Question 1 (30%)

Determine whether or not each of the following signals is periodic. In case a signal is periodic, specify its fundamental period.

(a) Discrete-time signal: $x(n) = \dfrac{\sqrt{1+\cos(\frac{\pi n}{8})}}{\sin(\frac{\pi n}{65})+1.2}$ (10%)

(b) Discrete-time signal: $x(n) = \sin(\frac{\pi n}{3}) + \cos(\frac{\pi n}{5}) + \delta(n-3)$,

where $\delta(n-3) = \begin{cases} 1, & n=3 \\ 0, & otherwise \end{cases}$ (10%)

(c) Analog signal: $x_a(t) = \cos(3.5t)rect(\frac{t}{8.2})$,

where $rect(\frac{t}{8.2}) = \begin{cases} 1, & -4.1 \le t \le 4.1 \\ 0, & elsewhere \end{cases}$ (10%)

Answer to Question 1:

(a) $\sqrt{1+\cos(\frac{\pi n}{8})} = \sqrt{2}\left|\cos(\frac{\pi n}{16})\right|$

$\frac{\pi}{8} N_p = 2\pi m \Rightarrow m=1, N=16$

$\Rightarrow \left|\cos(\frac{\pi n}{16})\right|$ has the period of $N_{p_1} = \frac{16}{2} = 8$

$\frac{5\pi N_{p_2}}{65} = 2\pi m \Rightarrow m=1, N_{p_2} = 130$

$N_p = LCM(N_{p_1}, N_{p_2}) = LCM(8, 130) = 520$ #

(b) Since $\delta(n-3)$ is not periodic,

$x(n)$ is not periodic.

(c) $x_a(t)$ is not periodic.

Question 2 (25%)

An analog band-limited signal $x_a(t)$ is sampled at the rate of F_s through an A/D converter to construct a discrete-time signal $x(n)$. $x(n)$ is assumed to be a periodic signal such that $x(n+M) = x(n)$, $\forall n$, where M is a large integer (M>>2) and it is not necessary the fundamental period.

(a) Since M may not be the fundamental period, we want to search for a smaller integer N than M from M given signal samples $x(0), x(1), x(2), \ldots, x(M-1)$ through computer. Two sets of positive integers are defined here, $D_k = \{k \mid M$ is divisible by k, $2 \le k \le M$, k is integer$\}$, $S_k = \{k \mid x(ik+n)=x(ik+k+n)$, $i=0,1,\ldots,M/k-2$, $n=0,1,\ldots,k-1$, $k \in D_k\}$. N can be determined by $N = \begin{cases} M/\max\{S_k\}, & S_k \ne \phi \\ M, & S_k = \phi \end{cases}$, where ϕ is the empty set. Is $\dfrac{N}{F_s}$ always the fundamental period T_p of the original analog signal? If your answer is no, show an example to justify your answer. (15%)

(b) Under what condition on the sampling frequency, $\dfrac{N}{F_s}$ is always the fundamental period T_p? (10%)

Answer to Question 2:

(a) $\frac{N}{F_s}$ is not necessary the fundamental period.

For example, $x_a(t) = A\cos(2\pi F_c t)$ is sinusoidal with frequency F_c Hz, and hence the fundamental period is $T_p = \frac{1}{F_c}$.

If we sample at $F_s = \frac{2}{3} F_c$ Hz,

$$x(n) = A\cos\left(2\pi \frac{F_c}{F_s} n\right) = A\cos(3\pi n).$$

$N = 2$ according to our algorithm

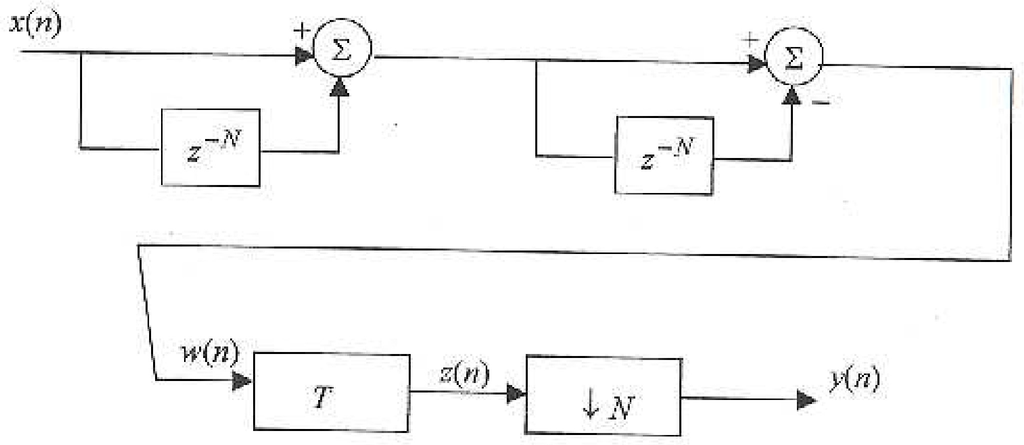
However $\frac{N}{F_s} = \frac{2}{\frac{2}{3}F_c} = \frac{3}{F_c} \neq \frac{1}{F_c} = T_p$ #.

(b) According to the sampling theorem,

$F_s > 2F_{max}$, where F_{max} is the largest frequency which this signal contains.

Question 3 (45%)

A digital communication system is depicted as below.



$x(n)$, $w(n)$, $z(n)$ and $y(n)$ are all discrete-time sequences. The system T is a moving average operation such that $z(n) = \sum_{k=0}^{N-1} w(n-k)$. $\downarrow N$ is a down-sampler where $y(n) = z(Nn)$ and N is an arbitrary integer greater than one.

(a) Is this communication system a linear system? Justify your answer through mathematics. (10%)

(b) Is this communication system a time-invariant system? Justify your answer through mathematics. (10%)

(c) Is this communication system BIBO stable? (10%)

(d) What is the output sequence $y(n)$ for a given input $x(n) = u(n)$ where $u(n)$ is a unit-step sequence? (15%)

Answer to Question 3:

(a)

$$w(n) = x(n) - x(n-N) - x(n-N) + x(n-2N)$$

$$= x(n) - 2x(n-N) + x(n-2N)$$

$$z(n) = \sum_{k=0}^{N-1} w(n-k) = \sum_{k=0}^{N-1} x(n-k) - 2\sum_{k=0}^{N-1} x(n-k-N) + \sum_{k=0}^{N-1} x(n-k-2N)$$

$$y(n) = z(Nn) = \sum_{k=0}^{N-1} x(Nn-k) - 2\sum_{k=0}^{N-1} x(Nn-k-N) + \sum_{k=0}^{N-1} x(Nn-k-2N)$$

(a) Assume $x_1(n) \longrightarrow y_1(n)$, $x_2(n) \longrightarrow y_2(n)$

$$x_3(n) = \alpha x_1(n) + \beta x_2(n) \longrightarrow y_3(n)$$

$$y_3(n) = \sum_{k=0}^{N-1} \{\alpha x_1(Nn-k) + \beta x_2(Nn-k)\} - 2\sum_{k=0}^{N-1} \{\alpha x_1(Nn-k-N) + \beta x_2(Nn-k-N)\} + \sum_{k=0}^{N-1} \{\alpha x_1(Nn-k-2N) + \beta x_2(Nn-k-2N)\}$$

$$= \alpha y_1(n) + \beta y_2(n)$$

$\therefore$ This is a linear system.

(b) $x(n) \rightarrow y(n)$

$x(n-l) \rightarrow y_2(n)$

$$y_2(n) = \sum_{k=0}^{N-1} x(Nn-k-l) - 2\sum_{k=0}^{N-1} x(Nn-k-N-l) + \sum_{k=0}^{N-1} x(Nn-k-2N-l)$$

$$y(n-l) = \sum_{k=0}^{N-1} x(Nn-Nl-k) - 2\sum_{k=0}^{N-1} x(Nn-Nl-k-N) + \sum_{k=0}^{N-1} x(Nn-Nl-k-2N)$$

Since $y_2(n) \neq y(n-l)$, this is not time-invariant. //

(c) Given a finite input such that

$|x(n)| \leq M_x < \infty$,

$$|y(n)| = \left| \sum_{k=0}^{N-1} x(Nn-k) - 2\sum_{k=0}^{N-1} x(Nn-k-N) + \sum_{k=0}^{N-1} x(Nn-k-2N) \right|$$

$$\leq \sum_{k=0}^{N-1} |x(Nn-k)| + 2\sum_{k=0}^{N-1} |x(Nn-k-N)| + \sum_{k=0}^{N-1} |x(Nn-k-2N)| \leq 4NM_x$$

$\therefore \; |y(n)| \le M_y < \infty$, where $M_y = 4NM_x$

$\Rightarrow$ This is a BIBO stable system #

(d)

$$y(n) = \sum_{k=0}^{N-1} u(Nn-k) - 2\sum_{k=0}^{N-1} u(Nn-k-N)$$

$$+ \sum_{k=0}^{N-1} u(Nn-k-2N)$$

Since $u(Nn-k-mN) = \begin{cases} 1, & (n-m)N \ge k \\ 0, & (n-m)N < k \end{cases}$,

$$\sum_{k=0}^{N-1} u(Nn-k-mN) = \begin{cases} 0, & n < m \\ 1, & n = m \\ N, & n > m \end{cases}$$

$y(n) = 0, \quad n < 0$

$y(n) = 1, \quad n = 0$

$y(n) = N-1, \quad n = 1$

$y(n) = N-2N+1 = -N+1, \quad n = 2$

$y(n) = N-2N+N = 0, \quad n > 2$

$$\therefore \; y(n) = \{0, \underset{\uparrow}{1}, (N-1), (-N+1), 0\}$$