Chapter 6

6.1 $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} x[n]z^{-n}$. Therefore, $\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \sum_{n=0}^{\infty} x[n]z^{-n}$

$= \lim_{z\to\infty} x[0] + \lim_{z\to\infty} \sum_{n=1}^{\infty} x[n]z^{-n} = x[0]$.

6.2 (a) $Z\delta\{[n]\} = \sum_{n=-\infty}^{\infty} \delta[n]z^{-n} = \delta[0] = 1$, which converges everywhere in the z-plane.

(b) $x[n] = \alpha^n \mu[n]$. From Table 6.1,

$Z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \frac{1}{1-\alpha z^{-1}}$, $|z| > |\alpha|$. Let $g[n] = nx[n]$. Then,

$Z\{g[n]\} = G(z) = \sum_{n=-\infty}^{\infty} nx[n]z^{-n}$. Now, $\frac{dX(z)}{dz} = -\sum_{n=-\infty}^{\infty} ng[n]z^{-n-1}$. Hence,

$z\frac{dX(z)}{dz} = -\sum_{n=-\infty}^{\infty} nx[n]z^{-n} = -G(z)$, or, $G(z) = -z\frac{dX(z)}{dz} = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$, $|z| > |\alpha|$.

(c) $x[n] = r^n \sin(\omega_o n)\mu[n] = \frac{r^n}{2j}(e^{j\omega_o n} - e^{-j\omega_o n})\mu[n]$. Using the results of Example 6.1 and the linearity property of the z-transform we get

$Z\{r^n \sin(\omega_o n)\mu[n]\} = \frac{1}{2j}\left(\frac{1}{1-re^{j\omega_o}z^{-1}}\right) - \frac{1}{2j}\left(\frac{1}{1-re^{-j\omega_o}z^{-1}}\right)$

$= \frac{\frac{r}{2j}(e^{j\omega_o} - e^{-j\omega_o})z^{-1}}{1-r(e^{j\omega_o}+e^{-j\omega_o})z^{-1}+r^2z^{-2}} = \frac{r\sin(\omega_o)z^{-1}}{1-2r\cos(\omega_o)z^{-1}+r^2z^{-2}}, \quad \forall |z| > |r|$.

6.3 (a) $x_1[n] = \alpha^n \mu[n-2]$. Note, $x_1[n]$ is a right-sided sequence. Hence, the ROC of its z-transform is exterior to a circle. Therefore, $X_1(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n-2]z^{-n} = \sum_{n=2}^{\infty} \alpha^n z^{-n}$

$= \sum_{n=0}^{\infty} \alpha^n z^{-n} - 1 - \alpha z^{-1}$, $|\alpha/z| < 1$. Simplifying we get

$X_1(z) = \frac{1}{1-\alpha z^{-1}} - 1 - \alpha z^{-1} = \frac{\alpha^2 z^{-2}}{1-\alpha z^{-1}}$ whose ROC is given by $|z| > |\alpha|$.

(b) $x_2[n] = -\alpha^n \mu[-n-3]$. Note, $x_2[n]$ is a left-sided sequence. Hence, the ROC of its z-transform is interior to a circle. Therefore,

$$X_2(z) = -\sum_{n=-\infty}^{\infty}\alpha^n\mu[-n-3]z^{-n} = -\sum_{n=-\infty}^{-3}\alpha^n z^{-n} = -\sum_{m=3}^{\infty}\alpha^{-m}z^m = -\sum_{m=3}^{\infty}(z/\alpha)^m$$

$= \sum_{m=0}^{\infty}(z/\alpha)^m - 1 - (z/\alpha) - (z/\alpha)^2$, $|z/\alpha| < 1$. Simplifying we get

$X_2(z) = \dfrac{(z/\alpha)^3}{1-(z/\alpha)}$ whose ROC is given by $|z| < |\alpha|$.

(c) $x_3[n] = \alpha^n\mu[n+4]$. Note, $x_3[n]$ is a right-sided sequence. Hence, the ROC of its z–transform is exterior to a circle. Therefore, $X_3(z) = \sum_{n=-\infty}^{\infty}\alpha^n\mu[n+4]z^{-n} = \sum_{n=-4}^{\infty}\alpha^n z^{-n}$

$= \sum_{n=0}^{\infty}(\alpha/z)^n + (\alpha/z)^{-1} + (\alpha/z)^{-2} + (\alpha/z)^{-3} = \dfrac{1}{1-(\alpha/z)} + (\alpha/z)^{-1} + (\alpha/z)^{-2} + (\alpha/z)^{-3}$,

$|\alpha/z| < 1$. Simplifying we get $X_3(z) = \dfrac{(\alpha/z)^{-3}}{1-(\alpha/z)}$ whose ROC is given by $|z| > |\alpha|$.

(d) $x_4[n] = \alpha^n\mu[-n]$. Note, $x_4[n]$ is a left-sided sequence. Hence, the ROC of its z–transform is interior to a circle. Therefore, $X_4(z) = \sum_{n=-\infty}^{\infty}\alpha^n\mu[-n]z^{-n} = \sum_{n=-\infty}^{0}\alpha^n z^{-n}$

$= \sum_{m=0}^{\infty}\alpha^{-m}z^m = \dfrac{1}{1-(z/\alpha)}$, $|z/\alpha| < 1$. Therefore the ROC of $X_4(z)$ is given by $|z| < |\alpha|$.

6.4 $Z\{(0.4)^n\mu[n]\} = \dfrac{1}{1-0.4z^{-1}}$, $|z| > 0.4$; $Z\{(-0.6)^n\mu[n]\} = \dfrac{1}{1+0.6z^{-1}}$, $|z| > 0.6$;

$Z\{(0.4)^n\mu[-n-1]\} = -\dfrac{1}{1-0.4z^{-1}}$, $|z| < 0.4$;

$Z\{(-0.6)^n\mu[-n-1]\} = -\dfrac{1}{1+0.6z^{-1}}$, $|z| < 0.6$;

(a) $Z\{x_1[n]\} = \dfrac{1}{1-0.4z^{-1}} + \dfrac{1}{1+0.6z^{-1}} = \dfrac{1+0.2z^{-1}}{(1-0.4z^{-1})(1+0.6z^{-1})}$, $|z| > 0.6$.

(b) $Z\{x_2[n]\} = \dfrac{1}{1-0.4z^{-1}} + \dfrac{1}{1+0.6z^{-1}} = \dfrac{1+0.2z^{-1}}{(1-0.4z^{-1})(1+0.6z^{-1})}$, $0.4 < |z| < 0.6$.

(c) $Z\{x_3[n]\} = \dfrac{1}{1-0.4z^{-1}} + \dfrac{1}{1+0.6z^{-1}} = \dfrac{1+0.2z^{-1}}{(1-0.4z^{-1})(1+0.6z^{-1})}$, $|z| < 0.4$.

$= \dfrac{(\alpha^{-2}+\beta^{-2})-(\alpha\beta^{-2}+\alpha^{-2}\beta)z^{-1}}{z^{-2}(1-\alpha z^{-1})(1-\beta z^{-1})}$ with its ROC given by $|z| > |\beta|$.

(b) $x_2[n] = \alpha^n \mu[-n-2] + \beta^n \mu[n-1]$ with $|\beta| > |\alpha|$. Note that $x_2[n]$ is a two-sided sequence. Now,

$\mathcal{Z}\{\alpha^n \mu[-n-2]\} = \sum_{n=-\infty}^{-2} \alpha^n z^{-n} = \sum_{m=2}^{\infty} \alpha^{-m} z^m = \sum_{m=0}^{\infty} (z/\alpha)^m - 1 - (z/\alpha) - (z/\alpha)^2$

$= \dfrac{(z/\alpha)^3}{1-(z/\alpha)}$ with its ROC given by $|z| < |\alpha|$. Likewise,

$\mathcal{Z}\{\beta^n \mu[n-1]\} = \sum_{n=1}^{\infty} \beta^n z^{-n} = \sum_{n=0}^{\infty} \beta^n z^{-n} - 1 = \dfrac{1}{1-\beta z^{-1}} - 1 = \dfrac{\beta z^{-1}}{1-\beta z^{-1}}$ with its ROC given by $|z| > |\beta|$. Since the two ROCs do not intersect, $\mathcal{Z}\{x_2[n]\}$ does not converge.

(c) $x_3[n] = \alpha^n \mu[n+1] + \beta^n \mu[-n-2]$ with $|\beta| > |\alpha|$. Note that $x_3[n]$ is a two-sided sequence. Now,

$\mathcal{Z}\{\alpha^n \mu[-n-2]\} = \sum_{n=-\infty}^{-2} \alpha^n z^{-n} = \sum_{m=2}^{\infty} \alpha^{-m} z^m = \sum_{m=0}^{\infty} (z/\alpha)^m - 1 - (z/\alpha) - (z/\alpha)^2$

$= \dfrac{(z/\alpha)^3}{1-(z/\alpha)}$ with its ROC given by $|z| < |\alpha|$. Likewise,

$\mathcal{Z}\{\beta^n \mu[n-1]\} = \sum_{n=1}^{\infty} \beta^n z^{-n} = \sum_{n=0}^{\infty} \beta^n z^{-n} - 1 = \dfrac{1}{1-\beta z^{-1}} - 1 = \dfrac{\beta z^{-1}}{1-\beta z^{-1}}$ with its ROC given by $|z| > |\beta|$.

6.8 The denominator factor $(z^2 + 0.3z - 0.18) = (z+0.6)(z-0.3)$ has poles at $z = -0.6$ and at $z = 0.3$, and the factor $(z^2 - 2z + 4)$ has poles with a magnitude 2. Hence, the four ROCs are defined by the regions: $\mathcal{R}_1$: $0 < |z| < 0.3$, $\mathcal{R}_2$: $0.3 < |z| < 0.6$, $\mathcal{R}_3 = 0.6 < |z| < 2$, and $\mathcal{R}_4$: $|z| > 2$. The inverse z-transform associated with the ROC $\mathcal{R}_1$ is a left-sided sequence, the inverse z-transforms associated with the ROCs $\mathcal{R}_2$ and $\mathcal{R}_3$ are two-sided sequences, and the inverse z-transform associated with the ROC $\mathcal{R}_4$ is a right-sided sequence.

6.9 $X(z) = \mathcal{Z}\{x[n]\}$ with an ROC given by $\mathcal{R}_x$. Using the conjugation property of the – transform given in Table 6.2, we observe that $\mathcal{Z}\{x^*[n]\} = X^*(z^*)$ whose ROC is given by $\mathcal{R}_x$. Now, $\text{Re}\{x[n]\} = \frac{1}{2}(x[n] + x^*[n])$. Hence, $\mathcal{Z}\{\text{Re}\, x[n]\}$ $= \frac{1}{2}\left(X(z) + X^*(z)\right)$ whose ROC is also $\mathcal{R}_x$. Likewise, $\text{Im}\{x[n]\} = \frac{1}{2j}(x[n] - x^*[n])$.

6.13 (a) $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$. Hence, $X(z^3) = \sum_{n=-\infty}^{\infty} x[n]z^{-3n} = \sum_{\substack{r=-\infty \\ m=3r}}^{\infty} x[m/3]z^{-m}$. Define a new sequence $g[m] = \begin{cases} x[m/3], & m = 0, \pm 3, \pm 6, \ldots, \\ 0, & \text{otherwise.} \end{cases}$ We can then express $X(z^3) = \sum_{n=-\infty}^{\infty} g[n]z^{-n}$. Thus, the inverse z–transform of $X(z^3)$ is given by $g[n]$.

For $x[n] = (-0.5)^n \mu[n]$, $g[n] = \begin{cases} (-0.5)^{n/3}, & n = 0, 3, 6, \ldots, \\ 0, & \text{otherwise.} \end{cases}$

(b) $Y(z) = (1 + z^{-1})X(z^3) = X(z^3) + z^{-1}X(z^3)$. Therefore,
$y[n] = Z^{-1}\{Y(z)\} = Z^{-1}X(z^3) + Z^{-1}z^{-1}X(z^3) = g[n] + g[n-1]$, where $g[n] = Z^{-1}X(z^3)$. From Part (a), $g[n] = \begin{cases} (-0.5)^{n/3}, & n = 0, 3, 6, \ldots, \\ 0, & \text{otherwise.} \end{cases}$ Hence,
$g[n-1] = \begin{cases} (-0.5)^{(n-1)/3}, & n = 1, 4, 7, \ldots, \\ 0, & \text{otherwise.} \end{cases}$ Therefore,

$$y[n] = \begin{cases} (-0.5)^{n/3}, & n = 0, 3, 6, \ldots, \\ (-0.5)^{(n-1)/3}, & n = 1, 4, 7, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

6.14 (a) $X_a(z) = Z\{\mu[n] - \mu[n-5]\} = \dfrac{1}{1-z^{-1}} - \dfrac{z^{-5}}{1-z^{-1}} = \dfrac{1-z^{-5}}{1-z^{-1}}$
$= 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$. Since has all poles at the origin, the ROC is the entire z–plane except the point $z = 0$, and hence includes the unit circle. On the unit circle,
$X_a(z)\big|_{z=e^{j\omega}} = X_a(e^{j\omega}) = 1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} + e^{-j4\omega} = \dfrac{1-e^{-j5\omega}}{1-e^{-j\omega}}$.

(b) $x_b[n] = \alpha^n \mu[n] - \alpha^n \mu[n-8]$, $|\alpha| < 1$. From Table 6.1,
$X_b(z) = \dfrac{1}{1-\alpha z^{-1}} - \dfrac{z^{-8}}{1-\alpha z^{-1}} = \dfrac{1-z^{-8}}{1-\alpha z^{-1}}$. The ROC is exterior to the circle at $|z| = |\alpha| < 1$. Hence, the ROC includes the unit circle. On the unit circle,
$X_b(z)\big|_{z=e^{j\omega}} = X_b(e^{j\omega}) = \dfrac{1-e^{j8\omega}}{1-\alpha e^{j\omega}}$.

(c) $x_c[n] = (n+1)\alpha^n \mu[n] = n\alpha^n \mu[n] + \alpha^n \mu[n]$, $|\alpha| < 1$. From Table 6.1,

$$X_c(z) = \frac{\alpha z^{-1}}{1-\alpha z^{-1}} + \frac{1}{1-\alpha z^{-1}} = \frac{1+\alpha z^{-1}}{1-\alpha z^{-1}}.$$ The ROC is exterior to the circle at $|z| = |\alpha| < 1$. Hence, the ROC includes the unit circle. On the unit circle,

$$X_c(e^{j\omega}) = \frac{1+\alpha e^{j\omega}}{1-e^{j\omega}}.$$

6.15 (a) $Y_1(z) = \sum_{n=-N}^{N} z^{-n} = z^N\left(\sum_{n=0}^{2N} z^{-n}\right) = \frac{1-z^{-(2N+1)}}{z^{-N}(1-z^{-1})}$. $Y_1(z)$ has N poles at $z = 0$ and N poles at $z = \infty$. Hence, the ROC is the entire z-plane excluding the points $z = 0$ and $z = \infty$, and includes the unit circle. On the unit circle,

$$Y_1(z)\big|_{z=e^{j\omega}} = Y_1(e^{j\omega}) = \frac{1-e^{j(2N+1)\omega}}{e^{-jN\omega}(1-e^{-j\omega})} = \frac{\sin\left(\omega(N+\frac{1}{2})\right)}{\sin(\omega/2)}.$$

(b) $Y_2(z) = \sum_{n=0}^{N} z^{-n} = \frac{1-z^{-(N+1)}}{1-z^{-1}}$. $Y_2(z)$ has N poles at $z = 0$. Hence, the ROC is the entire z-plane excluding the point $z = 0$. On the unit circle,

$$Y_2(z)\big|_{z=e^{j\omega}} = Y_2(e^{j\omega}) = \frac{1-e^{j(N+1)\omega}}{1-e^{-j\omega}} = e^{-N\omega/2}\frac{\sin\left(\frac{N+1}{2}\omega\right)}{\sin(\omega/2)}.$$

(c) $y_3[n] = \begin{cases} 1-\frac{|n|}{N}, & -N \le n \le N, \\ 0, & \text{otherwise.}\end{cases}$ Now, $y_3[n] = y_0[n] \circledast y_0[n]$ where

$y_0[n] = \begin{cases} 1, & -\frac{N}{2} \le n \le \frac{N}{2}, \\ 0, & \text{otherwise.}\end{cases}$ Therefore, $Y_3(z) = Y_0^2(z) = \frac{(1-z^{-(N+1)})^2}{z^{-N}(1-z^{-1})^2}$.

$Y_3(z)$ has $\frac{N}{2}$ poles at $z = 0$ and $\frac{N}{2}$ poles at $z = \infty$. Hence, the ROC is the entire z-plane excluding the points $z = 0$ and $z = \infty$, and includes the unit circle. On the unit circle, $Y_3(e^{j\omega}) = Y_0^2(e^{j\omega}) = \frac{\sin^2\left(\omega\left(\frac{N+1}{2}\right)\right)}{\sin^2(\omega/2)}$.

(d) $y_4[n] = \begin{cases} N+1-|n|, & -N \le n \le N, \\ 0, & \text{otherwise,}\end{cases} = y_1[n] + N \cdot y_3[n]$, where $y_1[n]$ is the sequence of Part (a) and $y_3[n]$ is the sequence of Part (c). Therefore,

$Y_4(z) = Y_1(z) + N \cdot Y_3(z) = \frac{1-z^{-(2N+1)}}{z^{-N}(1-z^{-1})} + N\frac{(1-z^{-(N+1)})^2}{z^{-N}(1-z^{-1})^2}$. Since the ROCs of both $Y_1(z)$ and $Y_3(z)$ include the unit circle, the ROC of $Y_4(z)$ also includes the unit

$\rho_\ell = \left.\dfrac{P(z)}{R(z)}\right|_{z=\lambda_\ell}$. Now,

$D'(z) = \dfrac{dD(z)}{dz^{-1}} = \dfrac{d[(1-\lambda_\ell z^{-1})R(z)]}{dz^{-1}} = -\lambda_\ell R(z) + (1-\lambda_\ell z^{-1})\dfrac{dR(z)}{dz^{-1}}$. Hence,

$D'(z)\big|_{z=\lambda_\ell} = -\lambda_\ell R(z)\big|_{z=\lambda_\ell}$. Therefore, $\rho_\ell = -\lambda_\ell \left.\dfrac{P(z)}{D'(z)}\right|_{z=\lambda_\ell}$.

6.20 (a) $X_a(z) = \dfrac{3z}{(z+0.6)(z-0.3)} = \dfrac{3z^{-1}}{(1+0.6z^{-1})(1-0.3z^{-1})} = \dfrac{\rho_1}{1+0.6z^{-1}} + \dfrac{\rho_2}{1-0.3z^{-1}}$,

where $\rho_1 = \left.\dfrac{3}{z-0.3}\right|_{z=-0.6} = \dfrac{3}{-0.9} = -\dfrac{10}{3}$, $\rho_2 = \left.\dfrac{3}{z+0.6}\right|_{z=0.3} = \dfrac{3}{0.9} = \dfrac{10}{3}$.

Therefore, $X_a(z) = -\dfrac{10/3}{1+0.6z^{-1}} + \dfrac{10/3}{1-0.3z^{-1}}$.

There are three ROCs - $\mathcal{R}_1: |z| < 0.3$, $\mathcal{R}_2: 0.3 < |z| < 0.6$, $\mathcal{R}_3: |z| > 0.6$.

The inverse z-transform associated with the ROC $\mathcal{R}_1$ is a left-sided sequence:

$\mathcal{Z}^{-1}\{X_a(z)\} = x_a[n] = \frac{10}{3}\left((-0.6)^n - (0.3)^n\right)\mu[-n-1]$.

The inverse z-transform associated with the ROC $\mathcal{R}_2$ is a two-sided sequence:

$\mathcal{Z}^{-1}\{X_a(z)\} = x_a[n] = -\frac{10}{3}(-0.6)^n\mu[-n-1] + \frac{10}{3}(0.3)^n\mu[n]$.

The inverse z-transform associated with the ROC $\mathcal{R}_3$ is a right-sided sequence:

$\mathcal{Z}^{-1}\{X_a(z)\} = x_a[n] = \frac{10}{3}\left(-(-0.6)^n + (0.3)^n\right)\mu[n]$.

(b) $X_b(z) = \dfrac{3z^{-1} + 0.1z^{-2} + 0.87z^{-3}}{(1+0.6z^{-1})(1-0.3z^{-1})^2} = K + \dfrac{\rho_1}{1+0.6z^{-1}} + \dfrac{\gamma_1}{1-0.3z^{-1}} + \dfrac{\gamma_2}{(1-0.3z^{-1})^2}$.

$K = X_b(0) = 0$, $\rho_1 = \left.\dfrac{3z^{-1} + 0.1z^{-2} + 0.87z^{-3}}{(1-0.3z^{-1})^2}\right|_{z=-0.6} = 2.7279$,

$\gamma_2 = \left.\dfrac{3z^{-1} + 0.1z^{-2} + 0.87z^{-3}}{1+0.6z^{-1}}\right|_{z=0.3} = 0.6190$,

$\gamma_1 = \dfrac{1}{-0.3}\cdot\dfrac{d}{dz^{-1}}\left.\left(\dfrac{3z^{-1} + 0.1z^{-2} + 0.87z^{-3}}{1+0.6z^{-1}}\right)\right|_{z=0.3} = -0.3469$. Hence,

$X_b(z) = \dfrac{2.7279}{1+0.6z^{-1}} - \dfrac{0.3469}{1-0.3z^{-1}} + \dfrac{0.6190}{(1-0.3z^{-1})^2}$.

There are three ROCs - $\mathcal{R}_1: |z| < 0.3$, $\mathcal{R}_2: 0.3 < |z| < 0.6$, $\mathcal{R}_3: |z| > 0.6$.

The inverse z-transform associated with the ROC $\mathcal{R}_1$ is a left-sided sequence:

$Z^{-1}\{X_b(z)\} = x_b[n] = 2.7279(n+1)(-0.6)^n \mu[-n-1]$

$\quad + \left(-0.3469 + 0.6190(n+1)\right)(0.3)^n \mu[-n-1].$

The inverse z-transform associated with the ROC $\mathcal{R}_2$ is a two-sided sequence:

$Z^{-1}\{X_b(z)\} = x_b[n] = 2.7279(n+1)(-0.6)^n \mu[-n-1]$

$\quad + \left(-0.3469 + 0.6190(n+1)\right)(0.3)^n \mu[n].$

The inverse z-transform associated with the ROC $\mathcal{R}_3$ is a right-sided sequence:

$Z^{-1}\{X_b(z)\} = x_b[n] = 2.7279(-0.6)^n \mu[n] + \left(-0.3469 + 0.6190(n+1)\right)(0.3)^n \mu[n].$

6.21 $G(z) = \dfrac{P(z)}{D(z)} = \dfrac{p_0 + p_1 z^{-1} + \cdots + p_M z^{-M}}{1 + d_1 z^{-1} + \cdots + d_N z^{-N}}$. Thus, $G(\infty) = \dfrac{P(\infty)}{D(\infty)}$. Now, a partial-fraction expansion of $G(z)$ in z^{-1} is given by $G(z) = \sum_{\ell=1}^{N} \dfrac{\rho_\ell}{1 - \lambda_\ell z^{-1}}$, from which we obtain

$G(\infty) = \sum_{\ell=1}^{N} \rho_\ell = \dfrac{p_0}{d_0}.$

6.22 $H(z) = \dfrac{1}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$, $|z| > r > 0$. By using partial-fraction expansion we write

$$H(z) = \frac{1}{(e^{j\theta} - e^{-j\theta})}\left(\frac{e^{j\theta}}{1 - re^{j\theta}z^{-1}} - \frac{e^{-j\theta}}{1 - re^{-j\theta}z^{-1}}\right) = \frac{1}{2\sin\theta}\left(\frac{e^{j\theta}}{1 - re^{j\theta}z^{-1}} - \frac{e^{-j\theta}}{1 - re^{-j\theta}z^{-1}}\right).$$

Thus, $h[n] = \dfrac{1}{j2\sin\theta}\left(r^n e^{j\theta} e^{jn\theta}\mu[n] - r^n e^{-j\theta} e^{-jn\theta}\mu[n]\right) = \dfrac{r^n}{\sin\theta}\left(\dfrac{e^{j\theta(n+1)} - e^{-j\theta(n+1)}}{2j}\right)\mu[n]$

$= \dfrac{r^n \sin\left((n+1)\theta\right)}{\sin\theta}\mu[n].$

6.23 (a) $X(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} \alpha^n z^{-n} = \sum_{m=1}^{\infty} \alpha^{-m} z^m = \sum_{m=1}^{\infty} (z/\alpha)^m = \sum_{m=0}^{\infty} (z/\alpha)^m - 1$

$= \dfrac{z/\alpha}{1 - (z/\alpha)} = \dfrac{1}{1 - \alpha z^{-1}}, \ |z| < |\alpha|.$

(b) Using the differentiation property, we obtain from Part (a),

$Z\{nx[n]\} = -z\dfrac{dX(z)}{dz} = \dfrac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}, \ |z| < |\alpha|.$ Therefore, $Z\{y[n]\} = Z\{nx[n] + x[n]\}$

$= \dfrac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2} + \dfrac{1}{1 - \alpha z^{-1}} = \dfrac{1}{(1 - \alpha z^{-1})^2}, \ |z| < |\alpha|.$

6.24 (a) Expanding $X_1(z)$ in a power series we get $X_1(z)=\sum_{n=0}^{\infty} z^{-3n}$, $|z|>1$. Thus,

$x_1[n]=\begin{cases}1, & \text{if } n=3k \text{ and } n\geq 0,\\ 0, & \text{otherwise.}\end{cases}$ Alternately, using partial-fraction expansion we get

$$X_1(z)=\frac{1}{1-z^{-3}}=\frac{\frac{1}{3}}{1-z^{-1}}+\frac{\frac{1}{3}}{1+(\frac{1}{2}+j\frac{\sqrt{3}}{2})z^{-1}}+\frac{\frac{1}{3}}{1+(\frac{1}{2}-j\frac{\sqrt{3}}{2})z^{-1}}.$$ Therefore,

$$x_1[n]=\tfrac{1}{3}\mu[n]+\tfrac{1}{3}\left(-\tfrac{1}{2}-j\tfrac{\sqrt{3}}{2}\right)\mu[n]+\tfrac{1}{3}\left(-\tfrac{1}{2}+j\tfrac{\sqrt{3}}{2}\right)\mu[n]$$

$=\frac{1}{3}\mu[n]+\frac{1}{3}e^{-j2\pi n/3}\mu[n]+\frac{1}{3}e^{j2\pi n/3}\mu[n]=\frac{1}{3}\mu[n]+\frac{2}{3}\cos(2\pi n/3)\mu[n]$. Thus,

$x_1[n]=\begin{cases}1, & \text{if } n=3k \text{ and } n\geq 0,\\ 0, & \text{otherwise.}\end{cases}$

(b) Expanding $X_2(z)$ in a power series we get $X_2(z)=\sum_{n=0}^{\infty} z^{-4n}$, $|z|>1$. Thus,

$x_2[n]=\begin{cases}1, & \text{if } n=4k \text{ and } n\geq 0,\\ 0, & \text{otherwise.}\end{cases}$ Alternately, using partial-fraction expansion we get

$$X_2(z)=\frac{\frac{1}{4}}{1-z^{-1}}+\frac{\frac{1}{4}}{1+z^{-1}}+\frac{\frac{1}{4}}{1+(\frac{1}{2}+j\frac{\sqrt{3}}{2})z^{-1}}+\frac{\frac{1}{4}}{1+(\frac{1}{2}-j\frac{\sqrt{3}}{2})z^{-1}}.$$ Thus,

$$x_2[n]=\tfrac{1}{4}\mu[n]+\tfrac{1}{4}(-1)^n\mu[n]+\tfrac{1}{4}\left(-\tfrac{1}{2}-j\tfrac{\sqrt{3}}{2}\right)\mu[n]+\tfrac{1}{4}\left(-\tfrac{1}{2}+j\tfrac{\sqrt{3}}{2}\right)\mu[n]$$

$=\frac{1}{4}\mu[n]+\frac{1}{4}(-1)^n\mu[n]+\frac{1}{4}e^{-j2\pi n/3}\mu[n]+\frac{1}{4}e^{j2\pi n/3}\mu[n]$

$=\frac{1}{4}\mu[n]+\frac{1}{4}(-1)^n\mu[n]+\frac{1}{2}\cos(2\pi n/3)\mu[n]$. Thus, $x_2[n]=\begin{cases}1, & \text{if } n=4k \text{ and } n\geq 0,\\ 0, & \text{otherwise.}\end{cases}$

6.25 (a) $X_1(z)=\log(1-\alpha z^{-1})$, $|z|>|\alpha|$. Expanding $X_1(z)$ in a power series we get

$X_1(z)=-\alpha z^{-1}-\frac{\alpha^2 z^{-2}}{2}-\frac{\alpha^3 z^{-3}}{3}-\ldots=-\sum_{n=1}^{\infty}\frac{\alpha^n}{n}z^{-n}$. Therefore,

$x_1[n]=-\frac{\alpha^n}{n}\mu[n-1]$.

(b) $X_2(z)=\log\left(\frac{\alpha-z^{-1}}{\alpha}\right)=\log\left(1-(\alpha z)^{-1}\right)$, $|z|<|\alpha|$. Expanding $X_2(z)$ in a power series

we get $X_2(z)=-(\alpha z)^{-1}-\frac{(\alpha z)^{-2}}{2}-\frac{(\alpha z)^{-3}}{3}-\ldots=-\sum_{n=1}^{\infty}\frac{(\alpha z)^{-n}}{n}$. Therefore,

$x_2[n] = -\frac{\alpha^{-n}}{n}\mu[n-1]$.

(c) $X_3(z) = \log\left(\frac{1}{1-\alpha z^{-1}}\right), |z| > |\alpha|$. Expanding $X_3(z)$ in a power series we get

$X_3(z) = \alpha z^{-1} + \frac{\alpha^2 z^{-2}}{2} + \frac{\alpha^3 z^{-3}}{3} \ldots = \sum_{n=1}^{\infty} \frac{\alpha^n}{n} z^{-n}$. Therefore, $x_3[n] = \frac{\alpha^n}{n}\mu[n-1]$.

(d) $X_4(z) = \log\left(\frac{\alpha}{\alpha - z^{-1}}\right) = -\log\left(1-(\alpha z)^{-1}\right), |z| < |\alpha|$. Expanding $X_4(z)$ in a power

series we get $X_4(z) = (\alpha z)^{-1} + \frac{(\alpha z)^{-2}}{2} + \frac{(\alpha z)^{-3}}{3} - \ldots = \sum_{n=1}^{\infty} \frac{(\alpha z)^{-n}}{n}$. Therefore,

$x_4[n] = \frac{\alpha^{-n}}{n}\mu[n-1]$.

6.26 $H(z) = \frac{z^{-1} + 1.7z^{-2}}{(1-0.3z^{-1})(1+0.5z^{-1})} = k + \frac{\rho_1}{1-0.3z^{-1}} + \frac{\rho_2}{1+0.5z^{-1}}$, where $k = H(0) = -\frac{34}{3}$,

$\rho_1 = \left.\frac{z^{-1} + 1.7z^{-2}}{1+0.5z^{-1}}\right|_{z=0.3} = \frac{25}{3}$, $\rho_2 = \left.\frac{z^{-1} + 1.7z^{-2}}{1-0.3z^{-1}}\right|_{z=-0.5} = 3$.

The statement `[r,p,k]=residuez([0 1 1.7],conv([1 -0.3],[1 0.6]);` yields

```
r =
 3.0000
 8.3333

p =
-0.5000
 0.3000

k =
-11.3333
```

Thus, $H(z) = -\frac{34}{3} + \frac{25/3}{1-0.3z^{-1}} + \frac{3}{1+0.5z^{-1}}$. Hence, its inverse z–transform is given by

$h[n] = -\frac{34}{3}\delta[n] + \frac{25}{3}(0.3)^n \mu[n] + 3(-0.5)^n \mu[n]$.

6.27 $G(z) = \mathcal{Z}\{g[n]\} = \sum_{n=-\infty}^{\infty} g[n]z^{-n}$ with a ROC given by $\mathcal{R}_g$ and $H(z) = \mathcal{Z}\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$ with a ROC given by $\mathcal{R}_h$.

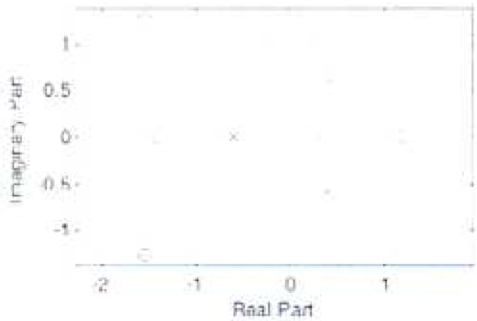
(a) $G^*(z) = \sum_{n=-\infty}^{\infty} g^*[n](z^*)^{-n}$ and $G^*(z^*) = \sum_{n=-\infty}^{\infty} g^*[n]z^{-n}$. Therefore,

```
   1.00000000000000    1.39999999999999                  0

Denominator factors
   1.00000000000000    0.59950918226500                  0
   1.00000000000000   -0.80131282790906   0.52021728677142
   1.00000000000000   -0.29819635435594                  0

Gain constant
      5
```

The factored form of $H(z)$ is thus

$$H(z)=\frac{5(1-1.2z^{-1})(1+3.1z^{-1}+4z^{-2})(1+1.4z^{-1})}{(1+0.5995z^{-1})(1-0.801313z^{-1}+0.520217z^{-2})(1-0.2982z^{-1})}$$

As all poles are inside the unit circle, $H(z)$ is BIBO stable.



6.39 A partial-fraction expansion of $H(z)$ in z^{-1} using the M-file `residuez` yields $H(z)=-\frac{1.21212}{1-0.4z^{-1}}+\frac{2.21212(1-0.81781z^{-1})}{1+0.5z^{-1}+0.3z^{-2}}$. Comparing the denominator of the quadratic factor with $1-2r\cos(\omega_o)z^{-1}+r^2z^{-2}$ we get $r=\sqrt{0.3}=0.54772$ and $\cos(\omega_o)=-\frac{0.5}{2\sqrt{0.3}}$, or $\omega_o=2.04478$. Hence, from Table 6.1 we have $h[n]=-1.21212(0.4)^n\mu[n]+(\sqrt{0.3})^n\cos(2.04478n)\mu[n]$.

6.40 (a) A partial-fraction expansion of $H(z)$ in z^{-1} using the M-file `residuez` yields $H(z)=-2+\frac{5}{1+0.6z^{-1}}-\frac{2}{1-0.3z^{-1}}$. Hence, from Table 6.1 we have $h[n]=-2\delta[n]+5(-0.6)^n\mu[n]-2(0.3)^n\mu[n]$.

(b) $x[n]=2.1(0.4)^n\mu[n]+0.3(-0.3)^n\mu[n]$. Its z–transform is thus given by

$X(z) = \frac{2.1}{1-0.4z^{-1}} + \frac{0.3}{1+0.3z^{-1}} = \frac{2.4+0.51z^{-1}}{(1-0.4z^{-1})(1+0.3z^{-1})}$, $|z| > 0.4$. The z-transform of the output is then given by $Y(z) = \left[\frac{2.4+0.51z^{-1}}{(1-0.4z^{-1})(1+0.3z^{-1})}\right]\cdot\left[\frac{1-3.3z^{-1}+0.36z^{-2}}{1+0.3z^{-1}-0.18z^{-2}}\right]$.

A partial-fraction expansion of $Y(z)$ in z^{-1} using the M-file `residuez` yields
$Y(z) = \frac{9.3}{1+0.6z^{-1}} - \frac{16.8}{1-0.4z^{-1}} + \frac{12.3}{1-0.3z^{-1}} - \frac{2.4}{1+0.3z^{-1}}$, $|z| > 0.6$. Hence, from Table 6.1 we have $y[n] = \left(9.3(-0.6)^n - 16.8(0.4)^n + 12.3(0.3)^n - 2.4(-0.3)^n\right)\mu[n]$.

6.41 (a) $H(z) = Z\{h[n]\} = \frac{1}{1+0.4z^{-1}}$, $|z| > 0.4$, $X(z) = Z\{x[n]\} = \frac{1}{1-0.2z^{-1}}$, $|z| > 0.4$. Thus, $Y(z) = H(z)X(z) = \frac{1}{(1+0.4z^{-1})(1-0.2z^{-1})}$, $|z| > 0.4$. A partial-fraction expansion of using the M-file `residuez` yields $Y(z) = \frac{2/3}{1+0.4z^{-1}} + \frac{1/3}{1-0.2z^{-1}}$. Hence, from Table 6.1
$y[n] = \frac{2}{3}(-0.4)^n \mu[n] + \frac{1}{3}(0.2)^n \mu[n]$.

(b) $H(z) = Z\{h[n]\} = \frac{1}{1+0.2z^{-1}}$, $|z| > 0.2$, $X(z) = Z\{x[n]\} = \frac{1}{1+0.2z^{-1}}$, $|z| > 0.4$. Thus, $Y(z) = H(z)X(z) = \frac{1}{(1+0.2z^{-1})^2}$, $|z| > 0.2$. Hence, from Table 6.1,
$y[n] = (n+1)(-0.2)^n \mu[n]$.

6.42 $Y(z) = Z\{y[n]\} = \frac{2}{1+0.3z^{-1}}$, $|z| > 0.3$, $X(z) = Z\{x[n]\} = \frac{4}{1-0.6z^{-1}}$, $|z| > 0.2$. Thus, $H(z) = \frac{Y(z)}{X(z)} = \frac{0.5(1-0.6z^{-1})}{1+0.3z^{-1}}$, $|z| > 0.3$. A partial-fraction expansion of using the M-file `residuez` yields $H(z) = -1 + \frac{1.5}{1+0.3z^{-1}}$. Hence, from Table 6.1,
$h[n] = -\delta[n] + 1.5(-0.3)^n \mu[n]$.

6.43 (a) Taking the -transform of both sides of the difference equation we get
$Y(z) = 0.2z^{-1}Y(z) + 0.08z^{-2}Y(z) + 2X(z)$. Hence, $H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1-0.2z^{-1}-0.08z^{-2}}$.