EE4150 Digital Signal Processing

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Final Examination, Spring of 2003

Time: 7:30~9:30 a.m., Saturday, May 17, 2003

Please ALWAYS work on the FRONT SIDE of each page. No answer on the BACK SIDE of any page will be GRADED! Ask for additional blank papers if necessary!

You may check any textbook, classnote or other references alone during the test. However, NO CHEATING or COLLABORATION is allowed and such violation of the university regulations will be reported.

Please write down your name and social security number here:

Full Name: _________________SOLUTION____________________

Social Security Number: ____

Partition	Score
Question 1	
Question 2	
Question 3	
Question 4	
Total	

Question 1 (30%)

The Z-transform of a discrete-time sequence $x(n)$ is $X(z) = \dfrac{z^2}{(z-a)^2(z-b)}$, $a > 1 > b > 0$.

(a) What is the partial fraction expansion of $X(z)$? (10%)

(b) What is $x(n)$ if the ROC is $a > |z| > b$? (10%)

(c) What is $x(n)$ if the ROC is $|z| > a$? (10%)

Answer to Question 1:

(a) Assume $\frac{X(z)}{z} = \frac{A_{11}}{(z-a)} + \frac{A_{12}}{(z-a)^2} + \frac{A_{21}}{z-b}$.

$$A_{11} = \frac{d}{dz}\left[(z-a)^2 \frac{X(z)}{z}\right]\Bigg|_{z=a} = \left.\frac{(z-b)-z}{(z-b)^2}\right|_{z=a} = \frac{-b}{(a-b)^2}$$

$$A_{12} = \left[(z-a)^2 \frac{X(z)}{z}\right]\Bigg|_{z=a} = \left.\frac{z}{(z-b)}\right|_{z=a} = \frac{a}{(a-b)}$$

$$A_{21} = \left[(z-b) \frac{X(z)}{z}\right]\Bigg|_{z=b} = \left.\frac{z}{(z-a)^2}\right|_{z=b} = \frac{b}{(a-b)^2}$$

$$\therefore X(z) = \frac{-b}{(a-b)^2}\frac{z}{z-a} + \frac{a}{a-b}\frac{z}{(z-a)^2} + \frac{b}{(a-b)^2}\frac{z}{z-b}.$$

(b) $x(n) = Z^{-1}\{X(z)\}$

$$= \frac{b}{(a-b)^2}a^n u(-n-1) - \frac{1}{a-b} na^n u(-n-1) + \frac{1}{(a-b)^2} b^{n+1} u(n)$$

(c) $x(n) = Z^{-1}\{X(z)\}$

$$= \frac{-b}{(a-b)^2}a^n u(n) + \frac{1}{a-b} na^n u(n) + \frac{1}{(a-b)^2} b^{n+1} u(n)$$

<u>Question 2</u> (20%)

A continuous-time function is stated as $x(t)=\begin{cases}1, & -a\le t\le a\\ 0, & otherwise\end{cases}$, where a>0.

(a) What is its Fourier transform $X(\Omega)$? (10%)

(b) Based on the result in (a), determine the inverse Fourier transform of $Y(\Omega)$, such that

$$Y(\Omega)=\begin{cases}1, & -\pi\le\Omega\le\pi\\ 0, & otherwise\end{cases}.\ (10\%)$$

Answer to Question 2:

(a) $X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt = \int_{-a}^{a} e^{-j\Omega t} dt = \left. \frac{e^{-j\Omega t}}{-j\Omega} \right|_{-a}^{a} = \frac{e^{-j\Omega a} - e^{j\Omega a}}{-j\Omega} = \left[\frac{-2j\sin\left(\frac{\Omega a}{2}\right)}{-j\Omega} \right]$

$= a \sin c\left(\frac{\Omega a}{2}\right)$, where $\sin c(\vartheta) = \frac{\sin(\vartheta)}{\vartheta}$

(b) Since $x(t) \xrightarrow{\text{Fourier Trasnform}} X(\Omega)$ exists,

$X(t) \xrightarrow{\text{Fourier Trasnform}} 2\pi x(-\Omega)$

$\because Y(\Omega) = x(-\Omega)|_{a=\pi}$

$\therefore IFT\{Y(\Omega)\} = IFT\{x(-\Omega)|_{a=\pi}\} = \frac{1}{2\pi} X(t)\Big|_{a=\pi} = \frac{1}{2\pi} \pi \sin c\left(\frac{\Omega\pi}{2}\right) = \frac{1}{2} \sin c\left(\frac{\Omega\pi}{2}\right)$

Question 3 (30%)

A discrete-time sequence $y(n)$ results from a convolution of three discrete-time sequences, namely $x_1(n) = x_2(n) = a^n u(n)$ and $x_3(n) = b^n u(n)$, where 0<a<1, 0<b<1, are constants and $u(n)$ is the unit-step sequence.

(a) Determine the DTFT of $x_1(n)$. (10%)

(b) Determine the DTFT of $x_3(n)$. (10%)

(c) Determine the sequence $y(n) = x_1(n) \otimes x_2(n) \otimes x_3(n)$. (10%)

Answer to Question 3:

(a) $X_1(\omega) = X_2(\omega) = DTFT[x_1(n)] = Z\{x_1(n)\}\Big|_{z=e^{j\omega}} = \dfrac{z}{z-a}\Big|_{z=e^{j\omega}} = \dfrac{e^{j\omega}}{e^{j\omega}-a}$

(b) $X_3(\omega) = DTFT[x_3(n)] = Z\{x_3(n)\}\Big|_{z=e^{j\omega}} = \dfrac{z}{z-b}\Big|_{z=e^{j\omega}} = \dfrac{e^{j\omega}}{e^{j\omega}-b}$

(c) $Y(z) = Z\{y(n)\} = Z\{x_1(n)\}Z\{x_2(n)\}Z\{x_3(n)\} = \dfrac{z^3}{(z-a)^2(z-b)}$

Assume $\dfrac{Y(z)}{z} = \dfrac{A_{11}}{z-a} + \dfrac{A_{12}}{(z-a)^2} + \dfrac{A_{21}}{z-b}$.

$$A_{11} = \frac{d}{dz}\left[(z-a)^2\frac{Y(z)}{z}\right]_{z=a} = \frac{d}{dz}\left[\frac{z^2}{z-b}\right]_{z=a} = \frac{2z(z-b)-z^2}{(z-b)^2}\Bigg|_{z=a} = \frac{a^2-2ab}{(a-b)^2}$$

$$A_{12} = \left[(z-a)^2\frac{Y(z)}{z}\right]_{z=a} = \left[\frac{z^2}{z-b}\right]_{z=a} = \frac{a^2}{(a-b)^2}$$

$$A_{21} = \left[(z-b)\frac{Y(z)}{z}\right]_{z=a} = \left[\frac{z^2}{(z-a)^2}\right]_{z=b} = \frac{b^2}{(a-b)^2}$$

$$\therefore Y(z) = \frac{a^2-2ab}{(a-b)^2}\frac{z}{z-a} + \frac{a^2}{(a-b)^2}\frac{z}{(z-a)^2} + \frac{b^2}{(a-b)^2}\frac{z}{z-b}$$

$$\Rightarrow y(n) = Z^{-1}\{Y(z)\} = \frac{a^2-2ab}{(a-b)^2}a^n u(n) + \frac{1}{(a-b)^2}na^{n+1}u(n) + \frac{b^2}{(a-b)^2}b^n u(n)$$

Question 4 (20%)

The Z-transform of a filter $h(n)$ is stated as $H(z) = \dfrac{z}{\left(z - 0.99e^{j\frac{3\pi}{4}}\right)\left(z - 0.99e^{-j\frac{3\pi}{4}}\right)}$,

where ROC is |z|>0.99.

(a) What is the DTFT of $h(n)$? (10%)

(b) What is the magnitude gain for a complex exponential signal $x(n) = e^{j\frac{\pi n}{4}}$ filtered by this $h(n)$? (10%)

Answer to Question 4:

(a) $H(\omega) = DTFT[h(n)] = Z\{x(n)\}\big|_{z=e^{j\omega}} = \left. \dfrac{z}{\left(z - 0.99e^{j\frac{3\pi}{4}}\right)\left(z - 0.99e^{-j\frac{3\pi}{4}}\right)} \right|_{z=e^{j\omega}}$

$$= \frac{e^{j\omega}}{\left(e^{j\omega} - 0.99e^{j\frac{3\pi}{4}}\right)\left(e^{j\omega} - 0.99e^{-j\frac{3\pi}{4}}\right)}$$

(b) The magnitude gain is

$$|H(\omega)|\big|_{\omega=\frac{\pi}{4}} = \left| \frac{e^{j\frac{\pi}{4}}}{\left(e^{j\frac{\pi}{4}} - 0.99e^{j\frac{3\pi}{4}}\right)\left(e^{j\frac{\pi}{4}} - 0.99e^{-j\frac{3\pi}{4}}\right)} \right|$$

$$= \left| \frac{1}{(1-0.99j)(1+0.99)} \right| = \frac{1}{\sqrt{1+0.99^2}\,(1+0.99)} \approx \frac{1}{2\sqrt{2}}$$