

The Transfer Function

- A generalization of the frequency response function
- The convolution sum description of an LTI discrete-time system with an impulse response $h[n]$ is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

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The Transfer Function

- Taking the z -transforms of both sides we get

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} h[k]x[n-k] \right) z^{-n} \\ &= \sum_{k=-\infty}^{\infty} h[k] \left(\sum_{n=-\infty}^{\infty} x[n-k]z^{-n} \right) \\ &= \sum_{k=-\infty}^{\infty} h[k] \left(\sum_{\ell=-\infty}^{\infty} x[\ell]z^{-(\ell+k)} \right) \end{aligned}$$

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The Transfer Function

- Or, $Y(z) = \sum_{k=-\infty}^{\infty} h[k] \underbrace{\left(\sum_{\ell=-\infty}^{\infty} x[\ell]z^{-\ell} \right)}_{X(z)} z^{-k}$
- Therefore, $Y(z) = \underbrace{\left(\sum_{k=-\infty}^{\infty} h[k]z^{-k} \right)}_{H(z)} X(z)$
- Thus, $Y(z) = H(z)X(z)$

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The Transfer Function

- Hence, $H(z) = Y(z)/X(z)$
- The function $H(z)$, which is the z -transform of the impulse response $h[n]$ of the LTI system, is called the **transfer function** or the **system function**
- The inverse z -transform of the transfer function $H(z)$ yields the impulse response $h[n]$

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The Transfer Function

- Consider an LTI discrete-time system characterized by a difference equation
- $$\sum_{k=0}^N d_k y[n-k] = \sum_{k=0}^M p_k x[n-k]$$
- Its transfer function is obtained by taking the z -transform of both sides of the above equation
 - Thus $H(z) = \frac{\sum_{k=0}^M p_k z^{-k}}{\sum_{k=0}^N d_k z^{-k}}$

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The Transfer Function

- Or, equivalently as
- $$H(z) = z^{(N-M)} \frac{\sum_{k=0}^M p_k z^{M-k}}{\sum_{k=0}^N d_k z^{N-k}}$$
- An alternate form of the transfer function is given by

$$H(z) = \frac{p_0}{d_0} \cdot \frac{\prod_{k=1}^M (1 - \xi_k z^{-1})}{\prod_{k=1}^N (1 - \lambda_k z^{-1})}$$

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The Transfer Function

- Or, equivalently as

$$H(z) = \frac{p_0}{d_0} z^{(N-M)} \frac{\prod_{k=1}^M (z - \xi_k)}{\prod_{k=1}^N (z - \lambda_k)}$$

- $\xi_1, \xi_2, \dots, \xi_M$ are the finite **zeros**, and $\lambda_1, \lambda_2, \dots, \lambda_N$ are the finite **poles** of $H(z)$
- If $N > M$, there are additional $(N - M)$ zeros at $z = 0$
- If $N < M$, there are additional $(M - N)$ poles at $z = 0$

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The Transfer Function

- For a causal IIR digital filter, the impulse response is a causal sequence
- The ROC of the causal transfer function

$$H(z) = \frac{p_0}{d_0} z^{(N-M)} \frac{\prod_{k=1}^M (z - \xi_k)}{\prod_{k=1}^N (z - \lambda_k)}$$

is thus exterior to a circle going through the pole furthest from the origin

- Thus the ROC is given by $|z| > \max_k |\lambda_k|$

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The Transfer Function

- Example** - Consider the M -point moving-average FIR filter with an impulse response

$$h[n] = \begin{cases} 1/M, & 0 \leq n \leq M-1 \\ 0, & \text{otherwise} \end{cases}$$

- Its transfer function is then given by

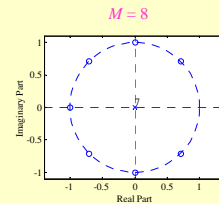
$$H(z) = \frac{1}{M} \sum_{n=0}^{M-1} z^{-n} = \frac{1 - z^{-M}}{M(1 - z^{-1})} = \frac{z^M - 1}{M[z^M(z - 1)]}$$

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The Transfer Function

- The transfer function has M zeros on the unit circle at $z = e^{j2\pi k/M}$, $0 \leq k \leq M-1$
- There are $M-1$ poles at $z = 0$ and a single pole at $z = 1$
- The pole at $z = 1$ exactly cancels the zero at $z = 1$
- The ROC is the entire z -plane except $z = 0$



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The Transfer Function

- Example** - A causal LTI IIR digital filter is described by a constant coefficient difference equation given by

$$y[n] = x[n-1] - 1.2x[n-2] + x[n-3] + 1.3y[n-1] - 1.04y[n-2] + 0.222y[n-3]$$

- Its transfer function is therefore given by

$$H(z) = \frac{z^{-1} - 1.2z^{-2} + z^{-3}}{1 - 1.3z^{-1} + 1.04z^{-2} - 0.222z^{-3}}$$

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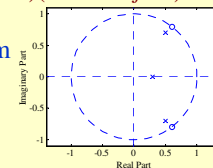
The Transfer Function

- Alternate forms:

$$H(z) = \frac{z^2 - 1.2z + 1}{z^3 - 1.3z^2 + 1.04z - 0.222} = \frac{(z - 0.6 + j0.8)(z - 0.6 - j0.8)}{(z - 0.3)(z - 0.5 + j0.7)(z - 0.5 - j0.7)}$$

- Note: Poles farthest from $z = 0$ have a magnitude $\sqrt{0.74}$

- ROC: $|z| > \sqrt{0.74}$



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Frequency Response from Transfer Function

- If the ROC of the transfer function $H(z)$ includes the unit circle, then the frequency response $H(e^{j\omega})$ of the LTI digital filter can be obtained simply as follows:

$$H(e^{j\omega}) = H(z) \Big|_{z=e^{j\omega}}$$

- For a real coefficient transfer function $H(z)$ it can be shown that

$$\begin{aligned} |H(e^{j\omega})|^2 &= H(e^{j\omega})H^*(e^{j\omega}) \\ &= H(e^{j\omega})H(e^{-j\omega}) = H(z)H(z^{-1}) \Big|_{z=e^{j\omega}} \end{aligned}$$

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Frequency Response from Transfer Function

- For a stable rational transfer function in the form

$$H(z) = \frac{p_0}{d_0} z^{(N-M)} \frac{\prod_{k=1}^M (z - \xi_k)}{\prod_{k=1}^N (z - \lambda_k)}$$

the factored form of the frequency response is given by

$$H(e^{j\omega}) = \frac{p_0}{d_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^M (e^{j\omega} - \xi_k)}{\prod_{k=1}^N (e^{j\omega} - \lambda_k)}$$

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Frequency Response from Transfer Function

- It is convenient to visualize the contributions of the **zero factor** $(z - \xi_k)$ and the **pole factor** $(z - \lambda_k)$ from the factored form of the frequency response
- The magnitude function is given by

$$|H(e^{j\omega})| = \left| \frac{p_0}{d_0} \right| e^{j\omega(N-M)} \left| \frac{\prod_{k=1}^M (e^{j\omega} - \xi_k)}{\prod_{k=1}^N (e^{j\omega} - \lambda_k)} \right|$$

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Frequency Response from Transfer Function

which reduces to

$$|H(e^{j\omega})| = \left| \frac{p_0}{d_0} \right| \frac{\prod_{k=1}^M |e^{j\omega} - \xi_k|}{\prod_{k=1}^N |e^{j\omega} - \lambda_k|}$$

- The phase response for a rational transfer function is of the form

$$\begin{aligned} \arg H(e^{j\omega}) &= \arg(p_0/d_0) + \omega(N-M) \\ &\quad + \sum_{k=1}^M \arg(e^{j\omega} - \xi_k) - \sum_{k=1}^N \arg(e^{j\omega} - \lambda_k) \end{aligned}$$

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Frequency Response from Transfer Function

- The magnitude-squared function of a real-coefficient transfer function can be computed using

$$|H(e^{j\omega})|^2 = \left| \frac{p_0}{d_0} \right|^2 \frac{\prod_{k=1}^M (e^{j\omega} - \xi_k)(e^{-j\omega} - \xi_k^*)}{\prod_{k=1}^N (e^{j\omega} - \lambda_k)(e^{-j\omega} - \lambda_k^*)}$$

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Geometric Interpretation of Frequency Response Computation

- The factored form of the frequency response

$$H(e^{j\omega}) = \frac{p_0}{d_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^M (e^{j\omega} - \xi_k)}{\prod_{k=1}^N (e^{j\omega} - \lambda_k)}$$

is convenient to develop a geometric interpretation of the frequency response computation from the pole-zero plot as ω varies from 0 to 2π on the unit circle

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Geometric Interpretation of Frequency Response Computation

- The geometric interpretation can be used to obtain a sketch of the response as a function of the frequency
- A typical factor in the factored form of the frequency response is given by

$$(e^{j\omega} - \rho e^{j\phi})$$

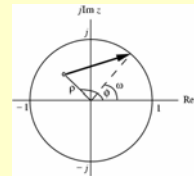
where $\rho e^{j\phi}$ is a zero if it is zero factor or is a pole if it is a pole factor

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Geometric Interpretation of Frequency Response Computation

- As shown below in the z -plane the factor $(e^{j\omega} - \rho e^{j\phi})$ represents a vector starting at the point $z = \rho e^{j\phi}$ and ending on the unit circle at $z = e^{j\omega}$



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Geometric Interpretation of Frequency Response Computation

- As ω is varied from 0 to 2π , the tip of the vector moves counterclockwise from the point $z = 1$ tracing the unit circle and back to the point $z = 1$

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Geometric Interpretation of Frequency Response Computation

- As indicated by

$$|H(e^{j\omega})| = \frac{|p_0| \prod_{k=1}^M |e^{j\omega} - \xi_k|}{|d_0| \prod_{k=1}^N |e^{j\omega} - \lambda_k|}$$

the magnitude response $|H(e^{j\omega})|$ at a specific value of ω is given by the product of the magnitudes of all zero vectors divided by the product of the magnitudes of all pole vectors

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Geometric Interpretation of Frequency Response Computation

- Likewise, from

$$\arg H(e^{j\omega}) = \arg(p_0/d_0) + \omega(N-M) + \sum_{k=1}^M \arg(e^{j\omega} - \xi_k) - \sum_{k=1}^N \arg(e^{j\omega} - \lambda_k)$$

we observe that the phase response at a specific value of ω is obtained by adding the phase of the term p_0/d_0 and the linear-phase term $\omega(N-M)$ to the sum of the angles of the zero vectors minus the angles of the pole vectors

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Geometric Interpretation of Frequency Response Computation

- Thus, an approximate plot of the magnitude and phase responses of the transfer function of an LTI digital filter can be developed by examining the pole and zero locations
- Now, a zero (pole) vector has the smallest magnitude when $\omega = \phi$

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Geometric Interpretation of Frequency Response Computation

- To highly attenuate signal components in a specified frequency range, we need to place zeros very close to or on the unit circle in this range
- Likewise, to highly emphasize signal components in a specified frequency range, we need to place poles very close to or on the unit circle in this range

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Stability Condition in Terms of the Pole Locations

- A causal LTI digital filter is BIBO stable if and only if its impulse response $h[n]$ is absolutely summable, i.e.,

$$S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- We now develop a stability condition in terms of the pole locations of the transfer function $H(z)$

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Stability Condition in Terms of the Pole Locations

- The ROC of the z -transform $H(z)$ of the impulse response sequence $h[n]$ is defined by values of $|z| = r$ for which $h[n]r^{-n}$ is absolutely summable
- Thus, if the ROC includes the unit circle $|z| = 1$, then the digital filter is stable, and vice versa

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Stability Condition in Terms of the Pole Locations

- In addition, for a stable and causal digital filter for which $h[n]$ is a right-sided sequence, the ROC will include the unit circle and entire z -plane including the point $z = \infty$
- An FIR digital filter with bounded impulse response is always stable

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Stability Condition in Terms of the Pole Locations

- On the other hand, an IIR filter may be unstable if not designed properly
- In addition, an originally stable IIR filter characterized by infinite precision coefficients may become unstable when coefficients get quantized due to implementation

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Stability Condition in Terms of the Pole Locations

- Example - Consider the causal IIR transfer function

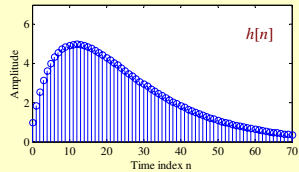
$$H(z) = \frac{1}{1 - 1.845z^{-1} + 0.850586z^{-2}}$$

- The plot of the impulse response coefficients is shown on the next slide

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Stability Condition in Terms of the Pole Locations



- As can be seen from the above plot, the impulse response coefficient $h[n]$ decays rapidly to zero value as n increases

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Stability Condition in Terms of the Pole Locations

- The absolute summability condition of $h[n]$ is satisfied
- Hence, $H(z)$ is a stable transfer function
- Now, consider the case when the transfer function coefficients are rounded to values with 2 digits after the decimal point:

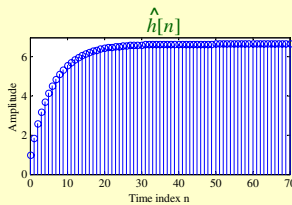
$$\hat{H}(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

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Stability Condition in Terms of the Pole Locations

- A plot of the impulse response of $\hat{h}[n]$ is shown below



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Stability Condition in Terms of the Pole Locations

- In this case, the impulse response coefficient $\hat{h}[n]$ increases rapidly to a constant value as n increases
- Hence, the absolute summability condition of is violated
- Thus, $\hat{H}(z)$ is an unstable transfer function

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Stability Condition in Terms of the Pole Locations

- The stability testing of a IIR transfer function is therefore an important problem
- In most cases it is difficult to compute the infinite sum

$$S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- For a causal IIR transfer function, the sum S can be computed approximately as

$$S_K = \sum_{n=0}^{K-1} |h[n]|$$

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Stability Condition in Terms of the Pole Locations

- The partial sum is computed for increasing values of K until the difference between a series of consecutive values of S_K is smaller than some arbitrarily chosen small number, which is typically 10^{-6}
- For a transfer function of very high order this approach may not be satisfactory
- An alternate, easy-to-test, stability condition is developed next

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Stability Condition in Terms of the Pole Locations

- Consider the causal IIR digital filter with a rational transfer function $H(z)$ given by

$$H(z) = \frac{\sum_{k=0}^M p_k z^{-k}}{\sum_{k=0}^N d_k z^{-k}}$$

- Its impulse response $\{h[n]\}$ is a right-sided sequence
- The ROC of $H(z)$ is exterior to a circle going through the pole furthest from $z = 0$

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Stability Condition in Terms of the Pole Locations

- But stability requires that $\{h[n]\}$ be absolutely summable
- This in turn implies that the DTFT $H(e^{j\omega})$ of $\{h[n]\}$ exists
- Now, if the ROC of the z -transform $H(z)$ includes the unit circle, then

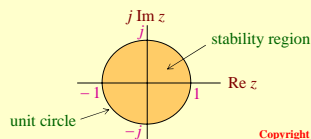
$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$$

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Stability Condition in Terms of the Pole Locations

- Conclusion: All poles of a causal stable transfer function $H(z)$ must be strictly inside the unit circle
- The stability region (shown shaded) in the z -plane is shown below



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Stability Condition in Terms of the Pole Locations

- Example - The factored form of

$$H(z) = \frac{1}{1 - 0.845z^{-1} + 0.850586z^{-2}}$$

is

$$H(z) = \frac{1}{(1 - 0.902z^{-1})(1 - 0.943z^{-1})}$$

which has a real pole at $z = 0.902$ and a real pole at $z = 0.943$

- Since both poles are inside the unit circle, $H(z)$ is BIBO stable

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Stability Condition in Terms of the Pole Locations

- Example - The factored form of

$$\hat{H}(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

is

$$\hat{H}(z) = \frac{1}{(1 - z^{-1})(1 - 0.85z^{-1})}$$

which has a real pole on the unit circle at $z = 1$ and the other pole inside the unit circle

- Since both poles are not inside the unit circle, $H(z)$ is unstable

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