$$f_{IR}(r_1, r_2, \cdots r_N) = \frac{\partial^N}{\partial r_1 \partial r_2 \cdots \partial r_N} P\left(\bigcap_{i=1}^{N} R_i \le r_i\right)$$

Let $P_i(r_i) = \int_{-\infty}^{\infty} \int_{-\infty}^{r_i - s_i} f_{SV}(s_i)\, f_{In}(n_i)\, dn_i\, ds_i$

$$P\left(\bigcap_{i=1}^{N} R_i \le r_i\right) = \prod_{i=1}^{N} P_i(r_i)$$

$$\frac{\partial}{\partial r_i} P_i(r_i) = \frac{\partial}{\partial r_i} \int_{-\infty}^{\infty} \int_{-\infty}^{r_i - s_i} f_{SV}(s_i)\, f_{In}(n_i)\, dn_i\, ds_i$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial r_i} \int_{-\infty}^{r_i - s_i} f_{SV}(s_i)\, f_{In}(n_i)\, dn_i\, ds_i$$

$$= \int_{-\infty}^{\infty} f_{SV}(s_i)\, f_{In}(r_i - s_i)\, ds_i$$

$$= f_{SV}(r_i) \circledast f_{In}(r_i)$$

$$= f_{IR_i}(r_i)$$

$$= f_{IR}(r_i) \quad (R_i\text{'s are identical})$$

$$\therefore \quad f_{IR}(r_1, r_2, \cdots r_N) = \prod_{i=1}^{N} f_{IR}(r_i)$$

(R_i's are statistially independent)

$\Rightarrow$ $r_1, r_2, \cdots r_N$ are i.i.d $\forall N$

$\Rightarrow$ r_p's are i.i.d.

(b). For any combinations of $\{r_{p_i}\}_{1\le i\le N}$, it is i.i.d according to (a), if we exclude $P_i = 3k+2$, $\forall k$

If r_{3k+2} is considered, the only two random processes related to r_{3k+2} are r_{3k} and r_{3k+1}.

We consider the sub set of the $\{r_{p_i}\}$ as $\{r_{3k}, r_{3k+1}, r_{3k+2}\}$, $\forall k$.

$$f_{\tilde{S}, n}(S_{3k}, S_{3k+1}, S_{3k+2}, n_{3k}, n_{3k+1}, n_{3k+2})$$

$$= f_{\tilde{S}}(S_{3k}, S_{3k+1}, S_{3k+2}) \prod_{i=3k}^{3k+2} f_n(n_i)$$

We know $f_{\tilde{S}_{3k}}(S) = f_{\tilde{S}_{3k+1}}(S) = \left[\frac{1}{2}\delta(S+1) + \frac{1}{2}\delta(S-1)\right]$

$$f_{\tilde{S}}(3k+2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\tilde{S}}(S_{3k}, S_{3k+1}, S_{3k+2})\, dS_{3k}\, dS_{3k+1}$$

$$= \left[\frac{1}{2}\delta(S_{3k+2}+1) + \frac{1}{2}\delta(S_{3k+2}-1)\right]$$

$\therefore f_{\tilde{S}_{3k}} = f_{\tilde{S}_{3k+1}} = f_{\tilde{S}_{3k+2}} \Rightarrow$ they have identical statistics

Since $f_{R_p}(r_p) = f_{\tilde{S}_p}(r_p) \circledast f_n(r_p)$

$\therefore$ r_p's have identical statistics

$$f_{\tilde{S}_{3k}}(S_{3k})\, f_{\tilde{S}_{3k+1}}(S_{3k+1})\, f_{\tilde{S}_{3k+2}}(S_{3k+2})$$

$$\begin{aligned}
= \frac{1}{8} [& \delta(S_{3k}+1)\,\delta(S_{3k+1}+1)\,\delta(S_{3k+2}+1) \\
& + \delta(S_{3k}+1)\,\delta(S_{3k+1}+1)\,\delta(S_{3k+2}-1) \\
& + \delta(S_{3k}+1)\,\delta(S_{3k+1}-1)\,\delta(S_{3k+2}+1) \\
& + \delta(S_{3k}+1)\,\delta(S_{3k+1}-1)\,\delta(S_{3k+2}-1) \\
& + \delta(S_{3k}-1)\,\delta(S_{3k+1}+1)\,\delta(S_{3k+2}+1) \\
& + \delta(S_{3k}-1)\,\delta(S_{3k+1}+1)\,\delta(S_{3k+2}-1) \\
& + \delta(S_{3k}-1)\,\delta(S_{3k+1}-1)\,\delta(S_{3k+2}+1) \\
& + \delta(S_{3k}-1)\,\delta(S_{3k+1}-1)\,\delta(S_{3k+2}-1)]
\end{aligned}$$

However, $f_{\tilde{S}}(S_{3k}, S_{3k+1}, S_{3k+2})$

$$\begin{aligned}
= \frac{1}{4} [& \delta(S_{3k}+1)\,\delta(S_{3k+1}+1)\,\delta(S_{3k+2}-1) \\
& + \delta(S_{3k}+1)\,\delta(S_{3k+1}-1)\,\delta(S_{3k+2}+1) \\
& + \delta(S_{3k}-1)\,\delta(S_{3k+1}+1)\,\delta(S_{3k+2}+1) \\
& + \delta(S_{3k}-1)\,\delta(S_{3k+1}-1)\,\delta(S_{3k+2}-1)]
\end{aligned}$$

Hence $f_{\tilde{S}}(S_{3k}, S_{3k+1}, S_{3k+2}) \neq \prod_{i=3k}^{3k+2} f_{\tilde{S}_i}(S_i)$

$S_{3k}, S_{3k+1}, S_{3k+2}$ are not statistically dependant.

Check the correlation between $r_{3k}, r_{3k+1}, r_{3k+2}$

$$E[r_{3k} r_{3k+1} r_{3k+2}]$$

$$= E[(S_{3k} + n_{3k})(S_{3k+1} + n_{3k+1})(S_{3k+2} + n_{3k+2})]$$

$$= E[S_{3k} S_{3k+1} S_{3k+2}] + E[S_{3k} S_{3k+1}] E[n_{3k+2}]$$
$$+ E[S_{3k} S_{3k+2}] E[n_{3k+1}] + E[S_{3k}] E[n_{3k+1}] E[n_{3k+2}]$$
$$+ E[S_{3k+1} S_{3k+2}] E[n_{3k}] + E[S_{3k+1}] E[n_{3k}] E[n_{3k+2}]$$
$$+ E[S_{3k+2}] E[n_{3k}] E[n_{3k+1}] + E[n_{3k}] E[n_{3k+1}] E[n_{3k+2}]$$

$$E[S_{3k} S_{3k+1} S_{3k+2}] = \frac{1}{4}(1) + \frac{1}{4}(1) + \frac{1}{4}(1) + \frac{1}{4}(1) = 1$$

$$\therefore E[r_{3k} r_{3k+1} r_{3k+2}] = 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 1$$

$$E[r_{3k}] = E[r_{3k+1}] = E[r_{3k+2}] = 0$$

$$\therefore E[r_{3k} r_{3k+1} r_{3k+2}] \neq \prod_{i=3k}^{3k+2} E[r_i]$$

$\Rightarrow$ r_p's have identical but not independent statistics.

2. $r(t) = s(t) \circledast h(t) + n(t)$

$= s(t) + 0.8\, s(t - 3.73\, T_b) + n(t)$

$r_p = r(t = pT_b) = s_p + 0.8\, s(pT_b - 3.73\, T_b) + n_p$

$= s_p + 0.8\, s_{p-3} + n_p$

Since s_p has identical statistics, r_p also has identical statistics. And $f_R(r)$

$= \left[\frac{1}{2}\delta(r+1) + \frac{1}{2}\delta(r-1)\right] \circledast \left[\frac{1}{2}\delta(r+0.8) + \frac{1}{2}\delta(r-0.8)\right] \circledast \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{r^2}{2\sigma^2}}$

$E[r_p] = E[s_p] + 0.8\, E[s_{p-3}] + E[n_p] = 0$

$E[r_p r_{p+3}] = E[(s_p + 0.8\, s_{p-3} + n_p)(s_{p+3} + 0.8\, s_p + n_{p+3})]$

$= E[s_p s_{p+3}] + 0.8\, E[s_p^2]$

$+ 0.8\, E[s_{p-3} s_{p+3}] + 0.64\, E[s_{p-3} s_p]$

$= 0.8\, E[s_p^2] = 0.8\left(\frac{1}{2}(-1)^2 + \frac{1}{2}(1)^2\right)$

$= 0.8$

and $E[r_{p+3}] = E[r_p] = 0$

$\therefore\ E[r_p r_{p+3}] \neq E[r_p]\, E[r_{p+3}]$

r_p's have identical but not independant statistics

3. D_1: region of the decision associated with $\hat{m} = 1$

D_{-1}: region of the decision associated with $\hat{m} = -1$

$$P_e = \frac{1}{2}\int_{D_{-1}} f_{R|1}(r)\,dr + \frac{1}{2}\int_{D_1} f_{R|-1}(r)\,dr$$

$$f_{R|1} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r-1)^2}{2\sigma^2}}, \quad f_{R|-1} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r+1)^2}{2\sigma^2}}$$

$$P_e = \frac{1}{2}\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r-1)^2}{2\sigma^2}}\,dr + \frac{1}{2}\int_{0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r+1)^2}{2\sigma^2}}\,dr$$

$$x = \frac{r-1}{\sigma} \qquad y = \frac{r+1}{\sigma}$$

$$= \frac{1}{2}\int_{-\infty}^{\frac{-1}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx + \frac{1}{2}\int_{\frac{1}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}}\,dy$$

$$= \frac{1}{2}\phi\left(\frac{-1}{\sigma}\right) + \frac{1}{2}\left[1 - \phi\left(\frac{1}{\sigma}\right)\right]$$

$$= \phi\left(\frac{-1}{\sigma}\right)$$

4.

$$f_{R|m}(r) = f_{NV}(r|m) \circledast f_{NV}(0.8r) \circledast f_{n}(n)$$

$$= \delta(r-m) \circledast \left[\frac{1}{2}\delta(r-0.8) + \frac{1}{2}\delta(r+0.8) \right] \circledast \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{r^2}{2\sigma^2}}$$

$$= \int_{-\infty}^{\infty} \delta(r-x-m) \left[\frac{1}{2}\delta(x-0.8) + \frac{1}{2}\delta(x+0.8) \right] dx \circledast \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{r^2}{2\sigma^2}}$$

$$= \left[\frac{1}{2}\delta(r-0.8-m) + \frac{1}{2}\delta(r+0.8-m) \right] \circledast \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{r^2}{2\sigma^2}}$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r-0.8-m)^2}{2\sigma^2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r+0.8-m)^2}{2\sigma^2}}$$

$$\therefore\ f_{R|1}(r) = \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r-1.8)^2}{2\sigma^2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r-0.2)^2}{2\sigma^2}}$$

$$f_{R|-1}(r) = \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r+0.2)^2}{2\sigma^2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r+1.8)^2}{2\sigma^2}}$$

$$P_e = \frac{1}{2}\int_{D_{-1}} f_{R|1}(r)\,dr + \frac{1}{2}\int_{D_1} f_{R|-1}(r)\,dr$$

$$= \frac{1}{4}\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r-1.8)^2}{2\sigma^2}}dr + \frac{1}{4}\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r-0.2)^2}{2\sigma^2}}dr$$

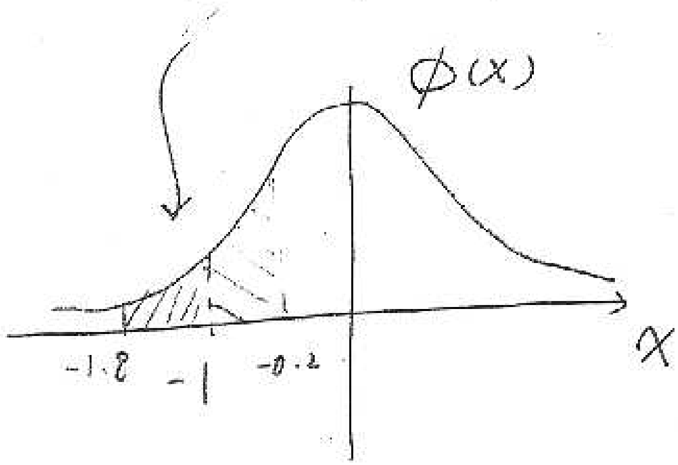
$$+ \frac{1}{4}\int_{0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r+0.2)^2}{2\sigma^2}}dr + \frac{1}{4}\int_{0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r+1.8)^2}{2\sigma^2}}dr$$

$$= \frac{1}{4}\phi\left(\frac{-1.8}{\sigma}\right) + \frac{1}{4}\phi\left(\frac{-0.2}{\sigma}\right)$$

$$+ \frac{1}{4} - \frac{1}{4}\phi\left(\frac{0.2}{\sigma}\right) + \frac{1}{4} - \frac{1}{4}\phi\left(\frac{1.8}{\sigma}\right)$$

$$= \frac{1}{2}\phi\left(-\frac{1.8}{\sigma}\right) + \frac{1}{2}\phi\left(\frac{-0.2}{\sigma}\right)$$

$$> \phi\left(\frac{-1}{\sigma}\right) \text{ in Prob. 3.}$$



5.

(a) $\Psi_{R_i}(j\omega) = E[e^{j\omega r_i}]$

$= E[e^{j\omega \sum_{k=0}^{N} a_k m_{i-k} + n_i}]$

Since m_i is i.i.d. and m_i is independent of n_i,

$$\Psi_{R_i}(j\omega) = \prod_{k=0}^{N} E[e^{j\omega a_k m_{i-k}}] \times E[e^{j\omega n_i}]$$

$E[e^{j\omega a_k m_{i-k}}]$

$$= \int_{-\infty}^{\infty} f_{M_{i-k}}(m_{i-k})\, e^{j\omega a_k m_{i-k}}\, dm_{i-k}$$

$$= \int_{-\infty}^{\infty} \left[\tfrac{1}{2}\delta(m_{i-k}+1) + \tfrac{1}{2}\delta(m_{i-k}-1)\right] \times e^{j\omega a_k m_{i-k}}\, dm_{i-k}$$

$$= \frac{1}{2}\left[e^{j\omega a_k} + e^{-j\omega a_k}\right]$$

$$= \cos(\omega a_k)$$

$$E[e^{j\omega n_i}]$$

$$= \int_{-\infty}^{\infty} f_{in}(n_i)\, e^{j\omega n_i}\, dn_i$$

$$= \mathcal{F}^*\left[\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{n_i^2}{2\sigma^2}}\right]$$

$$= e^{-\frac{\sigma^2\omega^2}{2}}$$

$$\Psi_{R_i}(j\omega) = \prod_{k=0}^{N} \cos(\omega a_k)\, e^{-\frac{\sigma^2\omega^2}{2}}$$

(b) By Taylor Expansion,

$$\cos(\omega a_k) = \left[1 - \frac{1}{2!}\omega^2 a_k^2 + \frac{1}{4!}\omega^4 a_k^4 - \cdots\right]$$

$$e^{-\frac{\sigma^2\omega^2}{2}} = 1 - \frac{\sigma^2\omega^2}{2} + \frac{\sigma^4\omega^4}{8} - \cdots$$

$$\Psi_{R_i}(j\omega) = \sum_{n=0}^{\infty} E[r_i^n]\, \frac{(j\omega)^n}{n!}$$

$\therefore$ $E[r_i^4]$ = the Taylor coeffiient associated with $\frac{(j\omega)^4}{4!}$ or $\frac{\omega^4}{4!}$

Hence the coefficient of ω^4 in

$1/R_i(j\omega)$ is

$$\sum_{k=0}^{N} \frac{a_k^4}{4!} + \sum_{k=0}^{N} \sum_{\substack{l=0 \\ l \neq k}}^{N} \frac{a_k^2}{2} \frac{a_l^2}{2}$$

$$+ \sum_{k=0}^{N} \left(-\frac{a_k^2}{2}\right)\left(-\frac{\sigma^2}{2}\right) + \frac{\sigma^4}{8}$$

$$= \sum_{k=0}^{N} \frac{a_k^4}{24} + \sum_{k=0}^{N} \sum_{l=0}^{N} \frac{a_k^2 a_l^2}{4} - \sum_{k=0}^{N} \frac{a_k^4}{4}$$

$$+ \sum_{k=0}^{N} \frac{a_k^2 \sigma^2}{4} + \frac{\sigma^4}{8}$$

$$= -\frac{5}{24} \sum_{k=0}^{N} a_k^4 + \sum_{k=0}^{N} \sum_{l=0}^{N} \frac{a_k^2 a_l^2}{4} + \sum_{k=0}^{N} \frac{a_k^2 \sigma^2}{4}$$

$$+ \frac{\sigma^4}{8} \quad \cdots (*)$$

$$E[r_i^4] = (*) \times 4! = -5 \sum_{k=0}^{N} a_k^4 + 6 \sum_{k=0}^{N} \sum_{l=0}^{N} a_k^2 a_l^2$$

$$+ 6 \sum_{k=0}^{N} a_k^2 \sigma^2 + 3\sigma^4$$

6.

(a) $H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$

$$= \int_{-\infty}^{\infty} \left[\delta(t) + 0.8\,\delta(t - 3T_b) \right] e^{-j\omega t}\, dt$$

$$= 1 + 0.8\, e^{-j3\omega T_b}$$

$$W(\omega) = \frac{1}{1 + 0.8\, e^{-j3\omega T_b}}$$

(b) In proakis book,

$$S(t) = Re\{ v(t)\, e^{j\omega_c t} \}$$

$$R_{SS}(\tau) = Re\{ R_{vv}(\tau)\, e^{j\omega_c \tau} \}$$

where $v(t) = \sum_{n=-\infty}^{\infty} I_n\, g(t - nT_b)$

$$\Rightarrow S_{SS}(\omega) = \mathcal{F}\{ R_{SS}(\tau) \}$$

$$= \frac{1}{2} \{ S_{vv}(\omega - \omega_c) + S_{vv}(-\omega - \omega_c) \}$$

$$R_{vv}(t, t+\tau) = \frac{1}{2} E[v^*(t)\, v(t+\tau)]$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \{ R_{II}(m-n)\, g^*(t - nT_b)$$

$$\times\, g(t + \tau - mT_b) \}$$

where $R_{II}(m-n) = \frac{1}{2} E[I_n^* I_m]$

$$E[I_n^* I_m] = \frac{1}{4}(1^2+1^2) \times 4 = 2, \quad m = n$$

$$= \frac{1}{16}(1^2+1^2) \times 4$$

$$+ \frac{1}{16}[(1-j)(1-j+1+j-1+j-1-j)$$

$$+(1+j)(1-j+1+j-1+j-1-j)$$

$$+(-1+j)(1-j+1+j-1+j-1-j)$$

$$+(-1-j)(1-j+1+j-1+j-1-j)]$$

$$= 0, \quad m \neq n$$

$$\therefore R_{II}(m-n) = \frac{1}{2} \times 2\, \delta(m-n)$$

$$= \delta(m-n)$$

$S_{vv}(\omega) = \frac{1}{T_b} |G(\omega)|^2$ according to Eq (4.4-18), p 205, Proakis book, where $G(\omega) = \mathcal{F}[g(t)]$

$$\therefore S_{ss}(\omega) = \frac{1}{2}\left\{ \frac{1}{T_b}|G(\omega-\omega_c)|^2 + \frac{1}{T_b}|G(-\omega-\omega_c)|^2\right.$$

Choose $g(t) = \begin{cases} 1, & 0 \le t \le T_b \\ 0, & \text{elsewhere} \end{cases}$

$$G(\omega) = \int_0^{T_b} e^{-j\omega t}\,dt$$

$$= \frac{e^{-j\omega T_b} - 1}{-j\omega} = \frac{e^{-j\frac{\omega T_b}{2}}\left[e^{-j\frac{\omega T_b}{2}} - e^{j\frac{\omega T_b}{2}}\right]}{-j\omega}$$

$$= e^{-j\frac{\omega T_b}{2}} \frac{(-2j)\ \sin\left(\frac{\omega T_b}{2}\right)}{-j\omega}$$

$$= T_b \operatorname{Sinc}\left(\frac{\omega T_b}{2}\right) e^{j\frac{\omega T_b}{2}}$$

$$|G(\omega)|^2 = T_b^2 \left|\operatorname{sinc}\left(\frac{\omega T_b}{2}\right)\right|^2$$

$\operatorname{Sinc}\left(\frac{\omega T_b}{2}\right) = 0$, when $\frac{\omega T_b}{2} = k\pi$

$\Rightarrow$ $\omega = \frac{2\pi}{T_b}$ is the first nullity in

$S_{ss}(\omega)$ $\Rightarrow$ the effative bandwidth is

$\left[\omega_c - \frac{2\pi}{T_b}, \omega_c + \frac{2\pi}{T_b}\right]$ and $\left[-\omega_c - \frac{2\pi}{T_b}, -\omega_c + \frac{2\pi}{T_b}\right]$

$$S_{RR}(\omega) = S_{SS}(\omega)\ \underbrace{|H(\omega)|^2 |W(\omega)|^2}_{1} + |W(\omega)|^2 \mathcal{N}(\omega)$$

$$= \frac{T_b}{2}\left\{ \operatorname{sinc}^2\left(\frac{(\omega-\omega_c)T_b}{2}\right) + \operatorname{sinc}^2\left(\frac{(\omega+\omega_c)T_b}{2}\right)\right\}$$

$$+ \frac{1}{|1+0.8e^{-j\omega T_b}|^2}\ \frac{\mathcal{N}_0}{2}$$

(c) Noise amplication factor $\equiv \dfrac{n_{Eq}}{n_E}$

$$n_{Eq} = \int_{-\omega_c-\frac{2\pi}{T_b}}^{-\omega_c+\frac{2\pi}{T_b}} \frac{1}{|1+0.8e^{-j\omega T_b}|^2}\ \frac{d\omega}{2\pi}$$

$$+ \int_{\omega_c-\frac{2\pi}{T_b}}^{\omega_c+\frac{2\pi}{T_b}} \frac{1}{|1+0.8e^{-j\omega T_b}|^2}\ \frac{d\omega}{2\pi}$$

$$n_E = 2\int_{\omega_c-\frac{2\pi}{T_b}}^{\omega_c+\frac{2\pi}{T_b}} \frac{\mathcal{N}_0}{2}\ \frac{d\omega}{2\pi}$$

$$= \frac{2\mathcal{N}_0}{T_b}$$