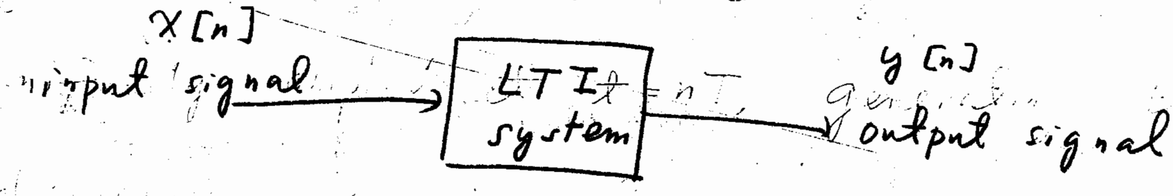
Chapter 6 Z-transform

The goal of signal processing is to analyze an LTI system (linear time-invariant) or design an LTI system ~~to~~ for a desired set of input/output signals.



The input $x[n]$, output $y[n]$ can be related via a difference equation, that is

$$\sum_{k=0}^{N} d_k\, y[n-k] = \sum_{k=0}^{M} p_k\, x[n-k]$$

Given an input sequence, $x[n]$, we would always like to compute $y[n]$ in terms of time index n. Chapter 2 introduces a recursive method to determine $y[n]$. However such a method (2.7.1 Total Solution Calculation) is complicated to apply. In this chapter, we like to discuss a neat general tool for any arbitrary LTI system characterized by a difference equation, Z-transform, to easily determine $y[n]$ in terms of $x[n]$. The ideal of Z-transform is to convert every sequence in an LTI system, $x[n]$, $y[n]$, $h[n]$ (system impulse response) into the corresponding Z-transform (a function with argument z). Thus, we can compute the sequence of interest using the inverse Z-transform (Look-up table).

6.1 Definition and Properties

For a given sequence $g[n]$, its z-transform $G(z)$ is defined as

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n},$$

where $z = Re(z) + j\,Im(z)$ is a continuous complex variable.

Very often, we can denote

$$\mathcal{Z}\{g[n]\} = G(z) = \sum_{n=-\infty}^{\infty} g[n]\, z^{-n}$$

or

$$g[n] \xleftrightarrow{\mathcal{Z}} G(z)$$

Since $z = re^{j\omega}$ (Euler formula),

$$G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]\, r^{-n} e^{-j\omega n}.$$

The z-transform $G(z)$ exists if $|G(z)| < \infty$

$$|G(z)| = |G(re^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} g[n]\, r^{-n} e^{-j\omega n} \right|$$

$$\le \sum_{n=-\infty}^{\infty} |g[n]\, r^{-n}| \, \cancelto{1}{|e^{-j\omega n}|}$$

$$\therefore \quad |G(z)| \le \sum_{n=-\infty}^{\infty} |g[n]\, r^{-n}|$$

If $\left| \sum_{n=-\infty}^{\infty} g[n]\, r^{-n} \right| < \infty$ (absolutely summable),

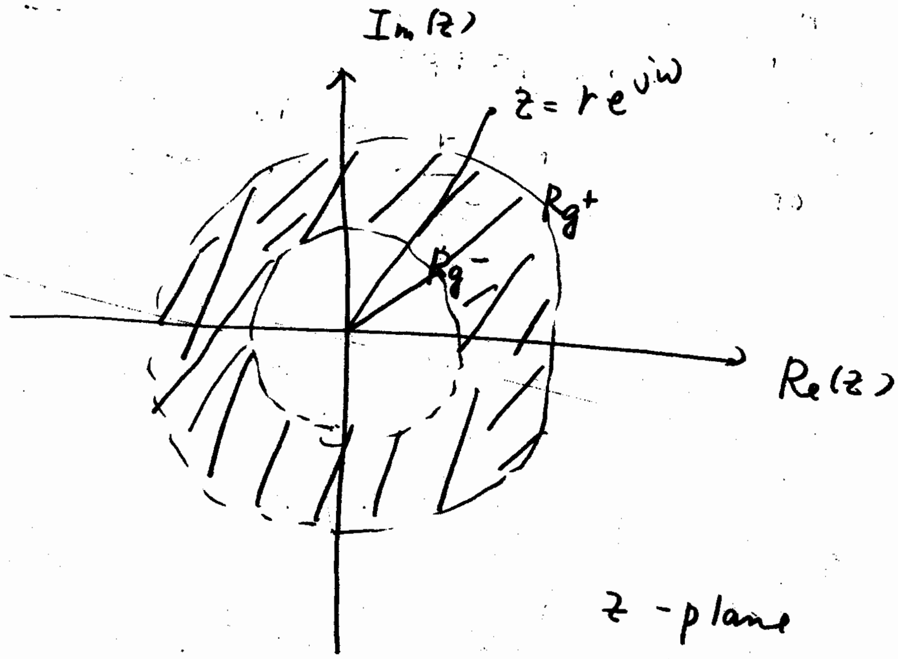
then $G(z)$ exists.

Consequently, we call the region of z to make $\left| \sum_{n=-\infty}^{\infty} g[n]\, r^{-n} \right| < \infty$ as the region of convergence (ROC).

Typical ROC can be expressed as

$$R_{g^-} < |z| < R_{g^+},$$

where $0 \le R_{g^-} < R_{g^+} < \infty$.



Shaded area: ROC

Example: Determine the z-transform of the causal sequence $x[n] = \alpha^n \mu[n] = \begin{cases} \alpha^n, & n \ge 0 \\ 0, & n < 0 \end{cases}$

and its ROC. $(\alpha \ne 0)$

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}} \quad \text{for } |\alpha z^{-1}| < 1$$

$\therefore$ $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$ is the ROC.

※ If $\alpha = 1$, $x[n] = \mu[n]$, the z-transform of the unit-step sequence $\mu[n]$ is

$$\mathcal{U}(z) = \frac{1}{1-z^{-1}} \quad \text{with ROC: } |z| > 1.$$

Example: Determine the z-transform of an anti-causal sequence $x[n] = -\alpha^n \mu[-n-1]$.

Solution:

$$X(z) = -\sum_{n=-\infty}^{-1} \alpha^n z^{-n} = -\sum_{\substack{m=1 \\ m=-n}}^{\infty} \alpha^{-m} z^m$$

$$= -\alpha^{-1} z \sum_{n=0}^{\infty} \alpha^{-m} z^m$$

$$= -\frac{\alpha^{-1} z}{1-\alpha^{-1} z} = \frac{1}{1-\alpha z^{-1}}, \quad |\alpha^{-1} z| < 1$$

$$\text{or } \underbrace{|z| < |\alpha|}_{\text{ROC}}$$

Example: Determine the z-transform of a finite-length sequence:

$$x[n] = \begin{cases} \alpha^n, & M \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Solution:

$$X(z) = \sum_{n=M}^{N-1} \alpha^n z^{-n} = z^{-M} \sum_{n=0}^{N-M-1} (\alpha z^{-1})^n$$

$$= z^{-M} \left(\frac{1 - \alpha^{N-M} z^{-(N-M)}}{1 - \alpha z^{-1}} \right)$$

$$= \frac{z^{-M} - \alpha^{N-M} z^{-N}}{1 - \alpha z^{-1}}$$

ROC: $z \neq 0$, $\cancel{z \neq \alpha}$

Table 6.1: Some commonly used z-transform pairs.

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n \mu[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$n\alpha^n \mu[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(n+1)\alpha^n \mu[n]$	$\frac{1}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(r^n \cos \omega_o n)\mu[n]$	$\frac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$
$(r^n \sin \omega_o n)\mu[n]$	$\frac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$

6.2 Rational z-transforms

Any LTI system can be characterized by a difference equation such that (pp. 149)

$$\sum_{k=0}^{N} d_k \underbrace{y[n-k]}_{\text{output}} = \sum_{k=0}^{M} p_k \underbrace{x[n-k]}_{\text{input}}$$

Hence,

$$\mathcal{Z}\left\{\sum_{k=0}^{N} d_k\, y[n-k]\right\} = \mathcal{Z}\left\{\sum_{k=0}^{M} p_k\, x[n-k]\right\}$$

$$\Rightarrow \sum_{k=0}^{N} d_k\, \mathcal{Z}\{y[n-k]\} = \sum_{k=0}^{M} p_k\, \mathcal{Z}\{x[n-k]\}$$

$$\mathcal{Z}\{y[n-k]\} = \sum_{n=-\infty}^{\infty} y[n-k]\, z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} y[m]\, z^{-m-k} = \sum_{m=-\infty}^{\infty} y[m]\, z^{-m} z^{-k}$$

$m = n-k$

$$= z^{-k} \underbrace{\sum_{m=-\infty}^{\infty} y[m]\, z^{-m}}_{Y(z)} = z^{-k}\, Y(z)$$

Similarly, $\mathcal{Z}\{x[n-k]\} = z^{-k} X(z)$,

where $X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

Thus,

$$\sum_{k=0}^{N} d_k z^{-k} Y(z) = \sum_{k=0}^{M} P_k z^{-k} X(z)$$

$$\Rightarrow \quad \frac{Y(z)}{X(z)} \triangleq \underset{\uparrow}{H(z)} = \frac{\sum_{k=0}^{M} P_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}}$$

transfer function

We define

$$P(z) \triangleq \sum_{k=0}^{M} P_k z^{-k}$$

$$D(z) \triangleq \sum_{k=0}^{N} d_k z^{-k}$$

Consequently,

$$H(z) = \frac{P(z)}{D(z)} = \frac{P_0 + P_1 z^{-1} + \cdots + P_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots + d_N z^{-N}}$$

$$= z^{(N-M)} \frac{P_0 z^{M} + P_1 z^{M-1} + \cdots + P_M}{d_0 z^{N} + d_1 z^{N-1} + \cdots + d_N}$$

(Eq. 6.13) (Eq. 6.14)

According to Eq. (6.14), we can factorize $H(z)$ as

$$H(z) = \frac{p_0 \prod_{\ell=1}^{M} (1-\xi_\ell z^{-1})}{d_0 \prod_{\ell=1}^{N} (1-\lambda_\ell z^{-1})}$$

$$= z^{(N-M)} \frac{p_0 \prod_{\ell=1}^{M} (z-\xi_\ell)}{d_0 \prod_{\ell=1}^{N} (z-\lambda_\ell)}$$

We denote ξ_ℓ, $\ell=1,2,\cdots,M$ as the "zeros" of the transfer function $H(z)$ and λ_ℓ, $\ell=1,2,\cdots,N$ as the "poles" of the $H(z)$. Such a factorization is called "pole-zero decomposition".

Example: Given a difference equation associated with an LTI system.

$$y[n] = 0.5\, y[n-1] + 0.2\, y[n-2] + 0.8\, x[n] - 0.6\, x[n-1]$$

Or $$y[n] - 0.5\, y[n-1] - 0.2\, y[n-2] = 0.8\, x[n] - 0.6\, x[n-1]$$

$p_0 = 0.8, \quad p_1 = -0.6, \quad d_0 = 1, \quad d_1 = -0.5, \quad d_2 = -0.2$

$P(z) = 0.8 - 0.6z^{-1}, \quad D(z) = 1 - 0.5z^{-1} - 0.2z^{-2}$

$$H(z) = \frac{P(z)}{D(z)} = \frac{0.8 - 0.6z^{-1}}{1 - 0.5z^{-1} - 0.2z^{-2}} = \frac{z(0.8z - 0.6)}{z^2 - 0.5z - 0.2}$$

zeros are the roots of $z(0.8z - 0.6) = 0$,

$\therefore$ $z = 0$, $z = 0.75$ are the roots. $\xi_1 = 0$, $\xi_2 = 0.75$

poles are the roots of $z^2 - 0.5z - 0.2 = 0$.

$\therefore$ $z = \dfrac{0.5 \pm \sqrt{0.25 + 0.8}}{2} = 0.76, -0.26$ are the roots

$\lambda_1 = 0.76$, $\lambda_2 = -0.26$

6.3 ROC of any Rational z-transform

The ROC of a rational z-transform is bounded by the pole locations. The ROC of a right-sided sequence associated with a pole $z = \alpha$ is the exterior area of the circle $|z| = |\alpha|$ while the ROC of a left-sided sequence associated with a pole $z = \alpha$

...is the interior area of $|z| = |a|$.

Example: Determine the ROC of $X(z)$ for

$$x[n] = (0.5)^n u[n].$$

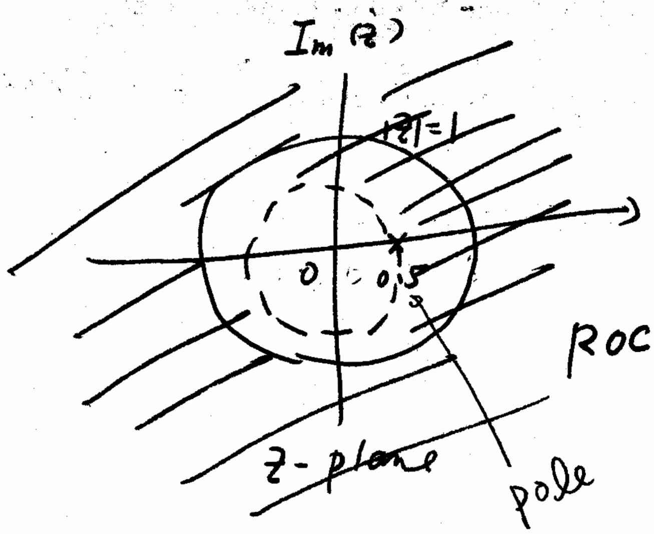
Solution

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} (0.5)^n z^{-n}$$

$$= \frac{1}{1-0.5z^{-1}} = \frac{z}{z-0.5},$$

where $|0.5z^{-1}| < 1 \Rightarrow \underbrace{|z| > 0.5}_{ROC}$



ROC: shaded area

Example: Determine the ROC of $X(z)$ for $x[n] = (0.5)^n \mu[-1-n]$.

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (0.5)^n \mu[-1-n] z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} (0.5)^{-1-m} \mu[m] z^{1+m}$$

$m = -1-n$

$$= \sum_{m=0}^{\infty} (0.5)^{-1-m} z^{1+m}$$

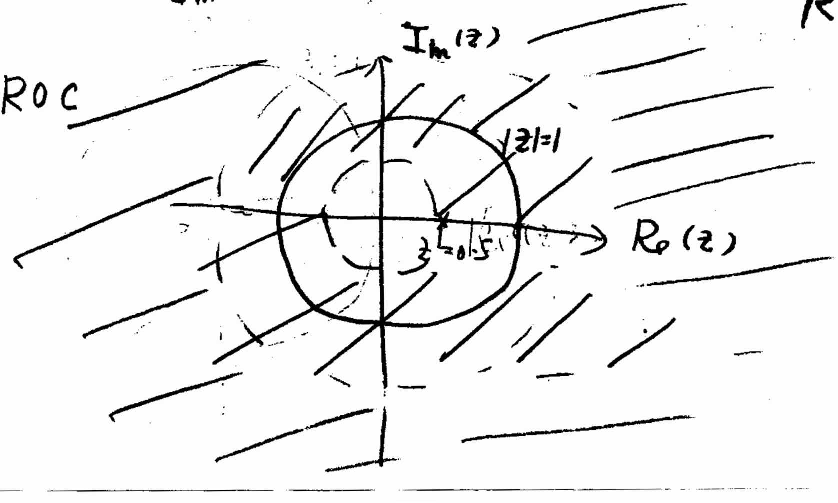
$$= (0.5)^{-1} z \sum_{m=0}^{\infty} (0.5)^{-m} z^{m}$$

$$= \frac{2z}{1-2z}$$

where $|2z| < 1 \Rightarrow \underbrace{|z| > 0.5}_{ROC}$

Im(z)

shaded area: ROC



Example: Determine the ROC of $G(z)$ for a finite-length sequence $g[n]$ defined for $-M \le n \le N$, where M, N are both nonnegative integers and $|g[n]| < \infty$.

Solution:

$$G(z) = \sum_{n=-M}^{N} g[n] z^{-n}$$

$$= \frac{\sum_{n=0}^{N+M} g[n-M] z^{N+M-n}}{z^{N}},$$

where $z \neq 0$ and $|z| \neq \infty$.

※ Any finite-length sequence has an ROC of $\{z \mid z \neq 0, |z| \neq \infty\}$.

Please read Example 6.10 by yourselves!

In summary, the ROCs of any rational z-transform can be given as:

(a) The ROC of the z-transform of a finite-length sequence defined for $M \le n \le N$ is the entire z-plane except possibly $z=0$ and/or $z=\infty$.

(b) The ROC of the z-transform of a right-sided sequence defined for $M \le n < \infty$ is the exterior area of a circle in the z-plane passing through the pole furthest from the origin.

(c) The ROC of the z-transform of a left-sided sequence defined for $-\infty < n < N$ is the interior area of a circle in the z-plane passing through the pole nearest to the origin.

(d) The ROC of the z-transform of a two-sided sequence of infinite length is a ring bounded by two circles in the z-plane passing through the two poles with no other poles inside such a ring.

(e) If the z-transform specifies a stable system, then the ROC should include the unit circle $|z|=1$.

6.4 Inverse Z-Transform

$$G(z) = \sum_{\ell=-\infty}^{\infty} g[\ell] z^{-\ell}$$

Multiply both sides by z^{n-1} and integrate over a closed contour inside the ROC encircling the origin $z=0$, we obtain

$$\oint_C G(z) z^{n-1} dz = \oint_C \sum_{\ell=-\infty}^{\infty} g[\ell] z^{-\ell} z^{n-1} dz$$

By Cauchy's integral theorem,

$$\frac{1}{2\pi j} \oint_C z^{n-1-\ell} dz = \delta[n-\ell]$$

Thus,

$$g[n] = \frac{1}{2\pi j} \oint_C G(z) z^{n-1} dz$$

According to the Cauchy's residue theorem,

$$g[n] = \sum \left[\text{residues of } G(z) z^{n-1} \text{ at the poles inside } C \right]$$

How to calulate the residues ?

If a pole at $z = \lambda_\phi$ of $G(z)\, z^{n-1}$ is of multiplicity m, we can define a function $P(z)$ as

$$P(z) \triangleq (z-\lambda_0)^m \, G(z)\, z^{n-1}$$

Then, the residue of $G(z)\, z^{n-1}$ at the pole $z = \lambda_0$ is given by

$$\text{Residue}\left[G(z)\, z^{n-1} \text{ at } z = \lambda_\phi \right]$$

$$= \frac{1}{(m-1)!} \left[\frac{d^{m-1} (z-\lambda_\phi)^m G(z)\, z^{n-1}}{d z^{m-1}} \right]_{z=\lambda_\phi}$$

$$= \frac{1}{(m-1)!} \left[\frac{d^{m-1} P(z)}{d z^{m-1}} \right]_{z=\lambda_\phi}$$

Example: Consider $X(z) = \dfrac{z}{(z-1)^2}$, $\underbrace{|z| > 1}_{\text{ROC}}$

Determine $x[n] = Z^{-1}\left[X(z) \right]$

Solution:

$X(z) z^{n-1} = \dfrac{z^n}{(z-1)^2}$ has only one pole at $z=1$ of multiplicity 2, where $n \ge 0$

$X(z) z^{n-1} = \dfrac{1}{(z-1)^2 z^k}$ has two poles at $z=1$ of multiplicity 2 and $z=0$ of multiplicity k ($k=-n$), where $n<0$.

For the pole $z=1$, $(m=2)$

$$(z-1)^2 X(z) z^{n-1} = z^n$$

The residue of $X(z) z^{n-1}$ at $z=1$

is $P_1 = \dfrac{d}{dz}\left[(z-1)^2 X(z) z^{n-1}\right]_{z=1}$

$$= \left.\frac{dz^n}{dz}\right|_{z=1} = \left. n z^{n-1}\right|_{z=1} = n, \quad -\infty < n < \infty$$

The residue of $X(z) z^{n-1}$ at $z=0$ (only when $n<0$) is $P_0 = \dfrac{1}{(k-1)!}\left[\dfrac{d^{k-1}}{dz^{k-1}}\left(z^k X(z) z^{-k-1}\right)\right]_{z=0}$

$$= \frac{1}{(k-1)!}\left[\frac{d^{k-1}}{dz^{k-1}}\left(\frac{1}{(z-1)^2}\right)\right]_{z=0}$$

$$= \left.\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left((z-1)^{-2}\right)\right|_{z=0}$$

$$= \frac{1}{(k-1)!} \quad k \ldots \Rightarrow k \ldots \quad n<0 \text{ or } k>0$$

$$\therefore \rho_\phi = -n, \quad n<0$$

Combining ρ_ϕ, ρ_1, we have

$$x[n] = \mathcal{Z}^{-1}\{X(z)\} = \begin{cases} n, & n \geq 0 \\ 0, & n<0 \end{cases}$$

X. Inverse z-transform by Look-up Table

Example: Determine the inverse z-transform $h[n]$ of $H(z)$ such that

$$H(z) = \frac{0.5z}{z^2 - z + 0.25}, \quad |z| > 0.5$$

Solution:

$$H(z) = \frac{0.5z}{(z-0.5)^2}, \quad |z| > 0.5$$

$$= \frac{0.5z^{-1}}{(1-0.5z^{-1})^2}, \quad |z| > 0.5$$

$h[n] = n(0.5)^n \mu[n]$ according to Table 6.1

※ Inverse z-transform by Partial-Fraction Expansion (PFE)

A rational z-transform $G(z)$ can be expressed as

$$G(z) = \frac{P(z)}{D(z)},$$

where $P(z)$ and $D(z)$ are polynomials of z.

$G(z)$ can always be re-expressed as

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \frac{P_1(z)}{D(z)},$$

where the degree of the polynomial $P_1(z)$ is less than that of $D(z)$ and $\frac{P_1(z)}{D(z)}$ is called a proper fraction.

Example:

$$G(z) = \frac{2 + 0.8z^{-1} + 0.5z^{-2} + 0.3z^{-3}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

$$= \frac{2z^3 + 0.8z^2 + 0.5z + 0.3}{z^3 + 0.8z^2 + 0.2z}$$

$$= 2 + \frac{(2z^3 + 0.8z^2 + 0.5z + 0.3) - 2(z^3 + 0.8z^2 + 0.2z)}{z^3 + 0.8z^2 + 0.2z}$$

$$= 2 + \frac{-0.8z^2 + 0.1z + 0.3}{z^3 + 0.8z^2 + 0.2z}$$

proper fraction

✗ PFE for simple poles

Since the z-transforms in Table 6.1 all involve a common factor "z", we can express $G(z)$ as $G(z) = z\,G'(z)$ and do the PFE for $G'(z) = \frac{G(z)}{z}$ such that $G'(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^\ell + \frac{P_1(z)}{D(z)}$,

where M is the degree of the numerator of $G'(z)$ and N is the degree of the denominator of $G'(z)$.

Then, we may express $\frac{P_1(z)}{D(z)}$ as

$$\frac{P_1(z)}{D(z)} = \sum_{\ell=1}^{N} \frac{p_\ell}{z-\lambda_\ell}\,,$$ λ_ℓ are the distinct roots of $D(z)$, $\ell = 1, 2, \cdots, N$.

We can observe that

$$(z-\lambda_k)\,\frac{P_1(z)}{D(z)}\bigg|_{z=\lambda_k} = \sum_{\substack{\ell=1\\ \ell\neq k}}^{N} \frac{p_\ell\,(z-\lambda_k)}{(z-\lambda_\ell)}\bigg|_{z=\lambda_k}$$

$$+\; p_k$$

$$= p_k\,, \quad 1 \le k \le N$$

Thus, we can determine the coefficients p_ℓ of the PFE of $\frac{P_1(z)}{D(z)}$, $\ell = 1, 2, \cdots, N$ as

$$p_\ell = (z-\lambda_\ell)\,\frac{P_1(z)}{D(z)}\bigg|_{z=\lambda_\ell}\,, \quad \ell = 1, 2, \cdots, N$$

Example: Determine the inverse z-transform $h[n]$ of $H(z)$ such that

$$H(z) = \frac{1 + 2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})}$$

$$= \frac{z(z+2)}{(z-0.2)(z+0.6)}$$

Solution: According to the PFE,

$$\frac{H(z)}{z} = \frac{z+2}{(z-0.2)(z+0.6)}$$ is proper,

and $$\frac{H(z)}{z} = \frac{\rho_1}{z-0.2} + \frac{\rho_2}{z+0.6}$$

$$\rho_1 = (z-0.2)\frac{H(z)}{z}\bigg|_{z=0.2} = \frac{z+2}{z+0.6}\bigg|_{z=0.2}$$

$$= \frac{2.2}{0.8} = \frac{11}{4} = 2.75$$

$$\rho_2 = (z+0.6)\frac{H(z)}{z}\bigg|_{z=-0.6} = \frac{z+2}{z-0.2}\bigg|_{z=-0.6}$$

$$= \frac{1.4}{-0.8} = -\frac{7}{4} = -1.75$$

$$\therefore H(z) = \frac{2.75z}{z-0.2} + \frac{(-1.75)z}{z+0.6}$$

$$h[n] = \mathcal{Z}^{-1}\{H(z)\}$$

$$= \mathcal{Z}^{-1}\left\{\frac{2.75z}{z-0.2}\right\} + \mathcal{Z}^{-1}\left\{\frac{-1.75z}{z+0.6}\right\}$$

$$= 2.75(0.2)^n \mu[n] - 1.75(-0.6)^n \mu[n].$$

※ PFE for multiple poles

Similarly, we may express $G(z)$ as $G(z) = zG'(z)$ and do the PFE for $G'(z) = \frac{G(z)}{z}$ such that

$$G'(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^\ell + \frac{P_1(z)}{D(z)}, \text{ where}$$

$\frac{P_1(z)}{D(z)}$ is proper.

Then, we may express $\frac{P_1(z)}{D(z)}$ as

$$\frac{P_1(z)}{D(z)} = \sum_{k_1=1}^{\ell_1} \frac{A_{1k_1}}{(z-P_1)^{k_1}} + \sum_{k_2=1}^{\ell_2} \frac{A_{2k_2}}{(z-P_2)^{k_2}} + \cdots$$

$$+ \sum_{k_i=1}^{\ell_i} \frac{A_{ik_i}}{(z-P_i)^{k_i}},$$

where the degree of $D(z) = \ell_1 + \ell_2 + \cdots + \ell_i$ and there are i poles in $D(z)$.

How to determine A_{mk_m}, $m = 1, 2, \cdots, i$?
$k_i = 1, 2, \cdots, l_i$

$$\frac{P_1(z)}{D(z)} (z - P_m)^{l_m} = \sum_{m' \neq m} \sum_{k_{m'}=1}^{l_{m'}} \left[\frac{A_{m'k_{m'}}}{(z - P_{m'})^{k_{m'}}} (z - P_m)^{l_m} \right] + \sum_{k_m''=1}^{l_m} A_{mk_m''} (z - P_m)^{l_m - k_m''}$$

Taking the $(l_m - k_m)^{th}$ derivative, we have

$$\frac{d^{(l_m - k_m)}}{dz^{(l_m - k_m)}} \left[\frac{P_1(z)}{D(z)} (z - P_m)^{l_m} \right]$$

$$= \sum_{m' \neq m} \sum_{k_{m'}=1}^{l_{m'}} \left\{ \frac{d^{(l_m - k_m)}}{dz^{(l_m - k_m)}} \left[\frac{A_{m'k_{m'}}}{(z - P_{m'})^{k_{m'}}} (z - P_m)^{l_m} \right] \right\}$$

$$+ \sum_{m' \neq m} \sum_{k_{m'}=1}^{l_{m'}} \left\{ \frac{A_{m'k_{m'}}}{(z - P_{m'})^{k_{m'}}} \frac{l_m!}{k_m!} (z - P_m)^{k_m} \right\}$$

$$+ \sum_{k_m''=1}^{k_m} \left\{ A_{mk_m''} \frac{(l_m - k_m'')!}{(k_m - k_m'')!} (z - P_m)^{k_m - k_m''} \right\}$$

$$\therefore \frac{d^{(l_m - k_m)}}{dz^{(l_m - k_m)}} \left[\frac{P_1(z)}{D(z)} (z - P_m)^{l_m} \right] \Bigg|_{z = P_m}$$

$$= A_{mk_m} (l_m - k_m)!$$

$$\therefore\ A_{mk_m} = \frac{1}{(l_m-k_m)!}\,\frac{d^{(l_m-k_m)}}{dz^{(l_m-k_m)}}\left[\frac{P_1(z)}{D(z)}(z-P_m)^{l_m}\right]\Bigg|_{z=P_m}$$

Example:

$$G(z) = \frac{z}{(z-1)^2(z-0.5)^2(z-0.6)}$$

Write $G(z)$ in PFE.

Solution: $i = 3$ (3 poles), $l_1 = 2$, $P_1 = 1$,

$l_2 = 2$, $P_2 = 0.5$, $l_3 = 1$, $P_3 = 0.6$

Determine

$$\frac{P_1(z)}{D(z)} = \frac{G(z)}{z} = \frac{1}{(z-1)^2(z-0.5)^2(z-0.6)}$$

$$\equiv \frac{A_{11}}{(z-1)} + \frac{A_{12}}{(z-1)^2} + \frac{A_{21}}{(z-0.5)} + \frac{A_{22}}{(z-0.5)^2} + \frac{A_{31}}{(z-0.6)}$$

$l_1 = 2$, $k_1 = 1$, $\Rightarrow$

$$A_{11} = \frac{1}{1!}\frac{d}{dz}\left[(z-1)^2\frac{G(z)}{z}\right]\Bigg|_{z=1}$$

$$= \frac{d}{dz}\,\frac{1}{(z-0.5)^2(z-0.6)}\Bigg|_{z=1}$$

$$= \frac{-2(z-0.5)(z-0.6)-(z-0.5)^2}{(z-0.5)^4(z-0.6)^2}\Bigg|_{z=1}$$

$$= -65$$

$$l_1 = 2,\ k_1 = 2 \Rightarrow A_{12} = \left[(z-1)^2 \frac{G(z)}{z} \right] \Big|_{z=1}$$

$$= \frac{1}{(z-0.5)^2 (z-0.6)} \Big|_{z=1}$$

$$= 10$$

$$l_2 = 2,\ k_2 = 1 \Rightarrow A_{21} = \frac{d}{dz} \left[(z-0.5)^2 \frac{G(z)}{z} \right] \Big|_{z=0.5}$$

$$= \frac{d}{dz} \frac{1}{(z-1)^2 (z-0.6)} \Big|_{z=0.5}$$

$$= \frac{-2(z-1)(z-0.6) - (z-1)^2}{(z-1)^4 (z-0.6)^2} \Big|_{z=0.5}$$

$$= -560$$

$$l_2 = 2,\ k_2 = 2 \Rightarrow A_{22} = \left[(z-0.5)^2 \frac{G(z)}{z} \right] \Big|_{z=0.5}$$

$$= \frac{1}{(z-1)^2 (z-0.6)} \Big|_{z=0.5}$$

$$= -40$$

$$l_3 = 1,\ k_3 = 1 \Rightarrow A_{31} = \left[(z-0.6) \frac{G(z)}{z} \right] \Big|_{z=0.6}$$

$$= \left[\frac{1}{(z-1)^2 (z-0.5)^2} \right] \Big|_{z=0.6}$$

$$= 625$$

$$\therefore \quad G(z) = z\,\frac{P_1(z)}{D(z)} = -65\,\frac{z}{z-1} + 10\,\frac{z}{(z-1)^2}$$

$$-560\,\frac{z}{z-0.5} - 40\,\frac{z}{(z-0.5)^2}$$

$$+625\,\frac{z}{z-0.6}$$

If the ROC of $G(z)$ is $|z| > 1$,

then according to Table 6.1, $g[n] = \mathcal{Z}^{-1}\{G(z)\}$

$$= -65\,\mu[n] + 10\,n\,\mu[n] - 560\,(0.5)^n\mu[n]$$

$$-80\,n\,(0.5)^n\mu[n] + 625\,(0.6)^n\mu[n]$$

6.5 Z-Transform Properties

Assume $g[n] \overset{\mathcal{Z}}{\longleftrightarrow} G(z)$, ROC $= R_g$.

We may obtain the z-transform properties as listed in Table 6.2.

Table 6.2: Some useful properties of the z-transform.

Property	Sequence	z-Transform	ROC
	$g[n]$	$G(z)$	$\mathcal{R}_g$
	$h[n]$	$H(z)$	$\mathcal{R}_h$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}_g$
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n - n_o]$	$z^{-n_o} G(z)$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\lvert\alpha\rvert \mathcal{R}_g$
Differentiation of $G(z)$	$ng[n]$	$-z\dfrac{dG(z)}{dz}$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Convolution	$g[n] \circledast h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j} \oint_C G(v)H(z/v)v^{-1}\,dv$	Includes $\mathcal{R}_g\mathcal{R}_h$
Parseval's relation		$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j}\oint_C G(v)H^*(1/v^*)v^{-1}\,dv$	

Note: If $\mathcal{R}_g$ denotes the region $R_{g-} < |z| < R_{g+}$ and $\mathcal{R}_h$ denotes the region $R_{h-} < |z| < R_{h+}$, then $1/\mathcal{R}_g$ denotes the region $1/R_{g+} < |z| < 1/R_{g-}$ and $\mathcal{R}_g\mathcal{R}_h$ denotes the region $R_{g-}R_{h-} < |z| < R_{g+}R_{h+}$.

Example: Verify the z-transform $X(z)$ of

$x[n] = r^n \cos(\omega_o n)\,\mu[n]$.

Solution:

$$x[n] = \frac{1}{2} r^n e^{j\omega_o n}\mu[n] + \frac{1}{2} r^n e^{-j\omega_o n}\mu[n]$$

According to Tables 6.1 and 6.2,

$$X(z) = \mathcal{Z}\left\{\tfrac{1}{2} r^n e^{j\omega_o n}\mu[n]\right\} + \mathcal{Z}\left\{\tfrac{1}{2} r^n e^{-j\omega_o n}\mu[n]\right\}$$

$$= \frac{z}{z - re^{jw_0}} + \frac{z}{z - re^{-jw_0}}, \quad |z| > r$$

$$= \frac{1}{2}\left[\frac{z^2 - zr\cos(w_0)}{z^2 - (2rz\cos(w_0)) + r^2}\right], \quad |z| > r$$

Example: $w[n] = [(-0.5)^{n-2} + (0.2)^{n-1}]\,\mu[n]$

Determine the z-transform of $w[n]$.

Solution:

$$w[n] = 4(-0.5)^n\mu[n] + 2(0.2)^n\mu[n]$$

$$(-0.5)^n\mu[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z+0.5}, \quad |z| > 0.5$$

$$(0.2)^n\mu[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-0.2}, \quad |z| > 0.2$$

$$\Rightarrow \mathcal{Z}\{w[n]\} = \frac{4z}{z+0.5} + \frac{2z}{z-0.2}, \quad \underbrace{|z| > 0.5 \cap |z| > 0.2}_{|z| > 0.5}$$

$$= \frac{6z^2 + 0.2z}{(z+0.5)(z-0.2)}, \quad |z| > 0.5$$

Example: $v[n] = \alpha^n\mu[n] - \beta^n\mu[-n-1]$

Determine $V(z) = \mathcal{Z}\{v[n]\}$.

Solution:

$$\mathcal{Z}\{\alpha^n \mu[n]\} = \frac{z}{z-\alpha}, \quad \text{for } |z| > |\alpha|$$

$$\mathcal{Z}\{\beta^n \mu[-n-1]\} = \frac{z}{z-\beta}, \quad \text{for } |z| < |\beta|$$

$$\therefore \quad V(z) = \frac{z}{z-\alpha} + \frac{z}{z-\beta} \quad \text{for } |\alpha| < |z| < |\beta|$$

and $|\alpha| < |\beta|$

Example: $d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$.

Take z-transforms of each term,

$$d_0 V(z) + d_1 z^{-1} V(z) = p_0 + p_1 z^{-1}$$

$$\Rightarrow V(z) = \frac{p_0 + p_1 z^{-1}}{d_0 + d_1 z^{-1}}$$

Example: $y[n] = (n+1)\alpha^n \mu[n]$. Determine $Y(z)$

$= \mathcal{Z}\{y[n]\}$

Solution: Let $x[n] = \alpha^n \mu[n] \overset{\mathcal{Z}}{\longleftrightarrow} \underbrace{\frac{z}{z-\alpha}}_{}$, $|z| > |\alpha|$ $\quad X(z)$

$$n\,x[n] = n\,\alpha^n \mu[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z \frac{dX(z)}{dz}, \quad |z| > |\alpha|$$

$$-z \frac{dX(z)}{dz} = -z \frac{d}{dz}\left(\frac{z}{z-\alpha}\right) = -z \frac{z-\alpha-z}{(z-\alpha)^2}$$

$$= \frac{\alpha z}{(z-\alpha)^2}, \quad |z| > |\alpha|$$

$$\therefore \quad Y(z) = \mathcal{Z}\{y[n]\} = X(z) - z\frac{dX(z)}{dz}$$

$$= \frac{z}{z-\alpha} + \frac{\alpha z}{(z-\alpha)^2}, \quad |z| > |\alpha|$$

Example: Consider two causal sequences, $g[n]$, $h[n]$ such that $y[n] = g[n] \circledast h[n]$

$$G(z) = \mathcal{Z}\{g[n]\} = \frac{2z+1.2}{z-0.2}, \quad |z| > 0.2$$

$$H(z) = \mathcal{Z}\{h[n]\} = \frac{3z}{z+0.6}, \quad |z| > 0.6$$

Determine $Y(z) = \mathcal{Z}\{y[n]\}$.

Solution: Since $y[n] = g[n] \circledast h[n]$,

$$Y(z) = G(z)\,H(z)$$

$$= \frac{2\cancel{(z+0.6)}}{z-0.2}\;\frac{3z}{\cancel{z+0.6}} = \frac{6z}{z-0.2},$$

$$|z| > 0.6$$

$$(\,|z| > 0.2 \cap |z| > 0.6 \Rightarrow |z| > 0.6\,)$$

6.6 Linear Convolution Using z-transform

Let $x[n]$ and $h[n]$ be the two causal and finite-length sequences. We can apply the z-transform to result in $y[n] = x[n] \circledast h[n]$

Assume that

$$X(z) = x[0] + x[1]z^{-1} + \cdots + x[L]\,x^{-L}$$
$$H(z) = h[0] + h[1]z^{-1} + \cdots + h[M]\,x^{-M}$$

If $y[n] = x[n] \circledast h[n]$, then $Y(z) = \mathcal{Z}\{y[n]\}$
$= X(z)\,H(z)$.

$$Y(z) = H(z)\,X(z)$$
$$= y[0] + y[1]z^{-1} + \cdots + y[L+M]\,z^{-(L+M)}.$$

where $y[n] = \sum_{k=0}^{L+M} x[k]\,h[n-k]$, $\quad 0 \le n \le L+M$

Example:

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = -2 + z^{-2} - z^{-3} + 3z^{-4}$$
$$h[n] \overset{\mathcal{Z}}{\longleftrightarrow} H(z) = 1 + 2z^{-1} - z^{-3}$$
$$y[n] = x[n] \circledast h[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z) = ?$$

Solution:

$$Y(z) = X(z)\,H(z)$$
$$= (-2 + z^{-2} - z^{-3} + 3z^{-4})\,(1 + 2z^{-1} - z^{-3})$$
$$= -2 - 4z^{-1} + z^{-2} + 3z^{-3} + z^{-4} + 5z^{-5}$$
$$+ z^{-6} - 3z^{-7}.$$

Using Matlab, we can calulate $y[n]$ as a vector.

```
>> x = [-2 0 -1 3];
>> h = [1 2 0 -1];
>> y = conv(x, h)
```

6.7 Transfer function

The transfer function of an LTI system is defined as

$$\underset{\substack{\uparrow \\ \text{transfer} \\ \text{function}}}{H(z)} \triangleq \frac{Y(z) \to z\text{-transform of output } y[n]}{X(z) \to z\text{-transform of input } x[n]}$$

In general, an LTI system can be described as a difference equation such that

$$\sum_{k=0}^{N} d_k \, y[n-k] = \sum_{k=0}^{M} p_k \, x[n-k]$$

Taking z-transform of each term, we have

$$\sum_{k=0}^{N} d_k \, z^{-k} Y(z) = \sum_{k=0}^{M} p_k \, z^{-k} X(z)$$

where $X(z) = \mathcal{Z}\{x[n]\}$ and $Y(z) = \mathcal{Z}\{y[n]\}$.

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}}$$

Example: an LTI system is speified as the difference equation

$$y[n] = x[n-1] - 1.2\,x[n-2] + x[n-3] + 1.3\,y[n-1] - 1.04\,y[n-2] + 0.222\,y[n-3]$$

Determine the system transfer function $H(z)$.

Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\mathcal{Z}\{y[n]\}}{\mathcal{Z}\{x[n]\}}$$

$$Y(z) = z^{-1}X(z) - 1.2\,z^{-2}X(z) + z^{-3}X(z) + 1.3\,z^{-1}Y(z) - 1.04\,z^{-2}Y(z) + 0.222\,z^{-3}Y(z)$$

$$\Rightarrow \quad \frac{Y(z)}{X(z)} = \frac{z^{-1} - 1.2\,z^{-2} + z^{-3}}{1 - 1.3\,z^{-1} + 1.04\,z^{-2} - 0.222\,z^{-3}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2 - 1.2z + 1}{z^3 - 1.3z^2 + 1.04z - 0.222}$$

6.7.5 Stability Condition in terms of Pole Locations

An LTI system is usually characterized as an impulse response $h[n]$. The LTI system is bounded-input bounded output (BIBO) stable if and only if $h[n]$ is absolutely summable such that

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty.$$

Theorem: If $h[n]$ specifies the impulse response of a BIBO stable LTI system, then the ROC of $H(z) = \mathcal{Z}\{h[n]\}$ should include the unit circle $|z| = 1$.

Proof:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

$$|H(z)| = \left| \sum_{n=-\infty}^{\infty} h[n] z^{-n} \right| \leq \sum_{n=-\infty}^{\infty} |h[n]| |z^{-n}|$$

On the unit circle, $z = e^{j\omega}$,

$$|H(z)|_{z=e^{j\omega}} = |H(e^{j\omega})| \leq \sum_{n=-\infty}^{\infty} |h[n]| |e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |h[n]|$$

Since $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ (BIBO stable),

$$|H(e^{j\omega})| \leq \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

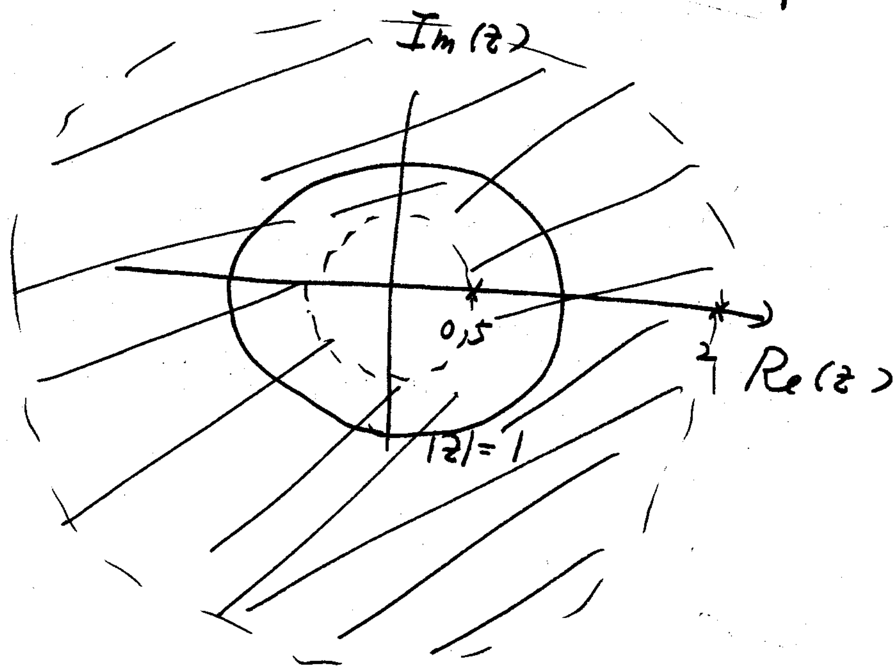
$\therefore$ $H(z)\Big|_{z=e^{j\omega}}$ exists, or the ROC of $H(z)$ includes the unit circle.

Example: If a BIBO stable LTI system has a transfer function

$$H(z) = \frac{z}{(z-0.5)(z-2)},$$

what is the appropriate ROC?

Solution: There are two poles, $z=0.5$, $z=2$. Depict them on the zero-pole plot as



The only combination of two individual ROCs associated with the two poles is the ring area $0.5<|z|<2$, which will include $|z|=1$.