

## Continuous-Time Fourier Transform

- **Definition** – The CTFT of a continuous-time signal  $x_a(t)$  is given by

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt$$

- Often referred to as the **Fourier spectrum** or simply the **spectrum** of the continuous-time signal

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## Continuous-Time Fourier Transform

- **Definition** – The inverse CTFT of a Fourier transform  $X_a(j\Omega)$  is given by

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega) e^{j\Omega t} d\Omega$$

- Often referred to as the **Fourier integral**
- A CTFT pair will be denoted as

$$x_a(t) \overset{\text{CTFT}}{\leftrightarrow} X_a(j\Omega)$$

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## Continuous-Time Fourier Transform

- $\Omega$  is real and denotes the continuous-time angular frequency variable in radians
- In general, the CTFT is a complex function of  $\Omega$  in the range  $-\infty < \Omega < \infty$
- It can be expressed in the polar form as

$$X_a(j\Omega) = |X_a(j\Omega)| e^{j\theta_a(\Omega)}$$

where

$$\theta_a(\Omega) = \arg\{X_a(j\Omega)\}$$

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## Continuous-Time Fourier Transform

- The quantity  $|X_a(j\Omega)|$  is called the **magnitude spectrum** and the quantity  $\theta_a(\Omega)$  is called the **phase spectrum**
- Both spectrums are real functions of  $\Omega$
- In general, the CTFT  $X_a(j\Omega)$  exists if  $x_a(t)$  satisfies the **Dirichlet conditions** given on the next slide

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## Continuous-Time Fourier Transform

### Dirichlet Conditions

- (a) The signal  $x_a(t)$  has a finite number of discontinuities and a finite number of maxima and minima in any finite interval
- (b) The signal is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |x_a(t)| dt < \infty$$

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## Continuous-Time Fourier Transform

- If the **Dirichlet conditions** are satisfied, then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega) e^{j\Omega t} d\Omega$$

converges to  $x_a(t)$  at values of  $t$  except at values of  $t$  where  $x_a(t)$  has discontinuities

- It can be shown that if  $x_a(t)$  is absolutely integrable, then  $|X_a(j\Omega)| < \infty$  proving the existence of the CTFT

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## Energy Density Spectrum

- The total energy  $\mathcal{E}_x$  of a finite energy continuous-time complex signal  $x_a(t)$  is given by

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x_a(t)|^2 dt = \int_{-\infty}^{\infty} x_a(t) x_a^*(t) dt$$

- The above expression can be rewritten as

$$\mathcal{E}_x = \int_{-\infty}^{\infty} x_a(t) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) e^{-j\Omega t} d\Omega \right] dt$$

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## Energy Density Spectrum

- Interchanging the order of the integration we get

$$\begin{aligned} \mathcal{E}_x &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) \left[ \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt \right] d\Omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) X_a(j\Omega) d\Omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 d\Omega \end{aligned}$$

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## Energy Density Spectrum

- Hence

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 d\Omega$$

- The above relation is more commonly known as the **Parseval's relation** for finite-energy continuous-time signals

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## Energy Density Spectrum

- The quantity  $|X_a(j\Omega)|^2$  is called the **energy density spectrum** of  $x_a(t)$  and usually denoted as

$$S_{xx}(\Omega) = |X_a(j\Omega)|^2$$

- The energy over a specified range of frequencies  $\Omega_a \leq \Omega \leq \Omega_b$  can be computed using

$$x_r = \frac{1}{2\pi} \int_{\Omega_a}^{\Omega_b} S_{xx}(\Omega) d\Omega$$

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## Band-limited Continuous-Time Signals

- A **full-band**, finite-energy, continuous-time signal has a spectrum occupying the whole frequency range  $-\infty < \Omega < \infty$
- A **band-limited** continuous-time signal has a spectrum that is limited to a portion of the frequency range  $-\infty < \Omega < \infty$

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## Band-limited Continuous-Time Signals

- An ideal band-limited signal has a spectrum that is zero outside a finite frequency range  $\Omega_a \leq |\Omega| \leq \Omega_b$ , that is

$$X_a(j\Omega) = \begin{cases} 0, & 0 \leq |\Omega| < \Omega_a \\ 0, & \Omega_a < |\Omega| < \infty \end{cases}$$

- However, an ideal band-limited signal cannot be generated in practice

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## Band-limited Continuous-Time Signals

- Band-limited signals are classified according to the frequency range where most of the signal's is concentrated
- A lowpass, continuous-time signal has a spectrum occupying the frequency range  $|\Omega| \leq \Omega_p < \infty$  where  $\Omega_p$  is called the bandwidth of the signal

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## Band-limited Continuous-Time Signals

- A highpass, continuous-time signal has a spectrum occupying the frequency range  $0 < \Omega_p \leq |\Omega| < \infty$  where the bandwidth of the signal is from  $\Omega_p$  to  $\infty$
- A bandpass, continuous-time signal has a spectrum occupying the frequency range  $0 < \Omega_L \leq |\Omega| \leq \Omega_H < \infty$  where  $\Omega_H - \Omega_L$  is the bandwidth

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## Discrete-Time Fourier Transform

- **Definition** - The discrete-time Fourier transform (DTFT)  $X(e^{j\omega})$  of a sequence  $x[n]$  is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- In general,  $X(e^{j\omega})$  is a complex function of the real variable  $\omega$  and can be written as

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + j X_{im}(e^{j\omega})$$

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## Discrete-Time Fourier Transform

- **Definition** - The discrete-time Fourier transform (DTFT)  $X(e^{j\omega})$  of a sequence  $x[n]$  is given by

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$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + j X_{im}(e^{j\omega})$$

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## Discrete-Time Fourier Transform

- $X_{re}(e^{j\omega})$  and  $X_{im}(e^{j\omega})$  are, respectively, the real and imaginary parts of  $X(e^{j\omega})$ , and are real functions of  $\omega$
- $X(e^{j\omega})$  can alternately be expressed as

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega)}$$

where

$$\theta(\omega) = \arg\{X(e^{j\omega})\}$$

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## Discrete-Time Fourier Transform

- $|X(e^{j\omega})|$  is called the magnitude function
- $\theta(\omega)$  is called the phase function
- Both quantities are again real functions of  $\omega$
- In many applications, the DTFT is called the Fourier spectrum
- Likewise,  $|X(e^{j\omega})|$  and  $\theta(\omega)$  are called the magnitude and phase spectra

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## Discrete-Time Fourier Transform

- For a real sequence  $x[n]$ ,  $|X(e^{j\omega})|$  and  $X_{re}(e^{j\omega})$  are even functions of  $\omega$ , whereas,  $\theta(\omega)$  and  $X_{im}(e^{j\omega})$  are odd functions of  $\omega$
- Note:**  $X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega+2\pi k)}$   
 $= |X(e^{j\omega})|e^{j\theta(\omega)}$   
 for any integer  $k$
-  The phase function  $\theta(\omega)$  cannot be uniquely specified for any DTFT

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## Discrete-Time Fourier Transform

- Unless otherwise stated, we shall assume that the phase function  $\theta(\omega)$  is restricted to the following range of values:

$$-\pi \leq \theta(\omega) < \pi$$

called the **principal value**

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## Discrete-Time Fourier Transform

- The DTFTs of some sequences exhibit discontinuities of  $2\pi$  in their phase responses
- An alternate type of phase function that is a continuous function of  $\omega$  is often used
- It is derived from the original phase function by removing the discontinuities of  $2\pi$

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## Discrete-Time Fourier Transform

- The process of removing the discontinuities is called “**unwrapping**”
- The continuous phase function generated by unwrapping is denoted as  $\theta_c(\omega)$
- In some cases, discontinuities of  $\pi$  may be present after unwrapping

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## Discrete-Time Fourier Transform

- Example** - The DTFT of the unit sample sequence  $\delta[n]$  is given by

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = \delta[0] = 1$$

- Example** - Consider the causal sequence

$$x[n] = \alpha^n \mu[n], \quad |\alpha| < 1$$

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## Discrete-Time Fourier Transform

- Its DTFT is given by

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \alpha^n \mu[n]e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1 - \alpha e^{-j\omega}} \end{aligned}$$

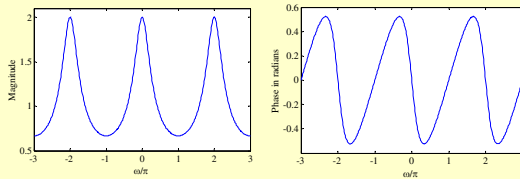
$$\text{as } |\alpha e^{-j\omega}| = |\alpha| < 1$$

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## Discrete-Time Fourier Transform

- The magnitude and phase of the DTFT  $X(e^{j\omega}) = 1/(1 - 0.5e^{-j\omega})$  are shown below



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## Discrete-Time Fourier Transform

- The DTFT  $X(e^{j\omega})$  of a sequence  $x[n]$  is a continuous function of  $\omega$
- It is also a periodic function of  $\omega$  with a period  $2\pi$ :

$$\begin{aligned} X(e^{j(\omega_o + 2\pi k)}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega_o + 2\pi k)n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega_o n} e^{-j2\pi kn} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega_o n} = X(e^{j\omega_o}) \end{aligned}$$

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## Discrete-Time Fourier Transform

- Therefore

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

represents the Fourier series representation of the periodic function

- As a result, the Fourier coefficients  $x[n]$  can be computed from  $X(e^{j\omega})$  using the Fourier integral

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

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## Discrete-Time Fourier Transform

- Inverse discrete-time Fourier transform:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- Proof:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega \ell} \right) e^{j\omega n} d\omega$$

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## Discrete-Time Fourier Transform

- The order of integration and summation can be interchanged if the summation inside the brackets converges uniformly, i.e.  $X(e^{j\omega})$  exists

- Then  $\frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega \ell} \right) e^{j\omega n} d\omega$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell] \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-\ell)} d\omega \right) = \sum_{\ell=-\infty}^{\infty} x[\ell] \frac{\sin \pi(n-\ell)}{\pi(n-\ell)}$$

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## Discrete-Time Fourier Transform

- Now  $\frac{\sin \pi(n-\ell)}{\pi(n-\ell)} = \begin{cases} 1, & n = \ell \\ 0, & n \neq \ell \end{cases}$   
 $= \delta[n-\ell]$

- Hence

$$\sum_{\ell=-\infty}^{\infty} x[\ell] \frac{\sin \pi(n-\ell)}{\pi(n-\ell)} = \sum_{\ell=-\infty}^{\infty} x[\ell] \delta[n-\ell] = x[n]$$

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## Discrete-Time Fourier Transform

- **Convergence Condition** - An infinite series of the form

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

may or may not converge

- Let

$$X_K(e^{j\omega}) = \sum_{n=-K}^K x[n]e^{-j\omega n}$$

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## Discrete-Time Fourier Transform

- Then for uniform convergence of  $X(e^{j\omega})$ ,

$$\lim_{K \rightarrow \infty} |X(e^{j\omega}) - X_K(e^{j\omega})| = 0$$

- Now, if  $x[n]$  is an absolutely summable sequence, i.e., if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

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## Discrete-Time Fourier Transform

- Then

$$|X(e^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- for all values of  $\omega$
- Thus, the absolute summability of  $x[n]$  is a sufficient condition for the existence of the DTFT  $X(e^{j\omega})$

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## Discrete-Time Fourier Transform

- **Example** - The sequence  $x[n] = \alpha^n \mu[n]$  for  $|\alpha| < 1$  is absolutely summable as

$$\sum_{n=-\infty}^{\infty} |\alpha^n| \mu[n] = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1-|\alpha|} < \infty$$

and its DTFT  $X(e^{j\omega})$  therefore converges to  $1/(1 - \alpha e^{-j\omega})$  uniformly

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## Discrete-Time Fourier Transform

- Since

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 \leq \left( \sum_{n=-\infty}^{\infty} |x[n]| \right)^2,$$

an absolutely summable sequence has always a finite energy

- However, a finite-energy sequence is not necessarily absolutely summable

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## Discrete-Time Fourier Transform

- **Example** - The sequence

$$x[n] = \begin{cases} 1/n, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

has a finite energy equal to

$$\mathcal{E}_x = \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)^2 = \frac{\pi^2}{6}$$

- But,  $x[n]$  is not absolutely summable

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## Discrete-Time Fourier Transform

- To represent a finite energy sequence  $x[n]$  that is not absolutely summable by a DTFT  $X(e^{j\omega})$ , it is necessary to consider a **mean-square convergence** of  $X(e^{j\omega})$ :

$$\lim_{K \rightarrow \infty} \int_{-\pi}^{\pi} |X(e^{j\omega}) - X_K(e^{j\omega})|^2 d\omega = 0$$

where

$$X_K(e^{j\omega}) = \sum_{n=-K}^K x[n] e^{-j\omega n}$$

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## Discrete-Time Fourier Transform

- Here, the total energy of the error

$$X(e^{j\omega}) - X_K(e^{j\omega})$$

must approach zero at each value of  $\omega$  as  $K$  goes to  $\infty$

- In such a case, the absolute value of the error  $|X(e^{j\omega}) - X_K(e^{j\omega})|$  may not go to zero as  $K$  goes to  $\infty$  and the DTFT is no longer bounded

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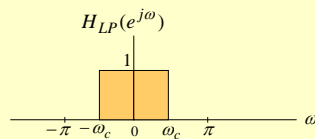
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## Discrete-Time Fourier Transform

- Example** - Consider the DTFT

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

shown below




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## Discrete-Time Fourier Transform

- The inverse DTFT of  $H_{LP}(e^{j\omega})$  is given by

$$\begin{aligned} h_{LP}[n] &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \left( \frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn} \right) = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty \end{aligned}$$

- The energy of  $h_{LP}[n]$  is given by  $\omega_c / \pi$
-   $h_{LP}[n]$  is a finite-energy sequence, but it is not absolutely summable

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## Discrete-Time Fourier Transform

- As a result

$$\sum_{n=-K}^K h_{LP}[n] e^{-j\omega n} = \sum_{n=-K}^K \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

does not uniformly converge to  $H_{LP}(e^{j\omega})$  for all values of  $\omega$ , but converges to  $H_{LP}(e^{j\omega})$  in the mean-square sense

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## Discrete-Time Fourier Transform

- The mean-square convergence property of the sequence  $h_{LP}[n]$  can be further illustrated by examining the plot of the function

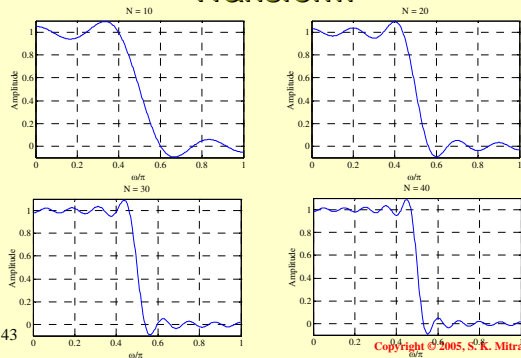
$$H_{LP,K}(e^{j\omega}) = \sum_{n=-K}^K \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

for various values of  $K$  as shown next

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## Discrete-Time Fourier Transform



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## Discrete-Time Fourier Transform

- As can be seen from these plots, independent of the value of  $K$  there are ripples in the plot of  $H_{LP,K}(e^{j\omega})$  around both sides of the point  $\omega = \omega_c$
- The number of ripples increases as  $K$  increases with the height of the largest ripple remaining the same for all values of  $K$

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## Discrete-Time Fourier Transform

- As  $K$  goes to infinity, the condition

$$\lim_{K \rightarrow \infty} \int_{-\pi}^{\pi} |H_{LP}(e^{j\omega}) - H_{LP,K}(e^{j\omega})|^2 d\omega = 0$$

holds indicating the convergence of  $H_{LP,K}(e^{j\omega})$  to  $H_{LP}(e^{j\omega})$

- The oscillatory behavior of  $H_{LP,K}(e^{j\omega})$  approximating  $H_{LP}(e^{j\omega})$  in the mean-square sense at a point of discontinuity is known as the **Gibbs phenomenon**

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## Discrete-Time Fourier Transform

- The DTFT can also be defined for a certain class of sequences which are neither absolutely summable nor square summable
- Examples of such sequences are the unit step sequence  $\mu[n]$ , the sinusoidal sequence  $\cos(\omega_o n + \phi)$  and the exponential sequence  $A\alpha^n$
- For this type of sequences, a DTFT representation is possible using the **Dirac delta function**  $\delta(\omega)$

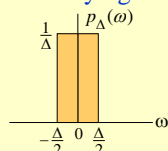
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## Discrete-Time Fourier Transform

- A Dirac delta function  $\delta(\omega)$  is a function of  $\omega$  with infinite height, zero width, and unit area
- It is the limiting form of a unit area pulse function  $p_{\Delta}(\omega)$  as  $\Delta$  goes to zero satisfying

$$\lim_{\Delta \rightarrow 0} \int_{-\infty}^{\infty} p_{\Delta}(\omega) d\omega = \int_{-\infty}^{\infty} \delta(\omega) d\omega$$



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## Discrete-Time Fourier Transform

- Example** - Consider the complex exponential sequence

$$x[n] = e^{j\omega_o n}$$

- Its DTFT is given by

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi \delta(\omega - \omega_o + 2\pi k)$$

where  $\delta(\omega)$  is an impulse function of  $\omega$  and

$$-\pi \leq \omega_o \leq \pi$$

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## Discrete-Time Fourier Transform

- The function

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$$

is a periodic function of  $\omega$  with a period  $2\pi$  and is called a **periodic impulse train**

- To verify that  $X(e^{j\omega})$  given above is indeed the DTFT of  $x[n] = e^{j\omega_o n}$  we compute the inverse DTFT of  $X(e^{j\omega})$

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## Discrete-Time Fourier Transform

- Thus

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k) e^{j\omega n} d\omega \\ &= \int_{-\pi}^{\pi} \delta(\omega - \omega_o) e^{j\omega n} d\omega = e^{j\omega_o n} \end{aligned}$$

where we have used the sampling property of the impulse function  $\delta(\omega)$

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## Commonly Used DTFT Pairs

Sequence DTFT

$$\delta[n] \leftrightarrow 1$$

$$1 \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$$

$$e^{j\omega_o n} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$$

$$\mu[n] \leftrightarrow \frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$$

$$\mu[n], (|\alpha| < 1) \leftrightarrow \frac{1}{1 - \alpha e^{-j\omega}}$$

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## DTFT Properties

- There are a number of important properties of the DTFT that are useful in signal processing applications
- These are listed here without proof
- Their proofs are quite straightforward
- We illustrate the applications of some of the DTFT properties

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**Table 3.1: DTFT Properties: Symmetry Relations**

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[n]$	$X^*(e^{j\omega})$
$\text{Re}\{x[n]\}$	$X_{cs}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\text{Im}\{x[n]\}$	$X_{ca}(e^{j\omega}) = \frac{1}{2j}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{cs}[n]$	$X_{cs}(e^{j\omega})$
$x_{ca}[n]$	$jX_{ca}(e^{j\omega})$

Note:  $X_{cs}(e^{j\omega})$  and  $X_{ca}(e^{j\omega})$  are the conjugate-symmetric and conjugate-antisymmetric parts of  $X(e^{j\omega})$ , respectively. Likewise,  $x_{cs}[n]$  and  $x_{ca}[n]$  are the conjugate-symmetric and conjugate-antisymmetric parts of  $x[n]$ , respectively.

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$x[n]$ : A complex sequence

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**Table 3.2: DTFT Properties: Symmetry Relations**

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$
$x_{ev}[n]$	$X_{re}(e^{j\omega})$
$x_{od}[n]$	$jX_{im}(e^{j\omega})$
Symmetry relations	$X(e^{j\omega}) = X^*(e^{-j\omega})$
	$X_{re}(e^{j\omega}) = X_{re}(e^{-j\omega})$
	$X_{im}(e^{j\omega}) = -X_{im}(e^{-j\omega})$
	$ X(e^{j\omega})  =  X(e^{-j\omega}) $
	$\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Note:  $x_{ev}[n]$  and  $x_{od}[n]$  denote the even and odd parts of  $x[n]$ , respectively.

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$x[n]$ : A real sequence

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**Table 3.4: General Properties of DTFT**

Type of Property	Sequence	Discrete-Time Fourier Transform
	$g[n]$ $h[n]$	$G(e^{j\omega})$ $H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time-shifting	$g[n - n_0]$	$e^{-j\omega n_0} G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_0 n} g[n]$	$G(e^{j(\omega - \omega_0)})$
Differentiation in frequency	$ng[n]$	$j \frac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] \otimes h[n]$	$G(e^{j\omega}) H(e^{j\omega})$
Modulation	$g[n] h[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta}) H(e^{j(\omega - \theta)}) d\theta$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n] h^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega}) H^*(e^{j\omega}) d\omega$	

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