EE7000 Adaptive Filter Theory

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Midterm Examination, Fall of 2003

Time: 10:40 a.m. ~ 11:30 p.m., Monday, October 13 of 2003

Please ALWAYS work on the FRONT SIDE of each page. No answer on the BACK SIDE of any page will be GRADED! Ask for additional blank papers if necessary!

You may check any textbook, classnote or other references alone during the test. However, NO CHEATING or COLLABORATION is allowed and such violation of the university regulations will be reported.

Please write down you name and social security number here:

Full Name: _________SOLUTION________________________

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Partition	Score
Question 1	
Question 2	
Total	

<u>Question 1</u>: (50%)

An adaptive filter is excited by the stationary input sequence $u(n)$. The desired response is $d(n)$. The autocorrelation function of the input is described as $r_u(0)=1$, $r_u(1)=\frac{1}{2}$, $r_u(2)=\frac{1}{4}$ and $r_u(3)=\frac{1}{8}$. The cross-correlation between the input and the desired responses can be described as $p(0)=p(-1)=p(-2)=p(-3)=\frac{1}{4}$. The variance of the desired response is $\sigma_d^2=1$.

(a) What is the optimal tap-weight vector for model order M=3? (15%)

(b) What is the mean-square error for such an optimal filter of model order M=3? (10%)

(c) What is the optimal tap-weight vector for model order M=4? (25%)

Answer to Question 1:

(a) $$\tilde{R}_3 = \begin{bmatrix} r_u(0) & r_u(1) & r_u(2) \\ r_u(1) & r_u(0) & r_u(1) \\ r_u(2) & r_u(1) & r_u(0) \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}, \quad \vec{P} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

$$\tilde{R}_3^{-1} = \frac{1}{(1+\frac{1}{16}+\frac{1}{16}-\frac{1}{16}-\frac{1}{4}-\frac{1}{4})} \begin{bmatrix} \frac{3}{4} & -\frac{3}{8} & 0 \\ -\frac{3}{8} & \frac{15}{16} & -\frac{3}{8} \\ 0 & -\frac{3}{4} & \frac{3}{4} \end{bmatrix} = \frac{16}{9} \begin{bmatrix} \frac{3}{4} & -\frac{3}{8} & 0 \\ -\frac{3}{8} & \frac{15}{16} & -\frac{3}{8} \\ 0 & -\frac{3}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & 0 \\ -\frac{2}{3} & \frac{5}{3} & -\frac{2}{3} \\ 0 & -\frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

$$\vec{w}_{opt} = \tilde{R}_3^{-1}\vec{P} = [\frac{1}{6} \quad \frac{1}{12} \quad \frac{1}{6}]^T$$

(b) $$J_{\min} = \sigma_d^2 - \vec{P}^H \vec{w}_{opt} = 1 - [\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4}][\frac{1}{6} \quad \frac{1}{12} \quad \frac{1}{6}]^T = 1 - \frac{5}{48} = \frac{43}{48}$$

(c) $$\tilde{R}_4^{-1} = \begin{bmatrix} R_3^{-1} & \vec{0} \\ \vec{0}^T & 0 \end{bmatrix} + \frac{1}{r_u(0) - \vec{r}^{BT} R_3^{-1} \vec{r}^{B*}} \begin{bmatrix} -R_3^{-1}\vec{r}^{B*} \\ 1 \end{bmatrix} [-\vec{r}^{BT} R_3^{-1} \quad 1],$$

where $\vec{r} = [r_u(1) \quad r_u(2) \quad r_u(3)]^T$ and $\vec{r}^{BT} = [r_u(3) \quad r_u(2) \quad r_u(1)]$.

$$r_u(0) - \vec{r}^{BT} R_3^{-1} \vec{r}^{B*} = 1 - [\frac{1}{8} \quad \frac{1}{4} \quad \frac{1}{2}] \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & 0 \\ -\frac{2}{3} & \frac{5}{3} & -\frac{2}{3} \\ 0 & -\frac{2}{3} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} = 1 - [0 \quad 0 \quad \frac{1}{2}] \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} = \frac{3}{4}$$

$$-R_3^{-1}\vec{r}^{B*} = \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & 0 \\ -\frac{2}{3} & \frac{5}{3} & -\frac{2}{3} \\ 0 & -\frac{2}{3} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} -R_3^{-1}\vec{r}^{B*} \\ 1 \end{bmatrix} [-\vec{r}^{BT} R_3^{-1} \quad 1] = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

$$\tilde{R}_4^{-1} = \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & 0 & 0 \\ -\frac{2}{3} & \frac{5}{3} & -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{4}{3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & 0 & 0 \\ -\frac{2}{3} & \frac{5}{3} & -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} & \frac{5}{3} & -\frac{2}{3} \\ 0 & 0 & -\frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

$$\vec{w}_{opt} = \tilde{R}_4^{-1}\vec{P} = \begin{bmatrix} \frac{1}{6} & \frac{1}{12} & \frac{1}{12} & \frac{1}{6} \end{bmatrix}^T$$

<u>Question 2</u>: (50%)

An adaptive filter of model order *M*=2 is excited by the stationary input sequence *u*(*n*). The desired response is *d*(*n*). The autocorrelation function of the input is described as $r_u(0) = 2$, $r_u(1) = 1$. The cross-correlation between the input and the desired responses can be described as $p(0) = \frac{1}{4}$, $p(-1) = \frac{1}{2}$. The variance of the desired response is $\sigma_d^2 = 1$.

(a) Write the Wiener-Hopf Equations. (5%)

(b) What is the mean-square error if an arbitrary tap-weight vector is given as $\vec{W} = [1 \quad 2]^T$? (10%)

(c) Specify the canonical form of Wiener-Hopf equations NUMERICALLY. (25%)

Answer to Question 2:

(a) $\tilde{R}\vec{w}_{opt} = \vec{P} \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} w_{o,0} \\ w_{o,1} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix}$.

(b) $J = \sigma_d^2 - \vec{w}^H \vec{P} - \vec{P}^H \vec{w} + \vec{w}^H \tilde{R}\vec{w} = 1 - 2\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 - \frac{5}{2} + 14 = \frac{25}{2}$

(c) $J_{\min} = \sigma_d^2 - \vec{P}^H \tilde{R}^{-1}\vec{P} = 1 - \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} = 1 - \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} = \frac{7}{8}$

Since $\tilde{R} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$, $\vec{w}_{opt} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}$,

$$\vec{v} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\left\{\begin{bmatrix} w_{o,0} \\ w_{o,1} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}\right\} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} w_{o,0} \\ w_{o,1} - \frac{1}{4} \end{bmatrix}.$$

Therefore, the canonical form of the Wiener-Hopf equations is

$$J = \frac{7}{8} + \begin{bmatrix} w_{o,0} & w_{o,1} - \frac{1}{4} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} w_{o,0} \\ w_{o,1} - \frac{1}{4} \end{bmatrix}.$$