CHAPTER 2

2.1 (a) Let

$w_k = x + jy$

$p(-k) = a + jb$

We may then write

$f = w_k p^*(-k)$

$= (x + jy)(a - jb)$

$= (ax + by) + j(ay - bx)$

Let

$f = u + jv$

with

$u = ax + by$

$v = ay - bx$

Hence,

$\frac{\partial u}{\partial x} = a$ $\frac{\partial u}{\partial y} = b$

$\frac{\partial v}{\partial y} = a$ $\frac{\partial v}{\partial x} = -b$

From these results we immediately see that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

In other words, the product term $w_k p^*(-k)$ satisfies the Cauchy-Rieman equations, and so this term is analytic.

(b) Let

$f = w_k{}^* p(-k)$

$= (x - jy)\,(a + jb)$

$= (ax + by) + j(bx - ay)$

Let

$f = u + jv$

with

$u = ax + by$

$v = bx - ay$

Hence,

$$\frac{\partial u}{\partial x} = a \qquad \frac{\partial u}{\partial y} = b$$

$$\frac{\partial v}{\partial x} = b \qquad \frac{\partial v}{\partial y} = -a$$

From these results we immediately see that

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

In other words, the product term $w_k^* p(\text{-k})$ does not satisfy the Cauchy-Rieman equations, and so this term is *not* analytic.

2.2 (a) From the Wiener-Hopf equation, we have

$$\mathbf{w}_o = \mathbf{R}^{-1}\mathbf{p} \qquad (1)$$

We are given

$$\mathbf{R} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

$$\mathbf{p} = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

Hence, the inverse matrix $\mathbf{R}^{-1}$ is

$$\mathbf{R}^{-1} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}^{-1}$$

$$= \frac{1}{0.75}\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

Using Eq. (1), we therefore get

$$\mathbf{w}_o = \frac{1}{0.75}\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

(b) The minimum mean-square error is

$$J_{\min} = \sigma_d^2 - \mathbf{p}^H \mathbf{w}_o$$

$$= \sigma_d^2 - [0.5,\ 0.25]\begin{bmatrix}0.5\\0\end{bmatrix}$$

$$= \sigma_d^2 - 0.25$$

(c) The eigenvalues of matrix **R** are roots of the characteristic equation

$$(1-\lambda)^2 - (0.5)^2 = 0$$

That is, the two roots are

$$\lambda_1 = 0.5 \text{ and } \lambda_2 = 1.5$$

The associated eigenvectors are defined by

$$\mathbf{R}\,\mathbf{q} = \lambda\,\mathbf{q}$$

For $\lambda_1 = 0.5$, we have

$$\begin{bmatrix}1 & 0.5\\0.5 & 1\end{bmatrix}\begin{bmatrix}q_{11}\\q_{12}\end{bmatrix} = 0.5\begin{bmatrix}q_{11}\\q_{12}\end{bmatrix}$$

Expanding

$$q_{11} + 0.5\,q_{12} = 0.5\,q_{11}$$

$$0.5\,q_{11} + q_{12} = 0.5\,q_{12}$$

Therefore,

$$q_{11} = -\,q_{12}$$

Normalizing the eigenvector $\mathbf{q}_1$ to unit length, we therefore have

$$\mathbf{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

Similarly, for the eigenvalue $\lambda_2 = 1.5$, we may show that

$$\mathbf{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

Accordingly, we may express the Wiener filter in terms of its eigenvalues and eigen-vectors as follows:

$$\mathbf{w}_o = \left(\sum_{i=1}^{2}\frac{1}{\lambda_i}\mathbf{q}_i\mathbf{q}_i^H\right)\mathbf{p}$$

$$= \left(\begin{bmatrix}1\\-1\end{bmatrix}[1, -1] + \frac{1}{3}\begin{bmatrix}1\\1\end{bmatrix}[1, 1]\right)\begin{bmatrix}0.5\\0.25\end{bmatrix}$$

$$= \Bigg(\underbrace{\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}}_{\frac{1}{\lambda_1}\mathbf{q}_1\mathbf{q}_1^H} + \underbrace{\frac{1}{3}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}}_{\frac{1}{\lambda_2}\mathbf{q}_2\mathbf{q}_2^H}\Bigg)\underbrace{\begin{bmatrix}0.5\\0.25\end{bmatrix}}_{\mathbf{p}}$$

2.3 (a) From the Wiener-Hopf equation we have

$$\mathbf{w}_o = \mathbf{R}^{-1}\mathbf{p} \tag{1}$$

We are given

$$\mathbf{R} = \begin{bmatrix}1 & 0.5 & 0.25\\0.5 & 1 & 0.5\\0.25 & 0.5 & 1\end{bmatrix}$$

and

$$\mathbf{p} = [0.5\ 0.25\ 0.125]^T$$

Hence, the use of these values in Eq. (1) yields

$$\mathbf{w}_o = \mathbf{R}^{-1}\mathbf{p}$$

$$= \begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 0.25 \\ 0.125 \end{bmatrix}$$

$$= \begin{bmatrix} 1.33 & -0.67 & 0 \\ -0.67 & 1.67 & -0.67 \\ 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.25 \\ 0.125 \end{bmatrix}$$

$$= [0.5\ 0\ 0]^T$$

(b) The minimum mean-square error is

$$J_{\min} = \sigma_d^2 - \mathbf{p}^H \mathbf{w}_o$$

$$= \sigma_d^2 - [0.5\ 0.25\ 0.125] \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix}$$

$$= \sigma_d^2 - 0.25$$

(c) The eigenvalues of matrix **R** are

$$\lambda = 0.4069, \quad 0.75, \quad 1.8431$$

The corresponding eigenvectors constitute the orthogonal matrix:

$$\mathbf{Q} = \begin{bmatrix} -0.4544 & -0.7071 & 0.5418 \\ 0.7662 & 0 & 0.6426 \\ -0.4544 & 0.7071 & 0.5418 \end{bmatrix}$$

Accordingly, we may express the Wiener filter in terms of its eigenvalues and eigenvectors as follows:

$$\mathbf{w}_o = \left(\sum_{i=1}^{3} \frac{1}{\lambda_i}\mathbf{q}_i\mathbf{q}_i^H\right)\mathbf{p}$$

$$= \left\{\left[\frac{1}{0.4069}\begin{bmatrix}-0.4544\\0.7662\\-0.4544\end{bmatrix}\begin{bmatrix}-0.4544 & 0.7662 & -0.4554\end{bmatrix}\right.\right.$$

$$+\frac{1}{0.75}\begin{bmatrix}-0.7071\\0\\0.7071\end{bmatrix}\begin{bmatrix}-0.7071 & 0 & 0.7071\end{bmatrix}$$

$$\left.+\frac{1}{1.8431}\begin{bmatrix}0.5418\\0.6426\\0.5418\end{bmatrix}\begin{bmatrix}0.5418 & 0.6426 & 0.5418\end{bmatrix}\right\}\times\begin{bmatrix}0.5\\0.25\\0.125\end{bmatrix}$$

$$= \left\{\frac{1}{0.4069}\begin{bmatrix}0.2065 & -0.3482 & 0.2065\\-0.3482 & 0.5871 & -0.3482\\0.2065 & -0.3482 & 0.2065\end{bmatrix}\right.$$

$$+\frac{1}{0.75}\begin{bmatrix}0.5 & 0 & -0.5\\0 & 0 & 0\\-0.5 & 0 & 0.5\end{bmatrix}$$

$$\left.+\frac{1}{1.8431}\begin{bmatrix}0.2935 & 0.3482 & 0.2935\\0.3482 & 0.4129 & 0.3482\\0.2935 & 0.3482 & 0.2935\end{bmatrix}\right\}\begin{bmatrix}0.5\\0.25\\0.125\end{bmatrix}$$

2.4 By definition, the correlation matrix

$$\mathbf{R} = E[\mathbf{u}(n)\mathbf{u}^H(n)]$$

where

$$= \left(\mathbf{s}(n)\mathbf{s}^H(n) + \frac{1}{\sigma_\alpha^2}\mathbf{R}_v \right)^{-1} \begin{bmatrix} e^{j\omega\tau} \\ e^{j\omega(\tau-1)} \\ \vdots \\ e^{j\omega(\tau-M+1)} \end{bmatrix}$$

2.6 The optimum filtering solution is defined by the Wiener-Hopf equation

$$\mathbf{R}\mathbf{w}_o = \mathbf{p} \tag{1}$$

for which the minimum mean-square error equals

$$J_{\min} = \sigma_d^2 - \mathbf{p}^H \mathbf{w}_o \tag{2}$$

Combine Eqs. (1) and (2) into a single relation:

$$\begin{bmatrix} \sigma_d^2 & \mathbf{p}^H \\ \mathbf{p} & \mathbf{R} \end{bmatrix} \begin{bmatrix} 1 \\ -\mathbf{w}_o \end{bmatrix} = \begin{bmatrix} J_{\min} \\ \mathbf{0} \end{bmatrix}$$

Define

$$\mathbf{A} = \begin{bmatrix} \sigma_d^2 & \mathbf{p}^H \\ \mathbf{p} & \mathbf{R} \end{bmatrix} \tag{3}$$

Since

$$\sigma_d^2 = E[d(n)d^*(n)],$$

$$\mathbf{p} = E[\mathbf{u}(n)d^*(n)] \text{ , and}$$

$$\mathbf{R} = E[\mathbf{u}(n)\mathbf{u}^*(n)],$$

we may rewrite Eq. (3) as

$$\mathbf{A} = \begin{bmatrix} E[d(n)d^*(n)] & E[d(n)]\mathbf{u}^H(n) \\ E[\mathbf{u}(n)d^*(n)] & E[\mathbf{u}(n)\mathbf{u}^H(n)] \end{bmatrix}$$

$$= E\left\{\begin{bmatrix} d(n) \\ \mathbf{u}(n) \end{bmatrix} \begin{bmatrix} d^*(n), & \mathbf{u}^H(n) \end{bmatrix}\right\}$$

The minimum mean-square error equals

$$J_{\min} = \sigma_d^2 - \mathbf{p}^H \mathbf{w}_o \tag{4}$$

Eliminate σ_d^2 between Eqs. (1) and (4):

$$J(\mathbf{w}) = J_{\min} + \mathbf{p}^H \mathbf{w}_o - \mathbf{p}^H \mathbf{w} - \mathbf{w}^H \mathbf{p} + \mathbf{w}^H \mathbf{R} \mathbf{w} \tag{5}$$

Eliminate **p** between (2) and (5):

$$J(\mathbf{w}) = J_{\min} + \mathbf{w}_o^H \mathbf{R} \mathbf{w}_o - \mathbf{w}_o^H \mathbf{R} \mathbf{w} - \mathbf{w}^H \mathbf{R} \mathbf{w}_o + \mathbf{w}^H \mathbf{R} \mathbf{w} \tag{6}$$

where we have used the property $\mathbf{R}^H = \mathbf{R}$. We may rewrite Eq. (6) simply as

$$J(\mathbf{w}) = J_{\min} + (\mathbf{w} - \mathbf{w}_o)^H \mathbf{R} (\mathbf{w} - \mathbf{w}_o)$$

which clearly show that $J(\mathbf{w}_o) = J_{\min}$.

2.7 The minimum mean-square error equals

$$J_{\min} = \sigma_d^2 - \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p} \tag{1}$$

Using the spectral theorem, we may express the correlation matrix **R** as

$$\mathbf{R} = \mathbf{Q} \Lambda \mathbf{Q}^H$$

$$= \sum_{k=1}^{M} \lambda_k \mathbf{q}_k \mathbf{q}_k^H$$

Hence, the inverse of **R** equals

$$\mathbf{R}^{-1} = \sum_{k=1}^{M} \frac{1}{\lambda_k} \mathbf{q}_k \mathbf{q}_k^H \tag{2}$$

$$\mathbf{a}_m = \begin{bmatrix} 0.8719 \\ -0.9129 \\ 0.2444 \end{bmatrix}$$

The last entry in the 4-by-1 vector $\mathbf{p}$ is therefore

$$\mathbf{r}^H_{M\text{-}m}\mathbf{a}_m = -0.0436 - 0.0912 + 0.1222$$

$$= -0.0126$$

2.10 $$J_{\min} = \sigma_d^2 - \mathbf{p}^H\mathbf{w}_o$$

$$= \sigma_d^2 - \mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}$$

When $m = 0$,

$$J_{\min} = \sigma_d^2$$

$$= 1.0$$

When $m = 1$,

$$J_{\min} = 1 - 0.5 \times \frac{1}{1.1} \times 0.5$$

$$= 0.9773$$

When $m = 2$,

$$J_{\min} = 1 - [0.5 \;\; -0.4]\begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}^{-1}\begin{bmatrix} 0.5 \\ -0.4 \end{bmatrix}$$

$$= 1 - 0.6781$$

$$= 0.3219$$

When $m = 3$,

$$J_{min} = 1 - \begin{bmatrix} 0.5 & -0.4 & -0.2 \end{bmatrix} \begin{bmatrix} 1.1 & 0.5 & 0.1 \\ 0.5 & 1.1 & 0.5 \\ 0.1 & 0.5 & 1.1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ -0.4 \\ -0.2 \end{bmatrix}$$

$= 1 - 0.6859$
$= 0.3141$

When $m = 4$,

$J_{min} = 1 - 0.6859$
$= 0.3141$

Thus any further increase in the filter order beyond $m = 3$ does not produce any meaningful reduction in the minimum mean-square error.

2.11 (a) $u(n) = x(n) + \nu_2(n)$ (1)

$d(n) = -d(n\text{-}1) \times 0.8458 + \nu_1(n)$ (2)

$x(n) = d(n) + 0.9458x(n\text{-}1)$ (3)

$d(n) = x(n) - 0.9458x(n\text{-}1)$

Using Eqs. (2) and (3):

$x(n) - 0.9458x(n\text{-}1) = 0.8458[-x(n\text{-}1) + 0.9458x(n\text{-}2)] + \nu_1(n)$

Rearranging terms:

$x(n) = (0.9458-8.8458)x(n\text{-}1) + 0.8x(n\text{-}2) + \nu_1(n)$

$x(n) = 0.1x(n\text{-}1) + 0.8x(n\text{-}2) + \nu_1(n)$

(b) $u(n) = x(n) + \nu_2(n)$

where $x(n)$ are $\nu_2(n)$ are uncorrelated

Therefore, $\mathbf{R} = \mathbf{R}_x + \mathbf{R}_{\nu_2}$