CHAPTER 1

1.1 Let

$$r_u(k) = E[u(n)u^*(n-k)] \qquad (1)$$

$$r_y(k) = E[y(n)y^*(n-k)] \qquad (2)$$

We are given that

$$y(n) = u(n+a) - u(n-a) \qquad (3)$$

Hence, substituting Eq. (3) into (2), and then using Eq. (1), we get

$$r_y(k) = E[(u(n+a) - u(n-a))(u^*(n+a-k) - u^*(n-a-k))]$$

$$= 2r_u(k) - r_u(2a+k) - r_u(-2a+k)$$

1.2 We know that the correlation matrix $\mathbf{R}$ is Hermitian; that is

$$\mathbf{R}^H = \mathbf{R}$$

Given that the inverse matrix $\mathbf{R}^{-1}$ exists, we may write

$$\mathbf{R}^{-1}\mathbf{R}^H = \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix. Taking the Hermitian transpose of both sides:

$$\mathbf{R}\mathbf{R}^{-H} = \mathbf{I}$$

Hence,

$$\mathbf{R}^{-H} = \mathbf{R}^{-1}$$

That is, the inverse matrix $\mathbf{R}^{-1}$ is Hermitian.

1.3 For the case of a two-by-two matrix, we may

$$\mathbf{R}_u = \mathbf{R}_s + \mathbf{R}_\nu$$

$$= \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} + \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} + \sigma^2 & r_{12} \\ r_{21} & r_{22} + \sigma^2 \end{bmatrix}$$

For $\mathbf{R}_u$ to be nonsingular, we require

$$\det(\mathbf{R}_u) = (r_{11} + \sigma^2)(r_{22} + \sigma^2) - r_{12}r_{21} > 0$$

With $r_{12} = r_{21}$ for real data, this condition reduces to

$$(r_{11} + \sigma^2)(r_{22} + \sigma^2) - r_{12}r_{21} > 0$$

Since this is quadratic in σ^2, we may impose the following condition on σ^2 for nonsingularity of $\mathbf{R}_u$:

$$\sigma^2 > \frac{1}{2}(r_{11} + r_{22})\left(\sqrt{1 - \frac{4\Delta_r}{(r_{11} + r_{22})^2 - 1}}\right)$$

where $\Delta_r = r_{11}r_{22} - r_{12}^2$

1.4 We are given

$$\mathbf{R} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

This matrix is positive definite because

$$a^T\mathbf{R}\mathbf{a} = [a_1, a_2]\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$= a_1^2 + 2a_1a_2 + a_2^2$$

$$= (a_1 + a_2)^2 > 0 \text{ for all nonzero values of } a_1 \text{ and } a_2$$

(Positive definiteness is stronger than nonnegative definiteness.)

But the matrix **R** is singular because

$$\det(\mathbf{R}) = (1)^2 - (1)^2 = 0$$

Hence, it is possible for a matrix to be positive definite and yet it can be singular.

1.5 (a)

$$\mathbf{R}_{M+1} = \begin{bmatrix} r(0) & \mathbf{r}^H \\ \mathbf{r} & \mathbf{R}_M \end{bmatrix} \quad (1)$$

Let

$$\mathbf{R}_{M+1}^{-1} = \begin{bmatrix} a & \mathbf{b}^H \\ \mathbf{b} & \mathbf{C} \end{bmatrix} \quad (2)$$

where a, **b** and **C** are to be determined. Multiplying (1) by (2):

$$\mathbf{I}_{M+1} = \begin{bmatrix} r(0) & \mathbf{r}^H \\ \mathbf{r} & \mathbf{R}_M \end{bmatrix} \begin{bmatrix} a & \mathbf{b}^H \\ \mathbf{b} & \mathbf{C} \end{bmatrix}$$

where $\mathbf{I}_{M+1}$ is the identity matrix. Therefore,

$$r(0)a + \mathbf{r}^H\mathbf{b} = 1 \quad (3)$$

$$\mathbf{r}a + \mathbf{R}_M\mathbf{b} = \mathbf{0} \quad (4)$$

$$\mathbf{r}\mathbf{b}^H + \mathbf{R}_M\mathbf{C} = \mathbf{I}_M \quad (5)$$

$$r(0)\mathbf{b}^H + \mathbf{r}^H\mathbf{C} = \mathbf{0}^T \quad (6)$$

From Eq. (4):

$$\mathbf{b} = -\mathbf{R}_M^{-1}\mathbf{r}a \qquad (7)$$

Hence, from (3) and (7):

$$a = \frac{1}{r(0) - \mathbf{r}^H\mathbf{R}_M^{-1}\mathbf{r}} \qquad (8)$$

Correspondingly,

$$\mathbf{b} = -\frac{\mathbf{R}_M^{-1}\mathbf{r}}{r(0) - \mathbf{r}^H\mathbf{R}_M^{-1}\mathbf{r}} \qquad (9)$$

From (5):

$$\mathbf{C} = \mathbf{R}_M^{-1} - \mathbf{R}_M^{-1}\mathbf{r}\mathbf{b}^H$$

$$= \mathbf{R}_M^{-1} + \frac{\mathbf{R}_M^{-1}\mathbf{r}\mathbf{r}^H\mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^H\mathbf{R}_M^{-1}\mathbf{r}} \qquad (10)$$

As a check, the results of Eqs. (9) and (10) should satisfy Eq. (6).

$$r(0)\mathbf{b}^H + \mathbf{r}^H\mathbf{C} = -\frac{r(0)\mathbf{r}^H\mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^H\mathbf{R}_M^{-1}\mathbf{r}} + \mathbf{r}^H\mathbf{R}_M^{-1} + \frac{\mathbf{r}^H\mathbf{R}_M^{-1}\mathbf{r}\mathbf{r}^H\mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^H\mathbf{R}_M^{-1}\mathbf{r}}$$

$$= \mathbf{0}^T$$

We have thus shown that

$$\mathbf{R}_{M+1}^{-1} = \left[\begin{array}{c|c} 0 & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{R}_M^{-1} \end{array}\right] + a\left[\begin{array}{c|c} 1 & -\mathbf{r}^H\mathbf{R}_M^{-1} \\ \hline \mathbf{R}_M^{-1}\mathbf{r} & \mathbf{R}_M^{-1}\mathbf{r}\mathbf{r}^H\mathbf{R}_M^{-1} \end{array}\right]$$

$$= \left[\begin{array}{c|c} 0 & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{R}_M^{-1} \end{array}\right] + a\left[\begin{array}{c} 1 \\ \hline -\mathbf{R}_M^{-1}\mathbf{r} \end{array}\right] \left[\begin{array}{c|c} 1 & -\mathbf{r}^H\mathbf{R}_M^{-1} \end{array}\right]$$

where the scalar a is defined by Eq. (8):

(b) $$\mathbf{R}_{M+1} = \begin{bmatrix} \mathbf{R}_M & \mathbf{r}^{B*} \\ \mathbf{r}^{BT} & r(0) \end{bmatrix} \tag{11}$$

Let

$$\mathbf{R}_{M+1}^{-1} = \begin{bmatrix} \mathbf{D} & \mathbf{e} \\ \mathbf{e}^{H} & f \end{bmatrix} \tag{12}$$

where $\mathbf{D}$, $\mathbf{e}$ and f are to be determined. Multiplying (11) by (12):

$$\mathbf{I}_{M+1} = \begin{bmatrix} \mathbf{R}_M & \mathbf{r}^{B*} \\ \mathbf{r}^{BT} & r(0) \end{bmatrix} \begin{bmatrix} \mathbf{D} & \mathbf{e} \\ \mathbf{e}^{H} & f \end{bmatrix}$$

Therefore

$$\mathbf{R}_M\mathbf{D} + \mathbf{r}^{B*}\mathbf{e}^{H} = \mathbf{I} \tag{13}$$

$$\mathbf{R}_M\mathbf{e} + \mathbf{r}^{B*}f = \mathbf{0} \tag{14}$$

$$\mathbf{r}^{BT}\mathbf{e} + r(0)f = 1 \tag{15}$$

$$\mathbf{r}^{BT}\mathbf{D} + r(0)\mathbf{e}^{H} = \mathbf{0}^{T} \tag{16}$$

From (14):

$$\mathbf{e} = -\mathbf{R}_M^{-1}\mathbf{r}^{B*}f \tag{17}$$

Hence, from (15) and (17):

$$f = \frac{1}{r(0) - \mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}} \tag{18}$$

Correspondingly,

$$\mathbf{e} = -\frac{\mathbf{R}_M^{-1}\mathbf{r}^{B*}}{r(0) - \mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}} \tag{19}$$

From (13):

$$\mathbf{D} = \mathbf{R}_M^{-1} - \mathbf{R}_M^{-1}\mathbf{r}^{B*}\mathbf{e}^H$$

$$= \mathbf{R}_M^{-1} + \frac{\mathbf{R}_M^{-1}\mathbf{r}^{B*}\mathbf{r}^{BT}\mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}} \tag{20}$$

As a check, the results of Eqs. (19) and (20) must satisfy Eq. (16). Thus

$$\mathbf{r}^{BT}\mathbf{D} + r(0)\mathbf{e}^H = \mathbf{r}^{BT}\mathbf{R}_M^{-1} + \frac{\mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}\mathbf{r}^{BT}\mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}} - \frac{r(0)\mathbf{r}^{BT}\mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}}$$

$$= \mathbf{0}^T$$

We have thus shown that

$$\mathbf{R}_{M+1}^{-1} = \begin{bmatrix} \mathbf{R}_M^{-1} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} + f\begin{bmatrix} \mathbf{R}_M^{-1}\mathbf{r}^{B*}\mathbf{r}^{BT}\mathbf{R}_M^{-1} & -\mathbf{R}_M^{-1}\mathbf{r}^{B*} \\ -\mathbf{r}^{BT}\mathbf{R}_M^{-1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R}_M^{-1} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} + f\begin{bmatrix} -\mathbf{R}_M^{-1}\mathbf{r}^{B*} \\ 1 \end{bmatrix} [-\mathbf{r}^{BT}\mathbf{R}_M^{-1} \quad 1]$$

where the scalar f is defined by Eq. (18).

1.6 (a) We express the difference equation describing the first-order AR process $u(n)$ as

$$u(n) = v(n) + w_1 u(n-1)$$

where $w_1 = -a_1$. Solving this equation by repeated substitution, we get

$$u(n) = v(n) + w_1 v(n-1) + w_1 u(n-2)$$

$$= \dots$$

$$= v(n) + w_1 v(n-1) + w_1^2 v(n-2) + \dots + w_1^{n-1} v(1) \quad (1)$$

Here we have used the initial condition

$$u(0) = 0$$

or equivalently

$$u(1) = v(1)$$

Taking the expected value of both sides of Eq. (1) and using

$E[v(n)] = \mu \qquad$ for all n,

we get the geometric series

$$E[u(n)] = \mu + w_1\mu + w_1^2\mu + \dots + w_1^{n-1}\mu$$

$$= \begin{cases} \mu\left(\dfrac{1-w_1^n}{1-w_1}\right), & w_1 \neq 1 \\ \mu n, & w_1 = 1 \end{cases}$$

This result shows that if $\mu \neq 0$, then $E[u(n)]$ is a function of time n. Accordingly, the AR process $u(n)$ is not stationary. If, however, the AR parameter satisfies the condition:

$$|a_1| < 1 \text{ or } |w_1| < 1$$

then

$$E[(n)] \to \frac{\mu}{1-w_1} \text{ as } n \to \infty$$

Under this condition, we say that the AR process is asymptotically stationary to order one.

(b) When the white noise process $v(n)$ has zero mean, the AR process $u(n)$ will likewise have zero mean. Then

$$\text{var}[v(n)] = \sigma_v^2$$

$$\text{var}[u(n)] = E[u^2(n)]. \tag{2}$$

Substituting Eq. (1) into (2), and recognizing that for the white noise process

$$E[v(n)v(k)] = \begin{cases} \sigma_v^2 & n = k \\ 0, & n \neq k \end{cases} \tag{3}$$

we get the geometric series

$$\text{var}[u(n)] = \sigma_v^2(1 + w_1^2 + w_1^4 + \cdots + w_1^{2n-2})$$

$$= \begin{cases} \sigma_v^2\left(\dfrac{1 - w_1^{2n}}{1 - w_1^2}\right), & w_1 \neq 1 \\ \sigma_v^2 n, & w_1 = 1 \end{cases}$$

When $|a_1| < 1$ or $|w_1| < 1$, then

$$\text{var}[u(n)] \approx \frac{\sigma_v^2}{1 - w_1^2} = \frac{\sigma_v^2}{1 - a_1^2} \text{ for large } n$$

(c) The autocorrelation function of the AR process $u(n)$ equals $E[u(n)u(n-k)]$. Substituting Eq. (1) into this formula, and using Eq. (3), we get

$$E[u(n)u(n-k)] = \sigma_v^2(w_1^k + w_1^{k+2} + \cdots + w_1^{k+2n-2})$$

$$= \begin{cases} \sigma_v^2 w_1^k\left(\dfrac{1 - w_1^{2n}}{1 - w_1^2}\right), & w_1 \neq 1 \\ \sigma_v^2 n, & w_1 = 1 \end{cases}$$

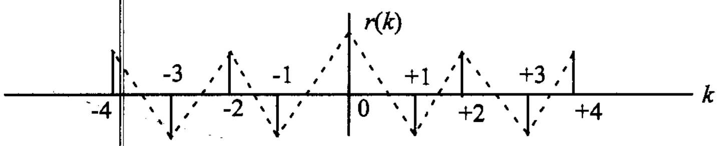
For $|a_1| < 1$ or $|w_1| < 1$, we may therefore express this autocorrelation function as

$$r(k) = E[u(n)u(n-k)]$$

$$\approx \frac{\sigma_v^2 w_1^k}{1 - w_1^2} \text{ for large } n$$

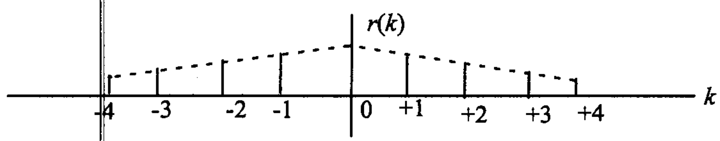
Case 1: $0 < a_1 < 1$

In this case, $w_1 = -a_1$ is negative, and $r(k)$ varies with k as follows:



Case 2: $-1 < a_1 < 0$

In this case, $w_1 = -a_1$ is positive and $r(k)$ varies with k as follows:



1.7 (a) The second-order AR process $u(n)$ is described by the difference equation:

$$u(n) = u(n-1) - 0.5u(n-2) + v(n)$$

Hence

$$w_1 = 1$$

$$w_2 = -0.5$$

and the AR parameters equal

$$a_1 = -1$$

$$a_2 = 0.5$$

Accordingly, we write the Yule-Walker equations as

$$\begin{bmatrix} r(0) & r(1) \\ r(1) & r(0) \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} r(1) \\ r(2) \end{bmatrix}$$

(b) Writing the Yule-Walker equations in expanded form:

$$r(0) - 0.5r(1) = r(1)$$

$$r(1) - 0.5r(0) = r(2)$$

Solving the first relation for $r(1)$:

$$r(1) = \frac{2}{3}r(0) \tag{1}$$

Solving the second relation for $r(2)$:

$$r(2) = \frac{1}{6}r(0) \tag{2}$$

(c) Since the noise $v(n)$ has zero mean, so will the AR process $u(n)$. Hence,

$$\begin{aligned} \text{var}[u(n)] &= E[(u^2 n)] \\ &= r(0). \end{aligned}$$

We know that

$$\begin{aligned} \sigma_v^2 &= \sum_{k=0}^{2} a_k r(k) \\ &= r(0) + a_1 r(1) + a_2 r(2) \end{aligned} \tag{3}$$

Substituting (1) and (2) into (3), and solving for $r(0)$, we get

$$r(0) = \frac{\sigma_v^2}{1 + \frac{2}{3}a_1 \frac{1}{6}a_2} = 1.2$$

1.8 By definition,

P_0 = average power of the AR process $u(n)$

$$= E[|u(n)|^2]$$

$$= r(0) \quad (1)$$

where $r(0)$ is the autocorrelation function of $u(n)$ for zero lag. We note that

$$\{a_1, a_2, \cdots, a_M\} \rightleftharpoons \left\{\frac{r(1)}{r(0)}, \frac{r(2)}{r(0)}, \cdots, \frac{r(M)}{r(0)}\right\}$$

Equivalently, except for the scaling factor r(0),

$$\{a_1, a_2, \cdots, a_M\} \rightleftharpoons \{r(1), r(2), \cdots, r(M)\} \quad (2)$$

Combining Eqs. (1) and (2):

$$\{P_0, a_1, a_2, \cdots, a_M\} \rightleftharpoons \{r(0), r(1), r(2), \cdots, r(M)\} \quad (3)$$

1.9 (a) The transfer function of the MA model of Fig. 2.3 is

$$H(z) = 1 + b_1^* z^{-1} + b_2^* z^{-2} + \dots + b_K^* z^{-K}$$

(b) The transfer function of the ARMA model of Fig. 2.4 is

$$H(z) = \frac{b_0^* + b_1^* z^{-1} + b_2^* z^{-2} + \dots + b_K^* z^{-K}}{1 + a_1^* z^{-1} + a_2^* z^{-2} + \dots + a_M^* z^{-M}}$$

(c) The ARMA model reduces to an AR model when

$$b_0 = b_1 = \dots = b_K = 0$$

It reduces to an MA model when

$$a_1 = a_2 = \dots = a_M = 0$$

1.10 We are given

$$x(n) = \upsilon(n) + 0.75\upsilon(n-1) + 0.25\upsilon(n-2)$$

Taking the z-transforms of both sides:

(c) $M = 10$

Finally, retaining terms in Eq. (2) up to z^{-10}, we obtain the following approximation in the form of an AR model of order ten:

$$\frac{X(z)}{V(z)} \approx \frac{1}{D(z)}$$

where $D(z)$ is given by the polynomial on the right-hand side of Eq. (2).

1.11 (a) The filter output is

$$x(n) = \mathbf{w}^H \mathbf{u}(n)$$

where $\mathbf{u}(n)$ is the tap-input vector. The average power of the filter output is therefore

$$E[|x(n)|^2] = E[\mathbf{w}^H \mathbf{u}(n)\mathbf{u}^H(n)\mathbf{w}]$$

$$= \mathbf{w}^H E[\mathbf{u}(n)\mathbf{u}^H(n)]\mathbf{w}$$

$$= \mathbf{w}^H \mathbf{R}\mathbf{w}$$

(b) If $\mathbf{u}(n)$ is extracted from a zero mean white noise of variance σ^2, we have

$$\mathbf{R} = \sigma^2 \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix. Hence,

$$E[|x(n)|^2] = \sigma^2 \mathbf{w}^H \mathbf{w}$$

1.12 (a) The process $u(n)$ is a linear combination of Gaussian samples. Hence, $u(n)$ is Gaussian.

(b) From inverse filtering, we recognize that $v(n)$ may also be expressed as a linear combination of samples represented by $u(n)$. Hence, if $u(n)$ is Gaussian, then $v(n)$ is also Gaussian.

1.13 (a) From the Gaussian moment factoring theorem:

$$E\left[(u_1^* u_2)^k\right] = E[u_1^* \cdots u_1^* u_2 \cdots u_2]$$

$$= k!\ E[u_1^* u_2] \cdots E[u_1^* u_2]$$

$$= k!\ (E[u_1^* u_2])^k \qquad (1)$$

(b) Putting $u_2 = u_1 = u$, Eq. (1) reduces to

$$E[|u|^{2k}] = k!\ (E[|u|^2])^k$$

1.14 It is not permissible to interchange the order of expectation and limiting operations in Eq. (1.113). The reason is that the expectation is a linear operation, whereas the limiting operation with respect to the number of samples N is nonlinear.

1.15 The filter output is

$$y(n) = \sum_i h(i)u(n-i)$$

Similarly, we may write

$$y(m) = \sum_k h(k)u(m-k)$$

Hence,

$$r_y(n, m) = E[y(n)y^*(m)]$$

$$= E\left[\sum_i h(i)u(n-i)\sum_k h^*(k)u^*(m-k)\right]$$

$$= \sum_i \sum_k h(i)h^*(k)E[u(n-i)u^*(m-k)]$$

$$= \sum_i \sum_k h(i)h^*(k)r_u(n-i, m-k)$$

1.16 The mean-square value of the filter output in response to white noise input is

$$P_o = \frac{2\sigma^2 \Delta\omega}{\pi}$$