c(40T_b - \tau)] d\tau = 0$$

$$l = 80,$$

$$C_{11}(80) = \int_{79T_b}^{80T_b} 0.1 \delta(\tau - 80T_b) \left[ \frac{\tau - 79T_b}{2} \right] \\ \times \cos[\omega_c(80T_b - \tau)] d\tau \\ = \frac{T_b}{2} \times 0.1$$

$$C_{12}(80) = 0$$

$$l = 120,$$

$$C_{11}(120) = \int_{119T_b}^{120T_b} 0.01 \delta(\tau - 120T_b) \left[ \frac{\tau - 119T_b}{2} \right] \\ \times \cos[\omega_c(120T_b - \tau)] d\tau \\ = \frac{T_b}{2} \times 0.01$$

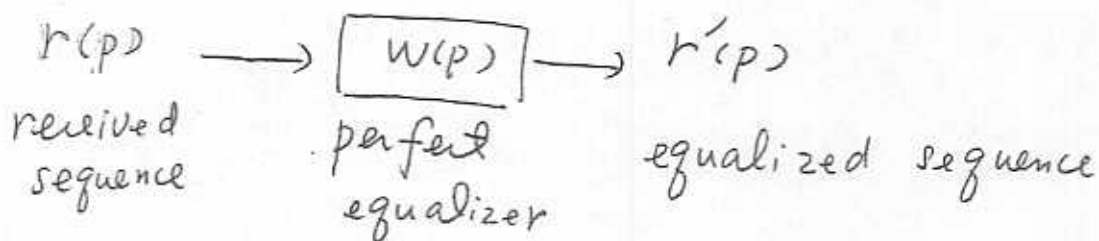
$$\therefore \zeta(l) = \frac{T_b}{2} \left[ \delta(l) + 0.631 \delta(l - 40) \right. \\ \left. + 0.1 \delta(l - 80) + 0.01 \delta(l - 120) \right]$$

(b)

The perfect equalizer's  $z$ -transform should be

$$W(z) = \frac{1}{\frac{T_b}{2} [1 + 0.6310z^{-40} + 0.1z^{-80} + 0.01z^{-120}]}$$

$$\text{or } r'(p) = -0.6310 r'(p-40) - 0.1 r'(p-80) - 0.01 r'(p-120) + r(p), \quad \forall p \in \mathbb{Z}$$



(c)

From Matlab,  $W(0) = 4 \times 10^7$ ,

$$W(1) = W(2) = \dots = W(15) \approx 0.$$

3.

$$(a) \quad \phi(x) = \cos(\omega_c x), \quad 0 \leq x < T_b$$

(b)

$$r(x) \rightarrow \begin{array}{c} \otimes \\ \downarrow \\ \phi(x - pT_b) \end{array} \rightarrow \boxed{\int_{pT_b}^{(p+1)T_b} dt} \rightarrow r_p = s_p' + n_p$$

(c)

$$r(x) = s(x) \otimes h(x) + n(x)$$

$$= \sum_{k=-\infty}^{\infty} m_k \cos[\omega_c(x - kT_b)] \text{rect}\left[\frac{x - kT_b}{T_b}\right]$$

$$\otimes h(x) + n(x) = s'(x) + n(x),$$

$$\text{where } \text{rect}(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$s'(x) = s(x) \otimes h(x)$$

$$= \sum_{k=-\infty}^{\infty} m_k \int_0^{B_f T_b} \cos[\omega_c(x - kT_b - \tau)]$$

$$\times \text{rect}\left[\frac{x - kT_b - \tau}{T_b}\right] h(\tau) d\tau$$

$$s_p' = \int_{pT_b}^{(p+1)T_b} s'(x) \cos[\omega_c(x - pT_b)] dx$$

$$= \sum_{k=p-B_f}^p m_k \int_{pT_b}^{(p+1)T_b} \int_{t-(k+1)T_b}^{t-kT_b} \cos[\omega_c(t-kT_b-\tau)] \\ \times \cos[\omega_c(t-pT_b)] h(\tau) d\tau dt$$

$$n_p = \int_{pT_b}^{(p+1)T_b} n(t) \cos[\omega_c(t-pT_b)] dt$$

$$r_p = s_p' + n_p, \quad \forall p \in \mathbb{Z}$$

(d)

$$E[n_p] = 0$$

$$E\{n_p n_q\} = \frac{N_0 T_b}{4} \delta(p-q)$$

(e)

$$a_k \rightarrow \boxed{C(k)} \rightarrow s_k'$$

$$C(k) = \int_{pT_b}^{(p+1)T_b} \int_0^{B_f T_b} \cos[\omega_c(t-kT_b-\tau)]$$

$$\times \text{rect}\left[\frac{t-kT_b-\tau}{T_b}\right] h(\tau) d\tau \cos[\omega_c(t-pT_b)]$$

$$\times \text{rect}\left[\frac{t-pT_b}{T_b}\right] dt$$



(f) Let  $l = p - k$ ,  $x' = x - pT_b$ ,

$$C(l) = \int_{(l-1)T_b}^{lT_b} h(\tau) \int_0^{\tau - (l-1)T_b} \cos[\omega_c(x' + lT_b - \tau)] \\ \times \cos[\omega_c x'] dx' d\tau$$

$$+ \int_{lT_b}^{(l+1)T_b} h(\tau) \int_{\tau - lT_b}^{T_b} \cos[\omega_c(x' + lT_b - \tau)] \\ \times \cos[\omega_c x'] dx' d\tau$$

$$\approx \int_{(l-1)T_b}^{lT_b} h(\tau) \left[ \frac{\tau - (l-1)T_b}{2} \right] \cos[\omega_c(lT_b - \tau)] d\tau$$

$$+ \int_{lT_b}^{(l+1)T_b} h(\tau) \left[ \frac{T_b - \tau + lT_b}{2} \right] \cos[\omega_c(lT_b - \tau)] d\tau$$

$$\hat{r}_p = \hat{s}_p \otimes C(p) + n_p, \quad \text{all are real}$$

4. (a) Similar to 2(a),

$$C(l) = \frac{T_b}{2} \left[ \delta(l) + 0.6310 \delta(l-40) \right. \\ \left. + 0.1 \delta(l-80) + 0.01 \delta(l-120) \right]$$

(b) Similar to 2 (b),

$$W(z) = \frac{2}{T_b [1 + 0.6310z^{-40} + 0.1z^{-80} + 0.01z^{-120}]}$$

or 
$$r'_p = -0.6310 r'_{p-40} - 0.1 r'_{p-80} - 0.01 r'_{p-120} + r_p, \quad \forall p \in \mathbb{Z}$$



(c) Similar to 2 (c),

$$W(0) = 4 \times 10^7, \quad W(1) = W(2) = \dots = W(15) = 0$$

---

```
%EE7000 Spring of 2003
```

```
%Homework 4
```

```
%Problem 1
```

```
%(c)
```

```
Tb = 0.05e-6;
```

```
c = zeros(1,121);
```

```
c(1) = Tb/2;
```

```
c(41) = Tb/2 * 0.6310;
```

```
c(81) = Tb/2 * 0.1;
```

```
c(121) = Tb/2 * 0.01;
```

```
a = c;
```

```
a(41) = -a(41);
```

```
a(81) = -a(81);
```

```
a(121) = -a(121);
```

```
w16 = filter(1,a,[1 zeros(1,15)]);
```

```
w256 = filter(1,a,[1 zeros(1,255)]);
```

```
subplot(311)
```

```
plot(0:1:120, c)
```

```
title('digital channel filter')
```

```
subplot(312);
```

```
plot(0:1:15, w16);
```

```
title('perfect equalizer approximation with 16 taps');
```

```
subplot(313);
```

```
plot(0:1:255, w256);
```

```
title('perfect equalizer approximation with 256 taps');
```

