Question 1 (30%)

Determine whether or not each of the following signals is periodic. In case a signal is periodic, specify its fundamental period.

(a) Discrete-time signal: \( x(n) = \sqrt{1 + \cos\left(\frac{\pi n}{8}\right)} \)
\( \frac{\sin\left(\frac{\pi n}{2}\right)}{65} + 1.2 \) (10%)

(b) Discrete-time signal: \( x(n) = \sin\left(\frac{\pi n}{3}\right) + \cos\left(\frac{\pi n}{5}\right) + \delta(n-3) \),
where \( \delta(n-3) = \begin{cases} 
1, & n = 3 \\
0, & \text{otherwise} 
\end{cases} \) (10%)

(c) Analog signal: \( x_a(t) = \cos(3.5t) \text{rect}\left(\frac{t}{8.2}\right) \),
where \( \text{rect}\left(\frac{t}{8.2}\right) = \begin{cases} 
1, & -4.1 \leq t \leq 4.1 \\
0, & \text{elsewhere} 
\end{cases} \) (10%)
Answer to Question 1:

(a) \[ \sqrt{1 + \cos \left( \frac{\pi n}{8} \right)} = \sqrt{2} \left| \cos \left( \frac{\pi n}{16} \right) \right| \]

\[ \frac{\pi N_p}{8} = 2\pi m \Rightarrow m = 1, \quad N = 16 \]

\[ \Rightarrow \left| \cos \left( \frac{\pi n}{16} \right) \right| \text{ has the period of } N_p = \frac{16}{2} = 8 \]

\[ \frac{\pi N_p}{65} = 2\pi m \Rightarrow m = 1, \quad N_p = \frac{130}{65} \]

\[ N_p = \text{LCM} (N_p_1, N_p_2) = \text{LCM} (8, 130) = 520 \]

(b) Since \( 8(n-3) \) is not periodic,

\[ \chi(n) \text{ is not periodic.} \]

(c) \[ \chi_\alpha(t) \text{ is not periodic.} \]
**Question 2 (25%)**

An analog band-limited signal $x_a(t)$ is sampled at the rate of $F_s$ through an A/D converter to construct a discrete-time signal $x(n)$. $x(n)$ is assumed to be a periodic signal such that $x(n+M) = x(n)$, $\forall n$, where $M$ is a large integer ($M >> 2$) and it is not necessary the fundamental period.

(a) Since $M$ may not be the fundamental period, we want to search for a smaller integer $N$ than $M$ from $M$ given signal samples $x(0), x(1), x(2), ..., x(M-1)$ through computer. Two sets of positive integers are defined here, $D_k = \{ k | M$ is divisible by $k$, $2 \leq k \leq M$, $k$ is integer $\}$, $S_k = \{ k | x((ik+n)=x((ik+k+n), i=0,1,...,M/k-2, n=0,1,...,k-1, k \in D_k \}$. $N$ can be determined by $N = \{ M/\max\{S_k\} , S_k \neq \phi \}$, where $\phi$ is the empty set. Is $\frac{N}{F_s}$ always the fundamental period $T_p$ of the original analog signal? If your answer is no, show an example to justify your answer. (15%)

(b) Under what condition on the sampling frequency, $\frac{N}{F_s}$ is always the fundamental period $T_p$? (10%)
(a) \( \frac{N}{F_s} \) is not necessary the fundamental period. For example, \( X_a(t) = A \cos(2\pi F_c t) \) is sinusoidal with frequency \( F_c \) Hz, and hence the fundamental period is \( T_p = \frac{1}{F_c} \).

If we sample at \( F_s = \frac{2}{3} F_c \) Hz, \( X(n) = A \cos\left(\frac{2\pi F_c n}{F_s}\right) = A \cos\left(3\pi n\right) \).

\( N = 2 \) according to our algorithm.

However \( \frac{N}{F_s} = \frac{2}{\frac{2}{3} F_c} = \frac{3}{F_c} \neq \frac{1}{F_c} = T_p \).

(b) According to the sampling theorem, \( F_s > 2F_{\text{max}} \), where \( F_{\text{max}} \) is the largest frequency which this signal contains.
Question 3 (45%)

A digital communication system is depicted as below.

\[ x(n) \rightarrow + \rightarrow \Sigma \rightarrow + \rightarrow \Sigma \rightarrow \downarrow N \rightarrow y(n) \]

\[ w(n) \rightarrow T \rightarrow z(n) \rightarrow \downarrow N \rightarrow y(n) \]

\[ z^{-N} \quad \Sigma \quad \Sigma \quad \downarrow N \]

\[ x(n), w(n), z(n) \text{ and } y(n) \text{ are all discrete-time sequences. The system } T \text{ is a moving} \]

average operation such that \[ z(n) = \sum_{k=0}^{x-1} w(n-k) . \downarrow N \text{ is a down-sampler where } y(n) = z(Nn) \text{ and } N \text{ is an arbitrary integer greater than one.} \]

(a) Is this communication system a linear system? Justify your answer through mathematics. (10%)

(b) Is this communication system a time-invariant system? Justify your answer through mathematics. (10%)

(c) Is this communication system BIDO stable? (10%)

(d) What is the output sequence \( y(n) \) for a given input \( x(n) = u(n) \) where \( u(n) \) is a unit-step sequence? (15%)
Answer to Question 3:

(a)
\[ W(n) = X(n) - X(n-N) - \alpha X(n-N) + \alpha X(n-2N) \]
\[ = X(n) - 2 \alpha X(n-N) + \alpha X(n-2N) \]
\[ Z(n) = \sum_{k=0}^{N-1} W(n-k) = \sum_{k=0}^{N-1} X(n-k) - 2 \sum_{k=0}^{N-1} \alpha X(n-k-N) \]
\[ + \sum_{k=0}^{N-1} \alpha X(n-k-2N) \]
\[ Y(n) = Z(Nn) = \sum_{k=0}^{N-1} X(Nn-k) - 2 \sum_{k=0}^{N-1} \alpha X(Nn-k-N) \]
\[ + \sum_{k=0}^{N-1} \alpha X(Nn-k-2N) \]

(a) Assume \( X_1(n) \to Y_1(n) \), \( X_2(n) \to Y_2(n) \)

\[ X_3(n) = \alpha X_1(n) + \beta X_2(n) \to Y_3(n) \]
\[ Y_3(n) = \sum_{k=0}^{N-1} \left\{ \alpha X_1(Nn-k) + \beta X_2(Nn-k) \right\} - 2 \sum_{k=0}^{N-1} \alpha X_1(Nn-k-N) \]
\[ + \beta X_2(Nn-k-N) \right\} + \sum_{k=0}^{N-1} \left\{ \alpha X_1(Nn-k-2N) + \beta X_2(Nn-k-2N) \right\} \]
\[ = \alpha Y_1(n) + \beta Y_2(n) \]

\[ \text{This is a linear system.} \]
(b) \[ x(n) \rightarrow y(n^{'}) \]
\[ x(n-l) \rightarrow y_{l}^{'}(n) \]

\[ y_{l}^{'}(n) = \sum_{k=0}^{N-1} x(Nn-k-l) - 2 \sum_{k=0}^{N-1} x(Nn-k-N-l) \]
\[ + \sum_{k=0}^{N-1} x(Nn-k-2N-l) \]

\[ y(n-l) = \sum_{k=0}^{N-1} x(Nn-Nl-k) - 2 \sum_{k=0}^{N-1} x(Nn-Nl-k-N) \]
\[ + \sum_{k=0}^{N-1} x(Nn-Nl-k-2N) \]

Since \( y_{l}^{'}(n) \neq y(n-l) \), this is not time-invariant.

(c) Given a finite input such that
\[ |x(n)| \leq M_{x} < \infty \]

\[ |y(n)| = \left| \sum_{k=0}^{N-1} x(Nn-k) - 2 \sum_{k=0}^{N-1} x(Nn-k-N) \right| \]
\[ + \sum_{k=0}^{N-1} x(Nn-k-2N) \]
\[ \leq \sum_{k=0}^{N-1} |x(Nn-k)| + 2 \sum_{k=0}^{N-1} |x(Nn-k-N)| \]
\[ + \sum_{k=0}^{N-1} |x(Nn-k-2N)| \leq 4NM_{x} \]
\[ \|y(n)\| \leq M_y < \infty, \text{ where } M_y = 4NMx \]

\( \Rightarrow \) This is a BIBO stable system.

\[ y(n) = \sum_{k=0}^{N-1} u(Nn-k) - 2 \sum_{k=0}^{N-1} u(Nn-k-N) \]
\[ + \sum_{k=0}^{N-1} u(Nn-k-2N) \]

Since \( u(Nn-k-mN) = \begin{cases} 1, & (n-m)N \geq k \\ 0, & (n-m)N < k \end{cases} \)

\[ \sum_{k=0}^{N-1} u(Nn-k-mN) = \begin{cases} 0, & n < m \\ 1, & n = m \\ N, & n > m \end{cases} \]

\[ y(n) = 0, \quad n < 0 \]
\[ y(n) = 1, \quad n = 0 \]
\[ y(n) = N-1, \quad n=1 \]
\[ y(n) = N-2N+1 = -N+1, \quad n=2 \]
\[ y(n) = N-2N+N = 0, \quad n > 2 \]

\[ \therefore y(n) = \{ 0, 1, (N-1), (-N+1), 0 \} \]