CHAPTER 1

1.1 Let

\[ r_u(k) = E[u(n)u^*(n-k)] \]  
\[ r_y(k) = E[y(n)y^*(n-k)] \]

We are given that

\[ y(n) = u(n+a) - u(n-a) \]

Hence, substituting Eq. (3) into (2), and then using Eq. (1), we get

\[ r_y(k) = E[(u(n+a) - u(n-a))(u^*(n+a-k) - u^*(n-a-k))] \]
\[ = 2r_u(k) - r_u(2a+k) - r_u(-2a+k) \]

1.2 We know that the correlation matrix \( R \) is Hermitian; that is

\[ R^H = R \]

Given that the inverse matrix \( R^{-1} \) exists, we may write

\[ R^{-1}R^H = I \]

where \( I \) is the identity matrix. Taking the Hermitian transpose of both sides:

\[ RR^{-H} = I \]

Hence,

\[ R^{-H} = R^{-1} \]

That is, the inverse matrix \( R^{-1} \) is Hermitian.

1.3 For the case of a two-by-two matrix, we may

\[ R_y = R_u + R_v \]
\[
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\]

For \( \mathbf{R}_p \) to be nonsingular, we require

\[
\det(\mathbf{R}_p) = (r_{11} + \sigma^2)(r_{22} + \sigma^2) - r_{12}r_{21} > 0
\]

With \( r_{12} = r_{21} \) for real data, this condition reduces to

\[
(r_{11} + \sigma^2)(r_{22} + \sigma^2) - r_{12}r_{21} > 0
\]

Since this is quadratic in \( \sigma^2 \), we may impose the following condition on \( \sigma^2 \) for nonsingularity of \( \mathbf{R}_p \):

\[
\sigma^2 > \frac{1}{2}(r_{11} + r_{22})\sqrt{1 - \frac{4\Delta_r}{(r_{11} + r_{22})^2 - 1}}
\]

where \( \Delta_r = r_{11}r_{22} - r_{12}^2 \)

1.4 We are given

\[
\mathbf{R} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\]

This matrix is positive definite because

\[
a^T \mathbf{R} a = [a_1, a_2] \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}
\]

\[
= a_1^2 + 2a_1a_2 + a_2^2
\]
\[(a_1 + a_2)^2 > 0\] for all nonzero values of \(a_1\) and \(a_2\).

(Positive definiteness is stronger than nonnegative definiteness.)

But the matrix \(R\) is singular because

\[\det(R) = (1)^2 - (1)^2 = 0\]

Hence, it is possible for a matrix to be positive definite and yet it can be singular.

\[1.5\]

(a)

\[R_{M+1} = \begin{bmatrix} r(0) & r^H \\ r & R_M \end{bmatrix}\]  \(\tag{1}\)

Let

\[R_{M+1}^{-1} = \begin{bmatrix} a & b^H \\ b & C \end{bmatrix}\]  \(\tag{2}\)

where \(a\), \(b\) and \(C\) are to be determined. Multiplying (1) by (2):

\[I_{M+1} = \begin{bmatrix} r(0) & r^H \\ r & R_M \end{bmatrix} \begin{bmatrix} a & b^H \\ b & C \end{bmatrix}\]

where \(I_{M+1}\) is the identity matrix. Therefore,

\[r(0)a + r^H b = 1\]  \(\tag{3}\)

\[ra + R_M b = 0\]  \(\tag{4}\)

\[rb^H + R_M C = I_M\]  \(\tag{5}\)

\[r(0)b^H + r^HC = 0^T\]  \(\tag{6}\)

From Eq. (4):
\[ b = -R_M^{-1} r a \] 

Hence, from (3) and (7):

\[ a = \frac{1}{r(0) - r^H R_M^{-1} r} \] 

Correspondingly,

\[ b = -\frac{R_M r}{r(0) - r^H R_M^{-1} r} \] 

From (5):

\[ C = R_M^{-1} - R_M^{-1} r b^E \]

\[ = R_M^{-1} + \frac{R_M^H r^H R_M^{-1}}{r(0) - r^H R_M^{-1} r} \] 

As a check, the results of Eqs. (9) and (10) should satisfy Eq. (6).

\[ r(0)b^E + r^H C = -\frac{r(0)r^H R_M^{-1}}{r(0) - r^H R_M^{-1} r} + \frac{r^H R_M^{-1} r r^H R_M^{-1}}{r(0) - r^H R_M^{-1} r} = 0^T \]

We have thus shown that

\[ R_{M+1}^{-1} = \begin{bmatrix} 0; 0^T \\ 0; R_M^{-1} \\ 0; R_M^{-1} r H R_M^{-1} \end{bmatrix} + \alpha \begin{bmatrix} 1; -r^H R_M^{-1} \\ R_M^H; R_M^{-1} r H R_M^{-1} \end{bmatrix} \]

\[ = \begin{bmatrix} 0; 0^T \\ 0; R_M^{-1} \\ 0; R_M^{-1} r H R_M^{-1} \end{bmatrix} + \alpha \begin{bmatrix} 1; -r^H R_M^{-1} \end{bmatrix} \]
where the scalar $\theta$ is defined by Eq. (8):

\[ \mathbf{R}_{M+1} = \begin{bmatrix} \mathbf{R}_M; \mathbf{f}^* \\ \mathbf{e}^T \end{bmatrix} \begin{bmatrix} \mathbf{f}^* \\ \mathbf{e} \end{bmatrix} \begin{bmatrix} \mathbf{H}^* \\ \mathbf{f} \end{bmatrix} \begin{bmatrix} \mathbf{e}^T \mathbf{f} \end{bmatrix} \begin{bmatrix} \mathbf{f} \mathbf{e} \end{bmatrix} \begin{bmatrix} \mathbf{H} \mathbf{f} \end{bmatrix} \begin{bmatrix} \mathbf{f}^* \mathbf{H}^* \end{bmatrix} = 1 \] (11)

Let

\[ \mathbf{R}_{M+1}^{-1} = \begin{bmatrix} \mathbf{D} \\ \mathbf{e} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{f} \end{bmatrix} \] (12)

where \( \mathbf{D}, \mathbf{e} \) and \( \mathbf{f} \) are to be determined. Multiplying (11) by (12):

\[ \mathbf{I}_{M+1} = \begin{bmatrix} \mathbf{R}_M; \mathbf{f}^* \\ \mathbf{e}^T \mathbf{f} \end{bmatrix} \begin{bmatrix} \mathbf{D} \\ \mathbf{e} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{f} \end{bmatrix} \begin{bmatrix} \mathbf{f} \mathbf{e} \end{bmatrix} \begin{bmatrix} \mathbf{H} \mathbf{f} \end{bmatrix} \begin{bmatrix} \mathbf{f}^* \mathbf{H}^* \end{bmatrix} = 1 \]

Therefore

\[ \mathbf{R}_M \mathbf{D} + r^* \mathbf{H} = \mathbf{I} \] (13)

\[ \mathbf{R}_M \mathbf{e} + r^* \mathbf{f} = \mathbf{0} \] (14)

\[ r^* \mathbf{e} + r(0) \mathbf{f} = \mathbf{1} \] (15)

\[ r^* \mathbf{D} + r(0) \mathbf{e}^T = \mathbf{0}^T \] (16)

From (14):

\[ \mathbf{e} = - \mathbf{R}_M^{-1} r^* \mathbf{f} \] (17)

Hence, from (15) and (17):

\[ \mathbf{f} = \frac{1}{r(0) - r^* \mathbf{D}} \mathbf{R}_M^{-1} r^* \mathbf{f} \] (18)

Correspondingly,
\[ e = - \frac{R_M^{-1} \beta^*}{r(0) - r^{BT} R_M^{-1} \beta^*} \]  

(19)

From (13):

\[ D = R_M^{-1} \beta^* e^{HT} \]

\[ = R_M^{-1} \beta^* + \frac{\beta^* R_M^{-1} \beta^*}{r(0) - r^{BT} R_M^{-1} \beta^*} \]

(20)

As a check, the results of Eqs. (19) and (20) must satisfy Eq. (16). Thus

\[ r^{BT} D + r(0) \beta^* = r^{BT} R_M^{-1} \beta^* + \frac{r^{BT} R_M^{-1} \beta^* r^{BT} R_M^{-1}}{r(0) - r^{BT} R_M^{-1} \beta^*} \]

\[ = 0^T \]

We have thus shown that

\[ R_{M+1}^{-1} = \begin{bmatrix} R_M^{-1} & 0 \\ \cdots & \cdots \\ 0^T & 0 \end{bmatrix} + f \begin{bmatrix} R_M^{-1} \beta^* R_M^{-1} \\ \cdots \\ r^{BT} R_M^{-1} \beta^* \end{bmatrix} \]

where the scalar \( f \) is defined by Eq. (18).

(a) We express the difference equation describing the first-order AR process \( u(n) \) as

\[ u(n) = v(n) + w_1 u(n-1) \]

where \( w_1 = -\alpha \). Solving this equation by repeated substitution, we get

\[ u(n) = v(n) + w_1 v(n-1) + w_1 u(n-2) \]
\[ u(n) = \ldots = v(n) + w_1 v(n-1) + w_1^2 v(n-2) + \ldots + w_1^{n-1} v(1) \] (1)

Here we have used the initial condition

\[ u(0) = 0 \]

or equivalently

\[ u(1) = v(1) \]

Taking the expected value of both sides of Eq. (1) and using

\[ E[v(n)] = \mu \quad \text{for all } n, \]

we get the geometric series

\[ E[u(n)] = \mu + w_1 \mu + w_1^2 \mu + \ldots + w_1^{n-1} \mu \]

\[ = \begin{cases} 
\mu \frac{1 - w_1^n}{1 - w_1} & w_1 \neq 1 \\
\mu n, & w_1 = 1 
\end{cases} \]

This result shows that if \( \mu \neq 0 \), then \( E[u(n)] \) is a function of time \( n \). Accordingly, the AR process \( u(n) \) is not stationary. If, however, the AR parameter satisfies the condition:

\[ |w_1| < 1 \quad \text{or} \quad |w| < 1 \]

then

\[ E[(n)] \rightarrow \frac{\mu}{1 - w_1} \quad \text{as} \quad n \rightarrow \infty \]

Under this condition, we say that the AR process is asymptotically stationary to order one.

(b) When the white noise process \( v(n) \) has zero mean, the AR process \( u(n) \) will likewise have zero mean. Then
\begin{align*}
\text{var}[v(n)] &= \sigma_v^2 \\
\text{var}[u(n)] &= E[u^2(n)].
\end{align*}

(2)

Substituting Eq. (1) into (2), and recognizing that for the white noise process

\begin{align*}
E[v(n)v(k)] &= \begin{cases} 
\sigma_v^2, & n = k \\
0, & n \neq k
\end{cases}
\end{align*}

(3)

we get the geometric series

\begin{align*}
\text{var}[u(n)] &= \sigma_v^2 \left( 1 + w_1^2 + w_1^4 + \cdots + w_1^{2n-2} \right) \\
&= \sigma_v^2 \left( \frac{1 - w_1^{2n}}{1 - w_1^2} \right) \quad w_1 \neq 1 \\
&= \sigma_v^2 n, \quad w_1 = 1
\end{align*}

When \(|a_1| < 1\) or \(|w_1| < 1\), then

\begin{align*}
\text{var}[u(n)] &= \frac{\sigma_v^2}{1 - w_1^2} = \frac{\sigma_v^2}{1 - a_1^2} \quad \text{for large } n
\end{align*}

(c) The autocorrelation function of the AR process \(u(n)\) equals \(E[u(n)u(n-k)]\). Substituting Eq. (1) into this formula, and using Eq. (3), we get

\begin{align*}
E[u(n)u(n-k)] &= \sigma_v^2 \left( w_1^k + w_1^{k+2} + \cdots + w_1^{2n-2k} \right) \\
&= \sigma_v^2 \left( \frac{1 - w_1^{2n}}{1 - w_1^2} \right) \quad w_1 \neq 1 \\
&= \sigma_v^2 n, \quad w_1 = 1
\end{align*}
For \(|a_1| < 1\) or \(|w_1| < 1\), we may therefore express this autocorrelation function as
\[
\begin{align*}
    r(k) &= E[u(n)u(n-k)] \\
    &= \frac{\sigma_u^2 w_1^k}{1 - w_1^2} \quad \text{for large } n
\end{align*}
\]

Case 1: \(0 < a_1 < 1\)

In this case, \(w_1 = -a_1\) is negative, and \(r(k)\) varies with \(k\) as follows:

\[
\begin{array}{ccccccccc}
    \ldots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \ldots \\
    r(k) & \ldots & 0 & 1 & 2 & 1 & 0 & -1 & -2 & \ldots
\end{array}
\]

Case 2: \(-1 < a_1 < 0\)

In this case, \(w_1 = -a_1\) is positive and \(r(k)\) varies with \(k\) as follows:

\[
\begin{array}{ccccccccc}
    \ldots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \ldots \\
    r(k) & \ldots & -1 & 0 & -1 & 0 & -1 & 0 & -1 & \ldots
\end{array}
\]

1.7 (a) The second-order AR process \(u(n)\) is described by the difference equation:

\[u(n) = u(n-1) - 0.5u(n-2) + v(n)\]

Hence

\[w_1 = 1\]
\[w_2 = -0.5\]

and the AR parameters equal

\[a_1 = -1\]
\[a_2 = 0.5\]

Accordingly, we write the Yule-Walker equations as
\[
\begin{bmatrix}
    r(0) & r(1) \\
    r(1) & r(0)
\end{bmatrix}
\begin{bmatrix}
    1 \\
    -0.5
\end{bmatrix}
= \begin{bmatrix}
    r(1) \\
    r(2)
\end{bmatrix}
\]

(b) Writing the Yule-Walker equations in expanded form:

\[ r(0) - 0.5r(1) = r(1) \]
\[ r(1) - 0.5r(0) = r(2) \]

Solving the first relation for \( r(1) \):

\[ r(1) = \frac{2}{3} r(0) \]  \hspace{1cm} (1)

Solving the second relation for \( r(2) \):

\[ r(2) = \frac{1}{6} r(0) \]  \hspace{1cm} (2)

(c) Since the noise \( v(n) \) has zero mean, so will the AR process \( u(n) \). Hence,

\[
\text{var}[u(n)] = E[u^2(n)] = r(0).
\]

We know that

\[
\sigma_u^2 = \sum_{k=0}^{2} a_k r(k)
\]
\[ = r(0) + a_1 r(1) + a_2 r(2) \]  \hspace{1cm} (3)

Substituting (1) and (2) into (3), and solving for \( r(0) \), we get

\[ r(0) = \frac{\sigma_u^2}{1 + \frac{2}{3} a_1 + \frac{1}{6} a_2} = 1.2 \]

By definition,

\[ P_0 = \text{average power of the AR process } u(n) \]
\[ E[|u(n)|^2] = r(0) \]

where \( r(0) \) is the autocorrelation function of \( u(n) \) for zero lag. We note that

\[
\begin{bmatrix}
\{a_1, a_2, \ldots, a_M\}
\end{bmatrix} = \begin{bmatrix}
r(1) & r(2) & \cdots & r(M) \\
r(0) & r(0) & \cdots & r(0)
\end{bmatrix}
\]

Equivalently, except for the scaling factor \( r(0) \),

\[
\{a_1, a_2, \ldots, a_M\} \equiv \{r(1), r(2), \ldots, r(M)\}
\]

Combining Eqs. (1) and (2):

\[
\{P, a_1, a_2, \ldots, a_M\} \equiv \{r(0), r(1), r(2), \ldots, r(M)\}
\]

1.9 (a) The transfer function of the MA model of Fig. 2.3 is

\[ H(z) = 1 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_K z^{-K} \]

(b) The transfer function of the ARMA model of Fig. 2.4 is

\[ H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_K z^{-K}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_M z^{-M}} \]

(c) The ARMA model reduces to an AR model when

\[ b_0 = b_1 = \cdots = b_K = 0 \]

It reduces to an MA model when

\[ a_1 = a_2 = \cdots = a_M = 0 \]

1.10 We are given

\[ x(n) = u(n) + 0.75u(n-1) + 0.25u(n-2) \]

Taking the z-transforms of both sides:
Finally, retaining terms in Eq. (2) up to $x^{10}$, we obtain the following approximation in the form of an AR model of order ten:

$$X(z) = \frac{1}{P(z)} D(z)$$

where $D(z)$ is given by the polynomial on the right-hand side of Eq. (2).

(a) The filter output is

$$x(n) = w^H u(n)$$

where $u(n)$ is the tap-input vector. The average power of the filter output is therefore

$$E[|x(n)|^2] = E[w^H u(n) u^H(n) w]$$

$$= w^H E[u(n) u^H(n)] w$$

$$= w^H R w$$

(b) If $u(n)$ is extracted from a zero mean white noise of variance $\sigma^2$, we have

$$R = \sigma^2 I$$

where $I$ is the identity matrix. Hence,

$$E[|x(n)|^2] = \sigma^2 w^H w$$

(a) The process $u(n)$ is a linear combination of Gaussian samples. Hence, $u(n)$ is Gaussian.

(b) From inverse filtering, we recognize that $v(n)$ may also be expressed as a linear combination of samples represented by $u(n)$. Hence, if $u(n)$ is Gaussian, then $v(n)$ is also Gaussian.

(a) From the Gaussian moment factoring theorem:

$$E\left[ \left( u_1, u_2 \right)^k \right] = E[ u_1^* \cdots u_1^* \cdots u_2^* ]$$
\[ = k! \ E[u_1 u_2] \cdots E[u_1 u_2] \]
\[ = k! \ (E[u_1 u_2])^k \]

(b) Putting \( u_2 = u_0 = u \), Eq. (1) reduces to
\[ E[u^{2k}] = k! \ (E[u^2])^k \]

1.14 It is not permissible to interchange the order of expectation and limiting operations in Eq. (1.113). The reason is that the expectation is a linear operation, whereas the limiting operation with respect to the number of samples \( N \) is nonlinear.

1.15 The filter output is
\[ y(n) = \sum_i h(i) u(n - i) \]
Similarly, we may write
\[ y(m) = \sum_k h(k) u(m - k) \]
Hence,
\[ r_x(n, m) = E[y(n) y^*(m)] \]
\[ = E\left[ \sum_i h(i) u(n - i) \sum_k h^*(k) u^*(m - k) \right] \]
\[ = \sum_i \sum_k h(i) h^*(k) E[u(n - i) u^*(m - k)] \]
\[ = \sum_i \sum_k h(i) h^*(k) r_x(n - i, m - k) \]

1.16 The mean-square value of the filter output in response to white noise input is
\[ P_o = \frac{2\sigma^2 \Delta \omega}{\pi} \]