

Computation of the DFT of Real Sequences

- In most practical applications, sequences of interest are real
- In such cases, the symmetry properties of the DFT given in Table 5.2 can be exploited to make the DFT computations more efficient

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N-Point DFTs of Two Length-N Real Sequences

- Let $g[n]$ and $h[n]$ be two length- N real sequences with $G[k]$ and $H[k]$ denoting their respective N -point DFTs
- These two N -point DFTs can be computed efficiently using a single N -point DFT
- Define a complex length- N sequence

$$x[n] = g[n] + j h[n]$$

- Hence, $g[n] = \text{Re}\{x[n]\}$ and $h[n] = \text{Im}\{x[n]\}$

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N-Point DFTs of Two Length-N Real Sequences

- Let $X[k]$ denote the N -point DFT of $x[n]$
- Then, from Table 5.1 we arrive at

$$G[k] = \frac{1}{2} \{X[k] + X^*[\langle -k \rangle_N]\}$$

$$H[k] = \frac{1}{2j} \{X[k] - X^*[\langle -k \rangle_N]\}$$

- Note that

$$X^*[\langle -k \rangle_N] = X^*[\langle N - k \rangle_N]$$

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N-Point DFTs of Two Length-N Real Sequences

- **Example** - We compute the 4-point DFTs of the two real sequences $g[n]$ and $h[n]$ given below

$$\{g[n]\} = \{1 \quad 2 \quad 0 \quad 1\}, \quad \{h[n]\} = \{2 \quad 2 \quad 1 \quad 1\}$$

- Then $\{x[n]\} = \{g[n]\} + j\{h[n]\}$ is given by

$$\{x[n]\} = \{1 + j2 \quad 2 + j2 \quad j \quad 1 + j\}$$

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N-Point DFTs of Two Length-N Real Sequences

- Its DFT $X[k]$ is

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 + j2 \\ 2 + j2 \\ j \\ 1 + j \end{bmatrix} = \begin{bmatrix} 4 + j6 \\ 2 \\ -2 \\ j2 \end{bmatrix}$$

- From the above

$$X^*[k] = [4 - j6 \quad 2 \quad -2 \quad -j2]$$

- Hence

$$X^*[\langle 4 - k \rangle_4] = [4 - j6 \quad -j2 \quad -2 \quad 2]$$

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N-Point DFTs of Two Length-N Real Sequences

- Therefore

$$\{G[k]\} = \{4 \quad 1 - j \quad -2 \quad 1 + j\}$$

$$\{H[k]\} = \{6 \quad 1 - j \quad 0 \quad 1 + j\}$$

verifying the results derived earlier

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2N-Point DFT of a Real Sequence Using an N-point DFT

- Let $v[n]$ be a length- $2N$ real sequence with an $2N$ -point DFT $V[k]$
- Define two length- N real sequences $g[n]$ and $h[n]$ as follows:
 $g[n] = v[2n], \quad h[n] = v[2n+1], \quad 0 \leq n \leq N-1$
- Let $G[k]$ and $H[k]$ denote their respective N -point DFTs

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2N-Point DFT of a Real Sequence Using an N-point DFT

- Define a length- N complex sequence
 $\{x[n]\} = \{g[n]\} + j\{h[n]\}$
with an N -point DFT $X[k]$
- Then as shown earlier

$$G[k] = \frac{1}{2} \{X[k] + X^*[\langle -k \rangle_N]\}$$

$$H[k] = \frac{1}{2j} \{X[k] - X^*[\langle -k \rangle_N]\}$$

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2N-Point DFT of a Real Sequence Using an N-point DFT

• Now $V[k] = \sum_{n=0}^{2N-1} v[n]W_{2N}^{nk}$

$$= \sum_{n=0}^{N-1} v[2n]W_{2N}^{2nk} + \sum_{n=0}^{N-1} v[2n+1]W_{2N}^{(2n+1)k}$$

$$= \sum_{n=0}^{N-1} g[n]W_N^{nk} + \sum_{n=0}^{N-1} h[n]W_N^{nk}W_{2N}^k$$

$$= \sum_{n=0}^{N-1} g[n]W_N^{nk} + W_{2N}^k \sum_{n=0}^{N-1} h[n]W_N^{nk}, \quad 0 \leq k \leq 2N-1$$

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2N-Point DFT of a Real Sequence Using an N-point DFT

- i.e.,
 $V[k] = G[\langle k \rangle_N] + W_{2N}^k H[\langle k \rangle_N], \quad 0 \leq k \leq 2N-1$
- Example - Let us determine the 8-point DFT $V[k]$ of the length-8 real sequence
 $\{v[n]\} = \{1 \quad 2 \quad 2 \quad 2 \quad 0 \quad 1 \quad 1 \quad 1\}$
 \uparrow
- We form two length-4 real sequences as follows

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2N-Point DFT of a Real Sequence Using an N-point DFT

$$\{g[n]\} = \{v[2n]\} = \{1 \quad 2 \quad 0 \quad 1\}$$

$$\uparrow$$

$$\{h[n]\} = \{v[2n+1]\} = \{2 \quad 2 \quad 1 \quad 1\}$$

$$\uparrow$$

- Now
 $V[k] = G[\langle k \rangle_4] + W_8^k H[\langle k \rangle_4], \quad 0 \leq k \leq 7$
- Substituting the values of the 4-point DFTs $G[k]$ and $H[k]$ computed earlier we get

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2N-Point DFT of a Real Sequence Using an N-point DFT

$$V[0] = G[0] + H[0] = 4 + 6 = 10$$

$$V[1] = G[1] + W_8^1 H[1]$$

$$= (1-j) + e^{-j\pi/4}(1-j) = 1 - j2.4142$$

$$V[2] = G[2] + W_8^2 H[2] = -2 + e^{-j\pi/2} \cdot 0 = -2$$

$$V[3] = G[3] + W_8^3 H[3]$$

$$= (1+j) + e^{-j3\pi/4}(1+j) = 1 - j0.4142$$

$$V[4] = G[0] + W_8^4 H[0] = 4 + e^{-j\pi} \cdot 6 = -2$$

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2N-Point DFT of a Real Sequence Using an N-point DFT

$$\begin{aligned}
 V[5] &= G[1] + W_8^5 H[1] \\
 &= (1-j) + e^{-j5\pi/4} (1-j) = 1 + j0.4142 \\
 V[6] &= G[2] + W_8^6 H[2] = -2 + e^{-j3\pi/2} \cdot 0 = -2 \\
 V[7] &= G[3] + W_8^7 H[3] \\
 &= (1+j) + e^{-j7\pi/4} (1+j) = 1 + j2.4142
 \end{aligned}$$

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Linear Convolution Using the DFT

- Linear convolution is a key operation in many signal processing applications
- Since a DFT can be efficiently implemented using FFT algorithms, it is of interest to develop methods for the implementation of linear convolution using the DFT

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Linear Convolution of Two Finite-Length Sequences

- Let $g[n]$ and $h[n]$ be two finite-length sequences of length N and M , respectively
- Denote $L = N + M - 1$
- Define two length- L sequences

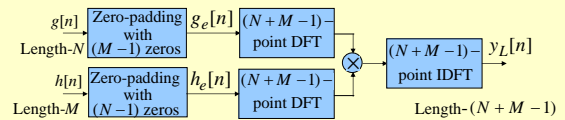
$$\begin{aligned}
 g_e[n] &= \begin{cases} g[n], & 0 \leq n \leq N-1 \\ 0, & N \leq n \leq L-1 \end{cases} \\
 h_e[n] &= \begin{cases} h[n], & 0 \leq n \leq M-1 \\ 0, & M \leq n \leq L-1 \end{cases}
 \end{aligned}$$

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Linear Convolution of Two Finite-Length Sequences

- Then
- The corresponding implementation scheme is illustrated below



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Linear Convolution of a Finite-Length Sequence with an Infinite-Length Sequence

- We next consider the DFT-based implementation of

$$y[n] = \sum_{\ell=0}^{M-1} h[\ell] x[n-\ell] = h[n] \otimes x[n]$$

where $h[n]$ is a finite-length sequence of length M and $x[n]$ is an infinite length (or a finite length sequence of length much greater than M)

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Overlap-Add Method

- We first segment $x[n]$, assumed to be a causal sequence here without any loss of generality, into a set of contiguous finite-length subsequences $x_m[n]$ of length N each:

$$x[n] = \sum_{m=0}^{\infty} x_m[n - mN]$$

where

$$x_m[n] = \begin{cases} x[n + mN], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

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Overlap-Add Method

- Thus we can write

$$y[n] = h[n] \otimes x[n] = \sum_{m=0}^{\infty} y_m[n - mN]$$

where

$$y_m[n] = h[n] \otimes x_m[n]$$

- Since $h[n]$ is of length M and $x_m[n]$ is of length N , the linear convolution $h[n] \otimes x_m[n]$ is of length $N + M - 1$

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Overlap-Add Method

- As a result, the desired linear convolution $y[n] = h[n] \otimes x[n]$ has been broken up into a sum of infinite number of short-length linear convolutions of length $N + M - 1$ each: $y_m[n] = x_m[n] \otimes h[n]$
- Each of these short convolutions can be implemented using the DFT-based method discussed earlier, where now the DFTs (and the IDFT) are computed on the basis of $(N + M - 1)$ points

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Overlap-Add Method

- There is one more subtlety to take care of before we can implement

$$y[n] = \sum_{m=0}^{\infty} y_m[n - mN]$$

using the DFT-based approach

- Now the first convolution in the above sum, $y_0[n] = h[n] \otimes x_0[n]$, is of length $N + M - 1$ and is defined for $0 \leq n \leq N + M - 2$

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Overlap-Add Method

- The second short convolution $y_1[n] = h[n] \otimes x_1[n]$, is also of length $N + M - 1$ but is defined for $N \leq n \leq 2N + M - 2$
- ➡ There is an overlap of $M - 1$ samples between these two short linear convolutions
- Likewise, the third short convolution $y_2[n] = h[n] \otimes x_2[n]$, is also of length $N + M - 1$ but is defined for $2N \leq n \leq 3N + M - 2$

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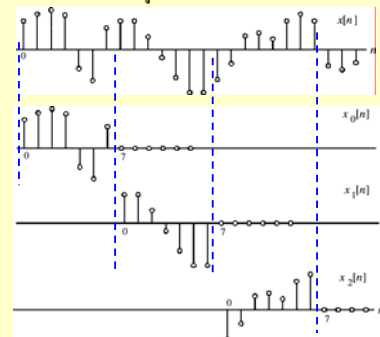
Overlap-Add Method

- Thus there is an overlap of $M - 1$ samples between $h[n] \otimes x_1[n]$ and $h[n] \otimes x_2[n]$
- In general, there will be an overlap of $M - 1$ samples between the samples of the short convolutions $h[n] \otimes x_{r-1}[n]$ and $h[n] \otimes x_r[n]$ for
- This process is illustrated in the figure on the next slide for $M = 5$ and $N = 7$

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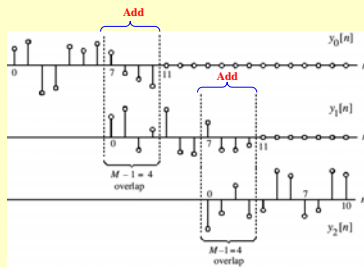
Overlap-Add Method



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Overlap-Add Method



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Overlap-Add Method

- Therefore, $y[n]$ obtained by a linear convolution of $x[n]$ and $h[n]$ is given by

$$\begin{aligned} y[n] &= y_0[n], & 0 \leq n \leq 6 \\ y[n] &= y_0[n] + y_1[n-7], & 7 \leq n \leq 10 \\ y[n] &= y_1[n-7], & 11 \leq n \leq 13 \\ y[n] &= y_1[n-7] + y_2[n-14], & 14 \leq n \leq 17 \\ y[n] &= y_2[n-14], & 18 \leq n \leq 20 \\ &\vdots & \end{aligned}$$

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Overlap-Add Method

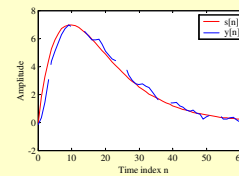
- The above procedure is called the **overlap-add method** since the results of the short linear convolutions overlap and the overlapped portions are added to get the correct final result
- The function `fftfilt` can be used to implement the above method

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Overlap-Add Method

- Program 5_5 illustrates the use of `fftfilt` in the filtering of a noise-corrupted signal using a length-3 moving average filter
- The plots generated by running this program is shown below



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Overlap-Save Method

- In implementing the overlap-add method using the DFT, we need to compute two $(N + M - 1)$ -point DFTs and one $(N + M - 1)$ -point IDFT since the overall linear convolution was expressed as a sum of short-length linear convolutions of length $(N + M - 1)$ each
- It is possible to implement the overall linear convolution by performing instead circular convolution of length shorter than $(N + M - 1)$

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Overlap-Save Method

- To this end, it is necessary to segment $x[n]$ into overlapping blocks $x_m[n]$, keep the terms of the circular convolution of $h[n]$ with $x_m[n]$ that corresponds to the terms obtained by a linear convolution of $h[n]$ and $x_m[n]$, and throw away the other parts of the circular convolution

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Overlap-Save Method

- To understand the correspondence between the linear and circular convolutions, consider a length-4 sequence $x[n]$ and a length-3 sequence $h[n]$
- Let $y_L[n]$ denote the result of a linear convolution of $x[n]$ with $h[n]$
- The six samples of $y_L[n]$ are given by

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Overlap-Save Method

$$\begin{aligned} y_L[0] &= h[0]x[0] \\ y_L[1] &= h[0]x[1] + h[1]x[0] \\ y_L[2] &= h[0]x[2] + h[1]x[1] + h[2]x[0] \\ y_L[3] &= h[0]x[3] + h[1]x[2] + h[2]x[1] \\ y_L[4] &= h[1]x[3] + h[2]x[2] \\ y_L[5] &= h[2]x[3] \end{aligned}$$

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Overlap-Save Method

- If we append $h[n]$ with a single zero-valued sample and convert it into a length-4 sequence $h_e[n]$, the 4-point circular convolution $y_C[n]$ of $h_e[n]$ and $x[n]$ is given by

$$\begin{aligned} y_C[0] &= h[0]x[0] + h[1]x[3] + h[2]x[2] \\ y_C[1] &= h[0]x[1] + h[1]x[0] + h[2]x[3] \\ y_C[2] &= h[0]x[2] + h[1]x[1] + h[2]x[0] \\ y_C[3] &= h[0]x[3] + h[1]x[2] + h[2]x[1] \end{aligned}$$

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Overlap-Save Method

- If we compare the expressions for the samples of $y_L[n]$ with the samples of $y_C[n]$, we observe that the first 2 terms of $y_C[n]$ do not correspond to the first 2 terms of $y_L[n]$, whereas the last 2 terms of $y_C[n]$ are precisely the same as the 3rd and 4th terms of $y_L[n]$, i.e.,

$$\begin{aligned} y_L[0] &\neq y_C[0], & y_L[1] &\neq y_C[1] \\ y_L[2] &= y_C[2], & y_L[3] &= y_C[3] \end{aligned}$$

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Overlap-Save Method

- General case: N -point circular convolution of a length- M sequence $h[n]$ with a length- N sequence $x[n]$ with $N > M$
- First $M - 1$ samples of the circular convolution are incorrect and are rejected
- Remaining $N - M + 1$ samples correspond to the correct samples of the linear convolution of $h[n]$ with $x[n]$

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Overlap-Save Method

- Now, consider an infinitely long or very long sequence $x[n]$
- Break it up as a collection of smaller length (length-4) overlapping sequences $x_m[n]$ as $x_m[n] = x[n + 2m]$, $0 \leq n \leq 3$, $0 \leq m \leq \infty$
- Next, form

$$w_m[n] = h[n] \oplus x_m[n]$$

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Overlap-Save Method

- Or, equivalently,

$$w_m[0] = h[0]x_m[0] + h[1]x_m[3] + h[2]x_m[2]$$

$$w_m[1] = h[0]x_m[1] + h[1]x_m[0] + h[2]x_m[3]$$

$$w_m[2] = h[0]x_m[2] + h[1]x_m[1] + h[2]x_m[0]$$

$$w_m[3] = h[0]x_m[3] + h[1]x_m[2] + h[2]x_m[1]$$

- Computing the above for $m = 0, 1, 2, 3, \dots$, and substituting the values of $x_m[n]$ we arrive at

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Overlap-Save Method

$$w_0[0] = h[0]x[0] + h[1]x[3] + h[2]x[2] \quad \leftarrow \text{Reject}$$

$$w_0[1] = h[0]x[1] + h[1]x[0] + h[2]x[3] \quad \leftarrow \text{Reject}$$

$$w_0[2] = h[0]x[2] + h[1]x[1] + h[2]x[0] = y[2] \quad \leftarrow \text{Save}$$

$$w_0[3] = h[0]x[3] + h[1]x[2] + h[2]x[1] = y[3] \quad \leftarrow \text{Save}$$

$$w_1[0] = h[0]x[2] + h[1]x[5] + h[2]x[4] \quad \leftarrow \text{Reject}$$

$$w_1[1] = h[0]x[3] + h[1]x[2] + h[2]x[5] \quad \leftarrow \text{Reject}$$

$$w_1[2] = h[0]x[4] + h[1]x[3] + h[2]x[2] = y[4] \quad \leftarrow \text{Save}$$

$$w_1[3] = h[0]x[5] + h[1]x[4] + h[2]x[3] = y[5] \quad \leftarrow \text{Save}$$

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Overlap-Save Method

$$w_2[0] = h[0]x[4] + h[1]x[5] + h[2]x[6] \quad \leftarrow \text{Reject}$$

$$w_2[1] = h[0]x[5] + h[1]x[4] + h[2]x[7] \quad \leftarrow \text{Reject}$$

$$w_2[2] = h[0]x[6] + h[1]x[5] + h[2]x[4] = y[6] \quad \leftarrow \text{Save}$$

$$w_2[3] = h[0]x[7] + h[1]x[6] + h[2]x[5] = y[7] \quad \leftarrow \text{Save}$$

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Overlap-Save Method

- It should be noted that to determine $y[0]$ and $y[1]$, we need to form $x_{-1}[n]$:

$$x_{-1}[0] = 0, \quad x_{-1}[1] = 0,$$

$$x_{-1}[2] = x[0], \quad x_{-1}[3] = x[1]$$

and compute $w_{-1}[n] = h[n] \circledast x_{-1}[n]$ for $0 \leq n \leq 3$
reject $w_{-1}[0]$ and $w_{-1}[1]$, and save $w_{-1}[2] = y[0]$
and $w_{-1}[3] = y[1]$

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Overlap-Save Method

- General Case: Let $h[n]$ be a length- N sequence
- Let $x_m[n]$ denote the m -th section of an infinitely long sequence $x[n]$ of length N and defined by

$$x_m[n] = x[n + m(N - M + 1)], \quad 0 \leq n \leq N - 1$$
with $M < N$

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Overlap-Save Method

- Let $w_m[n] = h[n] \circledast x_m[n]$
- Then, we reject the first $M - 1$ samples of $w_m[n]$ and “abut” the remaining $N - M + 1$ samples of $w_m[n]$ to form $y_L[n]$, the linear convolution of $h[n]$ and $x[n]$
- If $y_m[n]$ denotes the saved portion of $w_m[n]$, i.e.

$$y_m[n] = \begin{cases} 0, & 0 \leq n \leq M - 2 \\ w_m[n], & M - 1 \leq n \leq N - 2 \end{cases}$$

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Overlap-Save Method

- Then

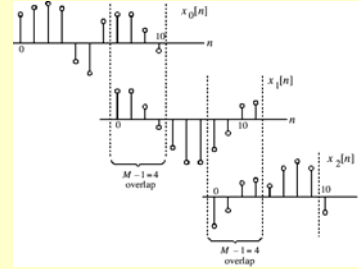
$$y_L[n + m(N - M + 1)] = y_m[n], \quad M - 1 \leq n \leq N - 1$$
- The approach is called **overlap-save method** since the input is segmented into overlapping sections and parts of the results of the circular convolutions are saved and abutted to determine the linear convolution result

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Overlap-Save Method

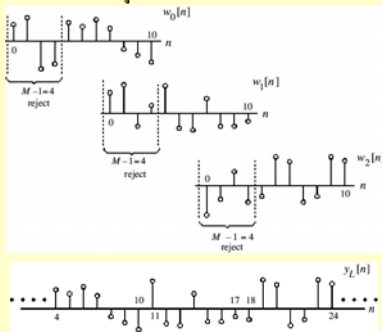
- Process is illustrated next



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Overlap-Save Method



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