5.9 **(a)** $Y_a[k] = \sum_{n=0}^{N-1} \alpha^n W_N^{kn} = \sum_{n=0}^{N-1} (\alpha W_N^k)^n = \dfrac{1-\alpha W_N^{kN}}{1-\alpha W_N^k} = \dfrac{1-\alpha}{1-\alpha W_N^k}.$

(b) $Y_b[k] = 2 \sum_{k \text{ even}} W_N^{kn} - 3 \sum_{k \text{ odd}} W_N^{kn}$. Assume first N is even, i.e., $N = 2L$. Then

$$Y_b[k] = 2\sum_{r=0}^{L-1} W_{2L}^{k2r} - 3\sum_{r=0}^{L-1} W_{2L}^{k(2r+1)} = 2\sum_{r=0}^{L-1} W_L^{kr} - 3W_{2L}^k \sum_{r=0}^{L-1} W_L^{kr} = \left(2-3W_N^k\right)\left(\frac{1-W_L^{kL}}{1-W_L^k}\right) = 0.$$

Next, assume N is odd, i.e., $N = 2L+1$. Then $Y_b[k] = 2\sum_{r=0}^{L} W_{2L}^{k2r} - 3\sum_{r=0}^{L-1} W_{2L}^{k(2r+1)}$

$$= 2\sum_{r=0}^{L} W_L^{kr} - 3W_{2L}^k \sum_{r=0}^{L-1} W_L^{kr} = 2\left(\frac{1-W_L^{k(L+1)}}{1-W_L^k}\right) - 3W_{2L}^k\left(\frac{1-W_L^{kL}}{1-W_L^k}\right) = 2\left(\frac{1-W_L^k}{1-W_L^k}\right) = 2.$$

5.12 $X[k] = \sum_{n=0}^{N-1} x[n]W_N^{nk} = \sum_{n=0}^{(N/2)-1} x[n]W_N^{nk} + \sum_{n=N/2}^{N-1} x[n]W_N^{nk}$

$$= \sum_{n=0}^{(N/2)-1} x[n]W_N^{nk} + W_N^{(N/2)k} \sum_{n=0}^{(N/2)-1} x[\frac{N}{2}+n]W_N^{nk}$$

$= \sum_{n=0}^{(N/2)-1} \left(x[n] + (-1)^k x[\frac{N}{2}+n]\right)W_N^{nk}$. For $k = 2\ell$, we get

$$X[2\ell] = \sum_{n=0}^{(N/2)-1} \left(x[n] + x[\frac{N}{2}+n]\right)W_N^{2n\ell} = \sum_{n=0}^{(N/2)-1} \left(x[n] + x[\frac{N}{2}+n]\right)W_{N/2}^{n\ell} = X_0[\ell]$$

and for $k = 2\ell+1$ we get $X[2\ell+1] = \sum_{n=0}^{(N/2)-1} \left(x[n] - x[\frac{N}{2}+n]\right)W_N^{(2\ell+1)n}$

$= \sum_{n=0}^{(N/2)-1} \left(x[n] + x[\frac{N}{2}+n]\right)W_N^{n} \cdot W_{N/2}^{n\ell} = X_1[\ell]$ where $0 \le \ell \le \frac{N}{2} - 1$.

5.13 $g[n] = \frac{1}{2}(x[2n] + x[2n+1]),\ h[n] = \frac{1}{2}(x[2n] - x[2n+1]),\ 0 \le n \le \frac{N}{2} - 1$. Solving for $x[2n]$ and $x[2n+1]$, we get $x[2n] = g[n] + h[n]$ and $x[2n+1] = g[n] - h[n]$. Therefore,

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{nk} = \sum_{r=0}^{(N/2)-1} x[2r]W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1]W_N^{(2r+1)k}$$

$$= \sum_{r=0}^{(N/2)-1} x[2r]W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1]W_{N/2}^{rk}$$

$$= \sum_{r=0}^{(N/2)-1} (g[n]+h[n])W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} (g[n]-h[n])W_{N/2}^{rk}$$

$$= (1+W_N^k)\sum_{r=0}^{(N/2)-1} g[n]W_{N/2}^{rk} + (1-W_N^k)\sum_{r=0}^{(N/2)-1} h[n]W_{N/2}^{rk}$$

$$= (1 + W_N^k)G[\langle k \rangle_{N/2}] + (1 - W_N^k)H[\langle k \rangle_{N/2}],\ 0 \le k \le N-1.$$

5.27 Since for a real sequence, $x[n] = x^*[n]$, taking the DFT of both sides we get $X[k] = X^*[\langle -k \rangle_N]$. This implies
$\mathrm{Re}\{X[k]\} + j\,\mathrm{Im}\{X[k]\} = \mathrm{Re}\{X[\langle -k \rangle_N]\} - j\,\mathrm{Im}\{X[\langle -k \rangle_N]\}$.
Comparing real and imaginary parts we get $\mathrm{Re}\{X[k]\} = \mathrm{Re}\{X[\langle -k \rangle_N]\}$ and $\mathrm{Im}\{X[k]\} = -\mathrm{Im}\{X[\langle -k \rangle_N]\}$.
Also, $|X[k]| = \sqrt{(\mathrm{Re}\{X[k]\})^2 + (\mathrm{Im}\{X[k]\})^2}$
$= \sqrt{(\mathrm{Re}\{X[\langle -k \rangle_N]\})^2 + (\mathrm{Im}\{X[\langle -k \rangle_N]\})^2} = |X[\langle -k \rangle_N]|$ and
$\arg\{X[k]\} = \tan^{-1}\left(\dfrac{\mathrm{Im}\{X[k]\}}{\mathrm{Re}\{X[k]\}}\right) = \tan^{-1}\left(\dfrac{-\mathrm{Im}\{X[\langle -k \rangle_N]\}}{\mathrm{Re}\{X[\langle -k \rangle_N]\}}\right) = -\arg\{X[\langle -k \rangle_N]\}$.

5.38 $X[1] = X^*[\langle -1 \rangle_9] = X^*[8] = 4.5 - j1.6$,
$X[4] = X^*[\langle -4 \rangle_9] = X^*[5] = -3.1 - j8.2$,
$X[6] = X^*[\langle -6 \rangle_9] = X^*[3] = -7.2 + 4.1$,
$X[7] = X^*[\langle -7 \rangle_9] = X^*[2] = 1.2 + j2.3$.

5.39 Since the DFT $X[k]$ is real-valued, $x[n]$ is a circularly even sequence, i.e.,
$x[n] = x[\langle -n \rangle_{12}]$. Therefore,
$x[1] = x[\langle -1 \rangle_{12}] = x[11] = -2$,
$x[4] = x[\langle -4 \rangle_{12}] = x[8] = 9.3$,
$x[7] = x[\langle -7 \rangle_{12}] = x[5] = 4.1$,
$x[9] = x[\langle -9 \rangle_{12}] = x[3] = -3.25$,
$x[10] = x[\langle -10 \rangle_{12}] = x[2] = 0.7$.

5.40 Since the DFT $X[k]$ is imaginary-valued, $x[n]$ is a circularly odd sequence, i.e.,
$x[n] = -x[\langle -n \rangle_{12}]$. Therefore,
$x[7] = -x[\langle -7 \rangle_{12}] = -x[5] = 9.3$,
$x[8] = -x[\langle -8 \rangle_{12}] = -x[4] = -2.87$,
$x[9] = -x[\langle -9 \rangle_{12}] = -x[3] = -4.1$,
$x[10] = -x[\langle -10 \rangle_{12}] = -x[2] = 3.25$,
$x[11] = -x[\langle -11 \rangle_{12}] = -x[1] = -0.7$.

5.41 $X[k] = X^*[\langle -k \rangle_{174}] = X^*[174-k]$.

$X[9] = X^*[174-9] = X^*[165] = -3.4 + j5.9 \quad \Rightarrow X[165] = -3.4 - j5.9$.

$X[51] = X^*[174-51] = X^*[123] = 5 - j1.6 \quad \Rightarrow X[123] = 5 + j1.6$.

$X[113] = X^*[174-113] = X^*[61] = 8.7 - j4.9 \quad \Rightarrow X[61] = 8.7 + j4.9$.

$X[162] = X^*[174-162] = X^*[12] = 7.1 - j2.4 \quad \Rightarrow X[12] = 7.1 + j2.4$.

$X[k_1] = 7.1 + j2.4,\ X[k_2] = 8.7 + j4.9,\ X[k_3] = 5 + j1.6,\ X[k_4] = -3.4 - j5.9$.

(a) Comparing these 4 DFT samples with the DFT samples given above we conclude $k_1 = 12,\ k_2 = 61,\ k_3 = 123,\ k_4 = 165$.

(b) dc value of $\{x[n]\} = X[0] = 11$.

(c)
$$x[n] = \frac{1}{174}\sum_{k=0}^{173} X[k] W_{174}^{-kn} = \frac{1}{174}\Big(X[0] + 2\,\mathrm{Re}\{X[9]W_{174}^{-9n}\} + 2\,\mathrm{Re}\{X[51]W_{174}^{-51n}\} + X[87]W_{174}^{-87n} + 2\,\mathrm{Re}\{X[113]W_{174}^{-113n}\} + 2\,\mathrm{Re}\{X[162]W_{174}^{-162n}\}\Big)$$

(d)
$$\sum_{n=0}^{173} |x[n]|^2 = \frac{1}{174}\sum_{k=0}^{173} |X[k]|^2 = 86.0279.$$