EE4150 Digital Signal Processing

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Final Examination, Spring of 2002

Time: 3~5 p.m., Tuesday, May 14, 2002

Please ALWAYS work on the FRONT SIDE of each page. No answer on the BACK SIDE of any page will be GRADED! Ask for additional blank papers if necessary!

You may check any textbook, classnote or other references alone during the test. However, NO CHEATING or COLLABORATION is allowed and such violation of the university regulations will be reported.

Please write down your name and social security number here:

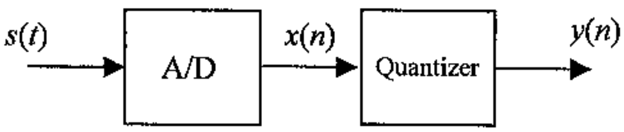
Full Name: Solution

Social Security Number: ______________________

Partition	Score
Question 1	
Question 2	
Question 3	
Question 4	
Total	

Question 1 (25%)

A periodic triangular train $s(t) = \sum_{k=-\infty}^{\infty} tri(\frac{t-kT}{T/2})$ needs to be sampled and then quantized, where $tri(\frac{t}{T/2}) = \begin{cases} \frac{T/2-|t|}{T/2}, & |t| \le T/2 \\ 0, & elsewhere \end{cases}$ as below.



We sample $s(t)$ at $t = 0, \pm\frac{T}{4}, \pm\frac{2T}{4}, \pm\frac{3T}{4}, ..., \frac{nT}{4}, ...,$ to form $x(n)$=$s(nT/4)$ where n is any integer.

(a) What is the fundamental period N_p of $x(n)$? (5%)

(b) What is the sequence $x(n)$ within the fundamental interval $0 \le n \le N_p - 1$? (5%)

(c) What is the mean squared quantization error between $x(n)$ and the quantized signal $y(n)$, if four quantization intervals are applied? {Hint: $\Delta = \frac{x_{\max} - x_{\min}}{4}$ and

$$y(n) = \begin{cases} x_{\min} + k\Delta + \frac{\Delta}{2}, & if\ x_{\min} + k\Delta \le x(n) < x_{\min} + (k+1)\Delta,\ k = 0,1,2,3 \\ x_{\max}, & if\ x(n) = x_{\max} \end{cases}\} \ (15\%)$$

Answer to Question 1:

(a) $x(n) = \{ \cdots 1, \frac{1}{2}, 0, \frac{1}{2}, 1, \frac{1}{2}, 0, \frac{1}{2} \cdots \}$

$$\uparrow$$

$$N_p = \frac{T}{\frac{T}{4}} = 4$$

(b) $x(n) = 1, \frac{1}{2}, 0, \frac{1}{2}, \quad 0 \le n \le 3$

(c) $X_{max} = 1, \quad X_{min} = 0, \quad \Delta = \frac{1}{4}$

$x(n)$	$y(n)$
1	1
$\frac{1}{2}$	$\frac{5}{8}$
0	$\frac{1}{8}$
$\frac{1}{2}$	$\frac{5}{8}$

$$MSE = \frac{1}{4} \sum_{n=0}^{3} [x(n) - y(n)]^2$$

$$= \frac{1}{4} \left(0 + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} \right)$$

$$= \frac{3}{256}$$

Question 2 (25%)

A linear-time-invariant system transfer function can be described as a rational Z-transform $H(z) = \frac{(z-0.5)}{(z^2+1.1z+0.3)}$. Initial conditions of the output response are given by

$y(-1) \neq 0, \quad y(-2) \neq 0, \quad y(-3) = y(-4) = y(-5) = \ldots = 0$.

(a) What is the corresponding difference equation for this LTI system? (5%)

(b) What is the zero-state response for a unit-step input sequence? (15%)

(c) What is the output response for a unit-step input sequence? (5%)

Answer to Question 2:

(a) $$H(z) = \frac{Y(z)}{X(z)} = \frac{(z-0.5)}{(z^2+1.1z+0.3)}$$

$$= \frac{z^{-1}-0.5z^{-2}}{1+1.1z^{-1}+0.3z^{-2}}$$

$$y(n) = -1.1y(n-1) - 0.3y(n-2) + x(n-1) - 0.5x(n-2)$$

(b) Assume $w(n) = aw(n-1) + x(n-1) - 0.5x(n-2)$

$$y(n) = by(n-1) + w(n)$$

$$y(n) - by(n-1) = ay(n-1) - ab\,y(n-2) + x(n-1) - 0.5x(n-2)$$

$$\therefore\ y(n) = (a+b)y(n-1) - ab\,y(n-2) + x(n-1) - 0.5x(n-2)$$

$$\Rightarrow\ a = -0.5,\ b = -0.6$$

$$w(-1) = y(-1) - by(-2)$$

$$w(-2) = y(-2)$$

$$W_{zs}(n) = \sum_{k=0}^{n} a^k x(n-k-1) - \sum_{k=0}^{n} 0.5a^k x(n-k-2)$$

$$= \sum_{k=0}^{n} a^k u(n-k-1) - \sum_{k=0}^{n} 0.5a^k u(n-k-2)$$

$$= \sum_{k=0}^{n-1} a^k u(n-1) - 0.5\sum_{k=0}^{n-2} a^k u(n-2)$$

$$= \frac{1-a^{n}}{1-a} u(n-1) - 0.5 \frac{1-a^{n-1}}{1-a} u(n-2)$$

$$W_{ZI}(n) = a^{n+1} W(-1) u(n)$$

$$W(n) = \frac{1-a^{n}}{1-a} u(n-1) - 0.5 \frac{1-a^{n-1}}{1-a} u(n-2) + a^{n+1} W(-1)$$

$$y'_{ZS}(n) = \sum_{k=0}^{n} b^{k} w(n-k)$$

$$= \sum_{k=0}^{n} \left(\frac{b^{k}}{1-a}\right) u(n-k-1) - \sum_{k=0}^{n} \frac{b^{k} a^{n-k}}{1-a} u(n-k-1)$$

$$- 0.5 \sum_{k=0}^{n} \frac{b^{k}}{1-a} u(n-k-2) + 0.5 \sum_{k=0}^{n} \frac{b^{k} a^{n-k-1}}{1-a} u(n-k-2)$$

$$+ \sum_{k=0}^{n} b^{k} a^{n-k+1} W(-1) u(n-k)$$

$$= \frac{(1-b^{n}) u(n-1)}{(1-a)(1-b)} - \frac{a^{n}}{1-a} \frac{1-\left(\frac{b}{a}\right)^{n}}{1-\frac{b}{a}} u(n-1)$$

$$-0.5 \frac{(1-b^{n-1})}{(1-a)(1-b)} u(n-2) + 0.5 \frac{a^{n-1}}{1-a} \frac{1-\left(\frac{b}{a}\right)^{n-1}}{1-\frac{b}{a}} u(n-2)$$

$$+ \frac{\left[1-\left(\frac{b}{a}\right)^{n+1}\right]}{1-\frac{b}{a}} a^{n+1} u(n) W(-1)$$, given input $w(n)$

(※ $y'_{ZS}(n)$ is not the actual $y_{ZS}(n)$)

$$= \frac{[1-(-0.6)^n]\,u(n-1)}{2.4} + \frac{(-0.5)^n[1-(1.2)^n]}{0.3}u(n-1)$$

$$- \frac{[1-(-0.6)^{n-1}]}{4.8}u(n-2) - \frac{(-0.5)^{n-1}[1-(1.2)^{n-1}]}{0.6}u(n-2)$$

$$- \frac{[1-(1.2)^{n+1}]}{0.2}(-0.5)^{n+1}[y(-1)+0.6\,y(-2)]\,u(n)$$

$$y_{ZS}(n) = y_{ZS}'(n)\Big|_{y(-1)=y(-2)=0} = \frac{[1-(-0.6)^n]\,u(n-1)}{2.4}$$

$$+ \frac{(-0.5)^n[1-(1.2)^n]}{0.3}u(n-1) - \frac{[1-(-0.6)^{n-1}]}{4.8}u(n-2)$$

$$- \frac{(-0.5)^{n-1}[1-(1.2)^{n-1}]}{0.6}u(n-2)$$

(c) $y_{ZI}'(n) = b^{n+1}y(-1)\,u(n) = (-0.6)^{n+1}y(-1)\,u(n)$

$$\therefore\ y(n) = y_{ZS}'(n) + y_{ZI}'(n)$$

$$= \frac{[1-(-0.6)^n]\,u(n-1)}{2.4} + \frac{(-0.5)^n[1-(1.2)^n]}{0.3}u(n-1)$$

$$- \frac{[1-(-0.6)^{n-1}]}{4.8}u(n-2) - \frac{(-0.5)^{n-1}[1-(1.2)^{n-1}]}{0.6}u(n-2)$$

$$- \frac{[1-(1.2)^{n+1}]}{0.2}(-0.5)^{n+1}[y(-1)+0.6\,y(-2)]\,u(n)$$

$$+ (-0.6)^{n+1}y(-1)\,u(n)$$

Question 3 (25%)

A linear-time-invariant BIBO stable system can be described as a rational Z-transform $H(z) = \dfrac{1}{(1-az^{-1})^p (1-bz^{-1})^q}$, where p, q can be any positive integers. The impulse response $h(n)$ can be derived through the inverse Z-transform.

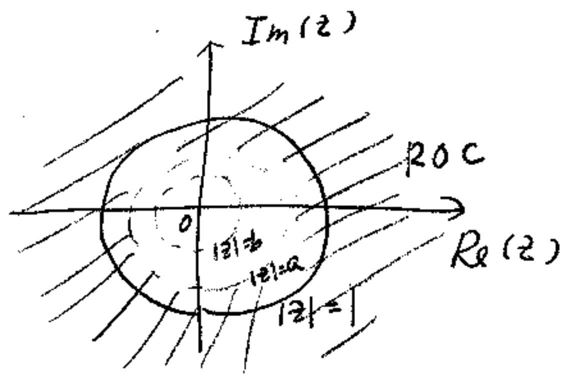
(a) If 1>a>b>0, what is the requirement for r where r is the radius of contour C: $|z| = r$ we need to do the contour integral? (5%)

(b) If a>1>b>0, what is the requirement for r where r is the radius of contour C: $|z| = r$ we need to do the contour integral? (5%)

(c) What is the impulse response $h(n)$ given that a>1>b>0 and p=q=2? (15%)

Answer to Question 3:

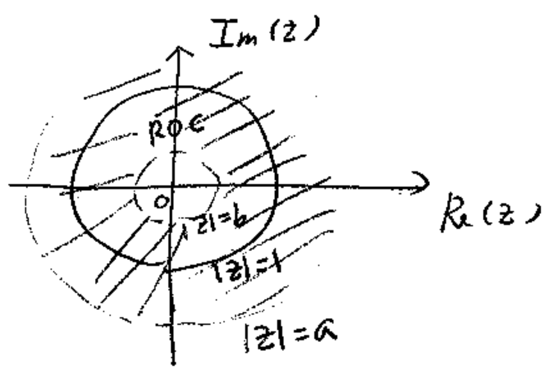
(a) $$H(z) = \frac{z^{p+q}}{(z-a)^p (z-b)^q}$$



ROC: $|z|>a \cap |z|>b \Rightarrow |z|>a$

$\therefore$ $|z|=r \subset |z|>a \Rightarrow r>a$ #

(b)



ROC: $|z|>b \cap |z|<a \Rightarrow b<|z|<a$

$\therefore$ $|z|=r \subset b<|z|<a \Rightarrow b<r<a$ #

(c) $$\frac{H(z)}{z} = \frac{z^3}{(z-a)^2(z-b)^2}$$

Assume $$\frac{H(z)}{z} = \frac{A_{11}}{(z-a)} + \frac{A_{12}}{(z-a)^2} + \frac{A_{21}}{(z-b)} + \frac{A_{22}}{(z-b)^2}$$

$$A_{11} = \frac{d}{dz}\left[\frac{z^3}{(z-b)^2}\right]\Bigg|_{z=a}$$

$$= \left[\frac{3z^2}{(z-b)^2} - \frac{2z^3(z-b)}{(z-b)^4}\right]\Bigg|_{z=a}$$

$$= \frac{3a^2}{(a-b)^2} - \frac{2a^3}{(a-b)^3}$$

$$= \frac{3a^3 - 3a^2b - 2a^3}{(a-b)^3} = \frac{a^3 - 3a^2b}{(a-b)^3}$$

$$A_{12} = \frac{z^3}{(z-b)^2}\Bigg|_{z=a} = \frac{a^3}{(a-b)^2}$$

$$A_{21} = \frac{d}{dz}\left[\frac{z^3}{(z-a)^2}\right]\Bigg|_{z=b}$$

$$= \frac{b^3 - 3ab^2}{(b-a)^3}$$

$$A_{22} = \frac{z^3}{(z-a)^2}\Bigg|_{z=b} = \frac{b^3}{(b-a)^2}$$

$$\therefore \quad H(z) = \frac{a^3-3a^2b}{(a-b)^3}\frac{z}{(z-a)} + \frac{a^3}{(a-b)^2}\frac{z}{(z-a)^2}$$

$$+ \frac{b^3-3ab^2}{(b-a)^3}\frac{z}{(z-b)} + \frac{b^3}{(b-a)^2}\frac{z}{(z-b)^2}$$

(d) $$\frac{1}{2\pi j}\oint_C \frac{z}{(z-\beta)^m} z^{n-1}\, dz = \frac{1}{2\pi j}\oint_C \frac{z^n}{(z-\beta)^m}\, dz,$$

$m = 1$ or 2

(i) if $n \geq 0$

$$\frac{1}{2\pi j}\oint_C \frac{z}{(z-\beta)^m} z^{n-1}\, dz$$

$$= \begin{cases} \dfrac{1}{(m-1)!}\dfrac{n!}{(n-m+1)!}\beta^{n-m+1} u(n-m+1), & \text{when } \beta \text{ is inside } C \\ 0, & \text{when } \beta \text{ is outside } C \end{cases}$$

(ii) if $n < 0$,

$$\frac{1}{2\pi j}\oint_C \frac{z}{(z-\beta)^m} z^{n-1}\, dz, \quad \text{let } l = -n > 0$$

$$= \frac{1}{2\pi j}\oint_C \frac{1}{z^l (z-\beta)^m}\, dz$$

Assume $\displaystyle \frac{1}{z^l (z-\beta)^m} = \sum_{k_1=1}^{l} \frac{A_{1k_1}}{z^{k_1}} + \sum_{k_2=1}^{m} \frac{A_{2k_2}}{(z-\beta)^{k_2}}$

$$A_{1k_1} = \frac{1}{(l-k_1)!} \frac{d^{(l-k_1)}}{dz^{(l-k_1)}} \left[(z-\beta)^{-m} \right] \Bigg|_{z=0}$$

$$= \frac{1}{(l-k_1)!} \frac{(-m)(-m-1)(-m-2)\cdots}{(-m-l+k_1)(-m-l+k_1-1)\cdots} (-\beta)^{-m-l+k_1}$$

$$A_{2k_2} = \frac{1}{(m-k_2)!} \frac{d^{(m-k_2)}}{dz^{(m-k_2)}} \left[z^{-l} \right] \Bigg|_{z=\beta}$$

$$= \frac{1}{(m-k_2)!} \frac{(-l)(-l-1)(-l-2)\cdots}{(-l-m+k_2)(-l-m+k_2-1)\cdots} \beta^{-l-m+k_2}$$

$$A_{11} = \frac{1}{(l-1)!} (-m)(-m-1)\cdots(-m-l+2) (-\beta)^{-m-l+1}$$

$$= \frac{1}{(l-1)!} (-1)^{(l-1)} \frac{(m+l-2)!}{(m-1)!} (-1)^{-m-l+1} \beta^{-m-l+1}$$

$$= C^{m+l-2}_{l-1} (-1)^{-m} \beta^{-m-l+1}$$

$$A_{21} = \frac{1}{(m-1)!} (-1)^{(m-1)} \frac{(m+l-2)!}{(l-1)!} \beta^{-l-m+1}$$

$$= -C^{m+l-2}_{l-1} (-1)^{m} \beta^{-m-l+1} = -A_{11}$$

$$\therefore \frac{1}{2\pi j} \oint_C \frac{z}{(z-\beta)^m} z^{n-1} dz = \begin{cases} 0, & \text{if } \beta \text{ is inside } C \\ A_{11}, & \text{if } \beta \text{ is outside } C \end{cases}$$

Since $a > r > b > 0$,

(i) $n \geq 0$, $h(n) = \mathcal{Z}^{-1}\{H(z)\}$

$$= \frac{(b^3 - 3ab^2)}{(b-a)^3} b^n u(n) + \frac{b^3}{(b-a)^2} n b^{n-1} u(n-1)$$

(ii) $n<0$, $h(n) = \mathcal{Z}^{-1}\{H(z)\}$

$$= -\frac{(a^3-3a^2b)}{(a-b)^3}\, a^n u(-1-n) - \frac{a^3}{(a-b)^2}\, n\, a^{n-1} u(-1-n)$$

$$\therefore \quad h(n) = \frac{(b^3-3ab^2)}{(b-a)^3}\, b^n u(n) + \frac{b^3}{(b-a)^2}\, n\, b^{n-1} u(n-1)$$

$$- \frac{(a^3-3a^2b)}{(a-b)^3}\, a^n u(-1-n) - \frac{a^3}{(a-b)^2}\, n\, a^{n-1} u(-1-n)$$

$$\text{or} \quad \frac{(b^3-3ab^2)}{(b-a)^3}\, b^n u(n) + \frac{b^3}{(b-a)^2}\, n\, b^{n-1} u(n)$$

$$- \frac{(a^3-3a^2b)}{(a-b)^3}\, a^n u(-1-n) - \frac{a^3}{(a-b)^2}\, n\, a^{n-1} u(-1-n)$$

Question 4 (25%)

A digital signal $x(n)$ is downsampled to form a new sequence $y(n)=x(2n)$. On the other hand, $x(n)$ is upsampled to form a new sequence $z(n)$ such that $z(Nn+k)=x(n)$, $k=0,1,2\ldots,(N-1)$, where N is an arbitrary positive integer. The discrete-time Fourier transforms for $x(n)$, $y(n)$, $z(n)$ are $X(\omega)$, $Y(\omega)$ and $Z(\omega)$ respectively.

(a) What is the relationship between $x(2n)$ and $x(2n+1)$ if $X(\omega)=Y(2\omega)[1+e^{-j\omega}]$? (15%)

(b) What is $Z(\omega)$ in terms of $X(\omega)$? (10%)

Answer to Question 4:

(a) $$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(2n+1) e^{-j\omega(2n+1)}$$

$$+ \sum_{n=-\infty}^{\infty} x(2n) e^{-j\omega(2n)}$$

Let $x_1(n) = x(2n+1)$ and $y(n) = x(2n)$

$$\therefore X(\omega) = \sum_{n=-\infty}^{\infty} x_1(n) e^{-j(2\omega)n} e^{-j\omega}$$

$$+ \sum_{n=-\infty}^{\infty} y(n) e^{-j(2\omega)n}$$

$$= X_1(2\omega) e^{-j\omega} + Y(2\omega)$$

where $X_1(\omega) = \sum_{n=-\infty}^{\infty} x_1(n) e^{-j\omega n}$

and $Y(\omega) = \sum_{n=-\infty}^{\infty} y(n) e^{-j\omega n}$

$$\therefore X(\omega) = X_1(2\omega) e^{-j\omega} + Y(2\omega)$$

$$= Y(2\omega) e^{-j\omega} + Y(2\omega)$$

$$\therefore \quad X_1(2\omega) = Y(2\omega) \Rightarrow X_1(\omega) = Y(\omega)$$

$$\Rightarrow \quad x_1(n) = y(n) \Rightarrow x(2n+1) = x(2n) \text{ *}$$

(b)

$$Z(\omega) = \sum_{n=-\infty}^{\infty} z(n) e^{-j\omega n}$$

$$= \sum_{k=0}^{N-1} \sum_{n=-\infty}^{\infty} z(Nn+k) \, e^{-j\omega(Nn+k)}$$

$$= \sum_{k=0}^{N-1} \sum_{n=-\infty}^{\infty} x(n) \, e^{-j(N\omega)n} \, e^{-j\omega k}$$

$$= \sum_{k=0}^{N-1} X(N\omega) \, e^{-j\omega k}$$