EE7000-6 Adaptive Filter Theory

Quiz 1 Solution

12:40~1:30 p.m., September 27, 2004

The input signal $x(n)$ of an adaptive filter with tap weight vector $\vec{w} = [w_0, w_1, \cdots w_{M-1}]^T$, can be modeled as a stationary process with a positive-definite correlation matrix $\tilde{R} > 0$. The input signal $x(n)$ and the desired signal $d(n)$ which is also a stationary process, have a correlation vector $\vec{P}$. The desired signal energy is $\sigma_d^2 = E\left\{|d(n)|^2\right\}$. Assume that signals and filter coefficients are complex-valued.

(a) Derive the Wiener-Hopf Equations from the minimization of mean-square error between the system output and desired signals, i.e., $\nabla J = 0$. (50%)

(b) Write the Hessian matrix,

$$\tilde{H} \equiv \begin{bmatrix} \left(\dfrac{\partial^2 J}{\partial w_{r,p} \partial w_{r,q}}\right)_{0 \le p \le M-1, 0 \le q \le M-1} & \left(\dfrac{\partial^2 J}{\partial w_{r,p} \partial w_{i,q}}\right)_{0 \le p \le M-1, 0 \le q \le M-1} \\ \left(\dfrac{\partial^2 J}{\partial w_{i,p} \partial w_{r,q}}\right)_{0 \le p \le M-1, 0 \le q \le M-1} & \left(\dfrac{\partial^2 J}{\partial w_{i,p} \partial w_{i,q}}\right)_{0 \le p \le M-1, 0 \le q \le M-1} \end{bmatrix}$$

in terms of $\tilde{R}$, where $\vec{w} = [w_p]_{0 \le p \le M-1} = \left[w_{r,p} + \sqrt{-1} w_{i,p}\right]_{0 \le p \le M-1}$. (20%)

(c) Show that $\tilde{H} > 0$. (30%)

(a) $J(\vec{w}) = \sigma_d^2 - \vec{w}^H \vec{P} - \vec{P}^H \vec{w} + \vec{w}^H \tilde{R} \vec{w}$

Let $\vec{w} = \vec{w}_r + \sqrt{-1}\vec{w}_i$, $\quad \tilde{R} = \tilde{R}_r + \sqrt{-1}\tilde{R}_i$, $\quad \vec{P} = \vec{P}_r + \sqrt{-1}\vec{P}_i$

$\tilde{R} = \tilde{R}^H \Rightarrow \tilde{R}_r = \tilde{R}_r^T$, $\quad \tilde{R}_i = -\tilde{R}_i^T$

$$
\begin{aligned}
J(\vec{w}) &= J(\vec{w}_r, \vec{w}_i) \\
&= \sigma_d^2 - 2\left(\vec{w}_r{}^T \vec{P}_r + \vec{w}_i{}^T \vec{P}_i\right) + \left(\vec{w}_r{}^T \tilde{R}_r \vec{w}_r - \vec{w}_r{}^T \tilde{R}_i \vec{w}_i + \vec{w}_i{}^T \tilde{R}_r \vec{w}_i + \vec{w}_i{}^T \tilde{R}_i \vec{w}_r\right)
\end{aligned}
$$

$$
\nabla J(\vec{w}_r, \vec{w}_i) = \begin{bmatrix} -2\vec{P}_r + 2\tilde{R}_r \vec{w}_r - 2\tilde{R}_i \vec{w}_i \\ -2\vec{P}_i + 2\tilde{R}_r \vec{w}_i + 2\tilde{R}_i \vec{w}_r \end{bmatrix} = \vec{0}
$$

$$
\Rightarrow \begin{cases} \tilde{R}_r \vec{w}_r - \tilde{R}_i \vec{w}_i = \vec{P}_r \\ \tilde{R}_r \vec{w}_i + \tilde{R}_i \vec{w}_r = \vec{P}_i \end{cases}.
$$

From the Wiener-Hopf Equations,

$$
\tilde{R}\vec{w} = \vec{P} \Rightarrow \left(\tilde{R}_r + \sqrt{-1}\tilde{R}_i\right)\left(\vec{w}_r + \sqrt{-1}\vec{w}_i\right) = \vec{P}_r + \sqrt{-1}\vec{P}_i
$$

$$
\Rightarrow \begin{cases} \tilde{R}_r \vec{w}_r - \tilde{R}_i \vec{w}_i = \vec{P}_r \\ \tilde{R}_r \vec{w}_i + \tilde{R}_i \vec{w}_r = \vec{P}_i \end{cases}.
$$

(b) $\dfrac{\partial J}{\partial w_{r,q}} = 2\left(\tilde{R}_r \vec{w}_r\right)_q - 2\left(\tilde{R}_i \vec{w}_i\right)_q$, $\quad \dfrac{\partial J}{\partial w_{i,q}} = 2\left(\tilde{R}_r \vec{w}_i\right)_q + 2\left(\tilde{R}_i \vec{w}_r\right)_q$

$$
\frac{\partial^2 J}{\partial w_{r,p} \partial w_{r,q}} = 2\tilde{R}_{r,pq}, \quad \frac{\partial^2 J}{\partial w_{r,p} \partial w_{i,q}} = 2\tilde{R}_{i,pq},
$$

$$
\frac{\partial^2 J}{\partial w_{i,p} \partial w_{r,q}} = -2\tilde{R}_{i,pq}, \quad \frac{\partial^2 J}{\partial w_{i,p} \partial w_{i,q}} = 2\tilde{R}_{r,pq}
$$

$$
\tilde{H} \equiv \begin{bmatrix} \left(\dfrac{\partial^2 J}{\partial w_{r,p} \partial w_{r,q}}\right)_{0 \le p \le M-1, 0 \le q \le M-1} & \left(\dfrac{\partial^2 J}{\partial w_{r,p} \partial w_{i,q}}\right)_{0 \le p \le M-1, 0 \le q \le M-1} \\ \left(\dfrac{\partial^2 J}{\partial w_{i,p} \partial w_{r,q}}\right)_{0 \le p \le M-1, 0 \le q \le M-1} & \left(\dfrac{\partial^2 J}{\partial w_{i,p} \partial w_{i,q}}\right)_{0 \le p \le M-1, 0 \le q \le M-1} \end{bmatrix}
$$

$$
= \begin{bmatrix} 2\tilde{R}_r & 2\tilde{R}_i \\ -2\tilde{R}_i & 2\tilde{R}_r \end{bmatrix}
$$

(c) Since $\tilde{R} > 0$, given any nontrivial $\vec{x} \equiv \vec{x}_r + \sqrt{-1}\vec{x}_i \neq \vec{0}$, $\vec{x}^H \tilde{R}\vec{x} > 0$.

$\because \tilde{R} = \tilde{R}^H$,

$$\vec{x}^H \tilde{R}\vec{x} = \vec{x}_r{}^T \tilde{R}_r \vec{x}_r - \vec{x}_r{}^T \tilde{R}_i \vec{x}_i + \vec{x}_i{}^T \tilde{R}_r \vec{x}_i + \vec{x}_i{}^T \tilde{R}_i \vec{x}_r$$

$$= \begin{bmatrix} \vec{x}_r{}^T & \vec{x}_i{}^T \end{bmatrix} \begin{bmatrix} \tilde{R}_r & -\tilde{R}_i \\ \tilde{R}_i & \tilde{R}_r \end{bmatrix} \begin{bmatrix} \vec{x}_r \\ \vec{x}_i \end{bmatrix} = \begin{bmatrix} \vec{x}_i{}^T & \vec{x}_r{}^T \end{bmatrix} \begin{bmatrix} \tilde{R}_r & \tilde{R}_i \\ -\tilde{R}_i & \tilde{R}_r \end{bmatrix} \begin{bmatrix} \vec{x}_i \\ \vec{x}_r \end{bmatrix} > 0$$

That is,

For an arbitrary nontrivial vector $\vec{a} = \sqrt{2}\begin{bmatrix} \vec{x}_i \\ \vec{x}_r \end{bmatrix} \neq \vec{0}$, it is certain that $\vec{a}^T \tilde{H}\vec{a} > 0$.