Chapter Five. Finite-length Discrete Transforms

5.2 Discrete Fourier Transform

The discrete Fourier transform (DFT) of length-N $x[n]$ is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}}, \quad 0 \le k \le N-1$$

Define $W_N = e^{-j\frac{2\pi}{N}}$

Thus,

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le N-1.$$

$$\frac{1}{N} \sum_{k=0}^{N-1} W_N^{kn} W_N^{-k\ell}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi k n}{N}} e^{-j\frac{2\pi k \ell}{N}}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi (k-\ell) n}{N}}$$

$$= \frac{1}{N} \frac{1 - e^{j2\pi(k-\ell)}}{1 - e^{j\frac{2\pi(k-\ell)}{N}}}$$

$$= \begin{cases} 1, & \text{for } k = l + rN, \quad r \in \mathbb{Z} \\ 0, & \text{for } k \neq l + rN \end{cases}$$

Hence,

$$\frac{1}{N} \sum_{k=0}^{N-1} X[k] \, W_N^{-kn} = x[n], \quad 0 \leq n \leq N-1$$

$$\Rightarrow \quad X[k] = \sum_{n=0}^{N-1} x[n] \, e^{-j\frac{2\pi n k}{N}} \quad \text{(DFT)}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, e^{j\frac{2\pi n k}{N}} \quad \text{(IDFT)}$$

Or the discrete Fourier transform pair will often be denoted as

$$x[n] \overset{DFT}{\longleftrightarrow} X[k].$$

Example: Consider the length-N sequence defined as

$$x[n] = \begin{cases} 1, & n = 0 \\ 0, & 1 \leq n \leq N-1. \end{cases}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi n k}{N}}$$

$$= 1, \quad 0 \le k \le N-1$$

Now consider the length-N sequence defined as

$$y[n] = \begin{cases} 1, & n = m, \quad 0 \le m \le N-1 \\ 0, & n \ne m \end{cases}$$

The DFT is given by

$$Y[k] = \sum_{n=0}^{N-1} y[n] e^{-j\frac{2\pi n k}{N}}$$

$$= e^{-j\frac{2\pi m k}{N}}$$

$$= W_N^{km}$$

5.2.2 Matrix Relations

The DFT samples $X[k]$, $0 \le k \le N-1$, can be expressed in a matrix form as

$$\vec{X} = \tilde{D}_N \vec{x},$$

where $\vec{X} = [\, X[0],\ X[1],\ \cdots,\ X[N-1] \,]^T$

$\vec{x} = [\, x[0],\ x[1],\ \cdots,\ x[N-1] \,]^T$

and $\tilde{D}_N$ is the $N \times N$ DFT matrix given by

$$\tilde{D}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & \cdots & & \vdots \\ \vdots & \vdots & & & \vdots \\ 1 & W_N^{N-1} & \cdots & & W_N^{(N-1)(N-1)} \end{bmatrix}$$

Likewise, the IDFT can be expressed as

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \tilde{D}_N^{-1} \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

where $\tilde{D}_N^{-1}$ is the $N \times N$ IDFT matrix given by

$$\tilde{D}_N^{-1} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & \cdots & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & \cdots & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & & & \vdots \\ 1 & W_N^{-(N-1)} & \cdots & \cdots & W_N^{-(N-1)(N-1)} \end{bmatrix}$$

and

$$\tilde{D}_N^{-1} = \frac{1}{N} D_N^*$$

5.3 Relation between DTFT (FT) and DFT

The DTFT or the Fourier transform $X(e^{j\omega})$ of the length-N sequence $x[n]$, defined for $0 \le n \le N-1$, is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n}$$

By uniformly sampling $X(e^{j\omega})$ at N equally spaced frequencies $\omega_k = \frac{2\pi k}{N}$, $0 \le k \le N-1$, on the ω-axis between $0 \le \omega < 2\pi$, we get

$$X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k}{N}},$$

$$0 \le k \le N-1.$$

It is observed that the N-point DFT sequence $X[k]$ is precisely the set of frequency samples of the Fourier transform $X(e^{j\omega})$ of the length-N sequence $x[n]$ at N equally spaced frequencies $\omega_k = \frac{2\pi k}{N}$, $0 \le k \le N-1$.

5.3.3

Given the N-point DFT $X[k]$ of a length-N sequence $x[n]$, it is also possible to determine its Fourier transform $X(e^{j\omega})$ uniquely by interpolating $X[k]$ at all values of ω.

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left[\sum_{k=0}^{N-1} X[k]\, W_N^{-kn} \right] e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \sum_{n=0}^{N-1} e^{j\frac{2\pi k n}{N}}\, e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \sum_{n=0}^{N-1} e^{-j(\omega - \frac{2\pi k}{N})n}$$

$$= \frac{1 - e^{-j(\omega N - 2\pi k)}}{1 - e^{-j[\omega - \frac{2\pi k}{N}]}}$$

$$= \frac{e^{-j\left[\frac{\omega N - 2\pi k}{2}\right]}}{e^{-j\left[\frac{\omega N - 2\pi k}{2N}\right]}} \frac{\sin\left(\frac{\omega N - 2\pi k}{2}\right)}{\sin\left(\frac{\omega N - 2\pi k}{2N}\right)}$$

$$= \frac{\sin\left(\frac{\omega N - 2\pi k}{2}\right)}{\sin\left(\frac{\omega N - 2\pi k}{2N}\right)} e^{-j\left[\omega - \frac{2\pi k}{N}\right]\left[\frac{N-1}{2}\right]}$$

Thus,

$$X(e^{j\omega}) = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \frac{\sin\left(\frac{\omega N - 2\pi k}{2}\right)}{\sin\left(\frac{\omega N - 2\pi k}{2N}\right)} e^{-j\left[\omega - \frac{2\pi k}{N}\right]\left[\frac{N-1}{2}\right]}$$

$$= \sum_{k=0}^{N-1} X[k] \Phi\left(\omega - \frac{2\pi k}{N}\right)$$

where $\Phi(\omega) = \frac{\sin\left(\frac{\omega N}{2}\right)}{N \sin\left(\frac{\omega}{2}\right)} e^{-j\omega\left[\frac{N-1}{2}\right]}$

5.3.4

Consider a sequence $\{x[n]\}$ with a discrete-time Fourier transform $X(e^{j\omega})$. We sample $X(e^{j\omega})$ at

N equally spaced points $\omega_k = \frac{2\pi k}{N}$, $0 \le k \le N-1$,

and develop the N frequency samples $\{X(e^{j\frac{2\pi k}{N}})\}$

$0 \le k \le N-1$. Thus,

$$Y[k] = X(e^{j\frac{2\pi k}{N}}) = \sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell}, \quad 0 \le k \le N$$

An N-point IDFT of $Y[k]$ yields

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} Y[k] W_N^{-kn}, \quad 0 \le n \le N-1.$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell} W_N^{-kn}$$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell] \left[\frac{1}{N} \sum_{k=0}^{N-1} W_N^{-k(n-\ell)} \right], \quad 0 \le n \le N-1.$$

$$\frac{1}{N} \sum_{k=0}^{N-1} W_N^{-k(n-\ell)} = \begin{cases} 1, & \text{for } \ell = n + mN \\ 0, & \text{otherwise} \end{cases}$$

Consequently,

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN], \quad 0 \le n \le N-1$$

Example: Illustration of Sampling Effect in the frequency domain

Let $x[n]$ be length-6 sequence for $0 \le n \le 5$ and it is given by

$$\{x[n]\} = \{0 \ 1 \ 2 \ 3 \ 4 \ 5\}$$

The DFT is simply sampling the DTFT

Said that

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \quad 0 \le n \le N-1$$

$$Y[k] = X(e^{j\frac{2\pi k}{N}})$$

$$= \sum_{n=-\infty}^{\infty} x[n] W_N^{kn}, \quad 0 \le k \le N-1$$

An N-point inverse DFT of $Y[k]$ yields

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} Y[k] W_N^{-kn}, \quad 0 \le n \le N-1$$

$$= \sum_{m=-\infty}^{\infty} x[n+mN], \quad 0 \le n \le N-1$$

That is, given a finite-length sequence $x[n]$, define for, $0 \le n \le L-1$, we can do the N-point DFT to construct $X[k]$. If $N \ge L$, according to Eq. (5.46), we have the unique $X(e^{j\omega})$ but different sample set $X[k]$, $0 \le k \le N-1$.

When we do the N-point IDFT, we have

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN], \quad 0 \le n \le N-1.$$

Hence, in this example, we apply 8-point DFT and 8-point IDFT to obtain

$$y[n] = IDFT_8\left\{DFT_8\left\{x[n]\right\}\right\}$$

$$= \sum_{m=-\infty}^{\infty} x[n+8m], \quad 0 \le n \le 7$$

$$= \{\cdots \underbrace{\underset{\underset{n=0}{\uparrow}}{0}\ 1\ 2\ 3\ 4\ 5\ 0\ 0}_{\text{one period}}\ 0\ 1\ 2\ 3\ 4\ 5 \cdots\}$$

$$y[n] = \underset{4}{IDFT}\left\{ \underset{4}{DFT}\left\{ x[n] \right\} \right\}$$

$$= \sum_{m=-\infty}^{\infty} x[n+4m], \qquad 0 \le n \le 3$$

$$= \underbrace{\{ \cdots \underset{\substack{\uparrow \\ n=0}}{0} \;\; 1 \;\; 2 \;\; \underset{\substack{\uparrow \\ n=3}}{3} \;\; 4 \;\; 5 \;\; 0 \cdots \}}_{x[n]}$$

$$+ \underbrace{\{ \cdots \underset{\substack{\uparrow \\ n=0}}{4} \;\; 5 \;\; 0 \;\; 0 \;\; \cdots \}}_{x[n+4]}$$

$$+ \underbrace{\{ \cdots \underset{\substack{\uparrow \\ n=0}}{0} \;\; 0 \;\; 0 \;\; \cdots \}}_{x[n-4]}$$

$$= \{ \cdots \underbrace{\underset{\substack{\uparrow \\ n=0}}{4}, 6, 2, 3}_{\text{One period}} \cdots \}$$

5.4 Operations of Finite-Length Sequences

5.4.1 Circular shift

Let $0, 1, \cdots, N$ be a set of N positive integer and let m be any integer. The integer r obtained by evaluating m modulo N is called the residue and is an integer with a value between 0 and $N-1$. The moduo operation is denoted by the notation

$$\langle m \rangle_N = m \text{ modulo } N.$$

If we let $r = \langle m \rangle_N$, then

$$r = m + \ell N, \quad \ell \in \mathbb{Z}.$$

where ℓ is an integer chosen to make $m + \ell N$ a number between 0 and $N-1$.

Example:

$$\langle 25 \rangle_7 = 4$$

$$\langle -16 \rangle_7 = 5$$

$$\langle 4 \rangle_4 = 0$$

Using the modulo operation, we define the circular shift of a length-N sequence $x[n]$ as

$$x_c[n] = x[\langle n-n_0\rangle_N],$$

where $x_c[n]$ is also a length-N sequence. If $n_0>0$, it is a right circular shift. If $n_0<0$, it is a left circular shift. It is obvious that

$$x_c[n] = \begin{cases} x[n-n_0], & \text{for } n_0 \le n \le N-1 \\ x[N-n_0+n], & \text{for } 0 \le n < n_0 \end{cases}$$

The circular time-reversal operation is defined as

$$x[\langle -n\rangle_N] = x[\langle N-n\rangle_N]$$

Since $\langle -n\rangle_N = N-n$ for $1 \le n < N-1$, this operation is formulated as

$$x[\langle -n\rangle_N] = \begin{cases} x[N-n], & 1 \le n \le N- \\ x[n], & n=0 \end{cases}$$

Similarly, in the frequency domain, the circular shifting operation by k_0 samples on the length-N DFT sequence $X[k]$ is defined as

$$X_c[k] = X[\langle k-k_0 \rangle_N],$$

where $X_c[k]$ is also a length-N DFT.

5.4.2 Circular Convolution

The circular convolution is defined as

$$y_c[n] = \sum_{m=0}^{N-1} g[m]\, h[\langle n-m \rangle_N]$$

$$= g[n] \circledN h[n]$$

The circular convolution is commutative, i.e.,

$$g[n] \circledN h[n] = h[n] \circledN g[n]$$

In a matrix form, we have

$$\begin{bmatrix} y_c[0] \\ y_c[1] \\ \vdots \\ y_c[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[1] \\ h[1] & h[0] & h[N-1] & \cdots & h[2] \\ \vdots & \vdots & & & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h[0] \end{bmatrix} \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ \vdots \\ g[N-1] \end{bmatrix}$$

Example:

$$g[n] = \{\underset{\substack{\uparrow \\ n=0}}{1}, 2, 0, 1\} = [1 \ 2 \ 0 \ 1]$$

$$h[n] = \{\underset{\substack{\uparrow \\ n=0}}{2}, 2, 1, 1\} = [2 \ 2 \ 1 \ 1]$$

$$y_c[n] = g[n] \ ④ \ h[n] = \sum_{m=0}^{3} g[m]\, h[\langle n-m \rangle_4]$$

$$y_c[0] = \sum_{m=0}^{3} g[m]\, h[\langle -m \rangle_4]$$

$$= g[0]h[0] + g[1]h[3] + g[2]h[2] + g[3]h[1] = 1\times 2 + 2\times 1 + 0\times 1 + 1\times 2$$

$$= 6$$

$$g[m] \qquad [1 \quad 2 \quad 0 \quad 1]$$

$$h[\langle -m \rangle_4] \quad [1 \quad 1 \quad 2 \quad 2]$$

$$\xrightarrow{+4}$$

$$= [2 \quad 1 \quad 1 \quad 2]$$

Similarly,

$$y_c[1] = g[0]\,h[1] + g[1]\,h[0] + g[2]\,h[3] + g[3]\,h[2]$$

$$= 1 \times 2 + 2 \times 2 + 0 \times 1 + 1 \times 1 = 7$$

$$y_c[2] = g[0]\,h[2] + g[1]\,h[1] + g[2]\,h[0] + g[3]\,h[3]$$

$$= 1 \times 1 + 2 \times 2 + 0 \times 2 + 1 \times 1 = 6$$

$$y_c[3] = g[0]\,h[3] + g[1]\,h[2] + g[2]\,h[1] + g[3]\,h[0]$$

$$= 1 \times 1 + 2 \times 1 + 0 \times 2 + 1 \times 2 = 5$$

5.5 Classification of Finite-Length Sequences.

The symmetry property must be reinvestigated due to the introduction of the circular shift

5.5.1 Classification Based on Conjugate Symmetry

For a length-N sequence $x[n]$, it can be expressed as

$$x[n] = x_{cs}[n] + x_{ca}[n], \quad 0 \le n \le N-1,$$

where the circular conjugate-symmetric part $x_{cs}[n]$ and the circular conjugate-antisymmetric part are defined by

$$x_{cs}[n] = \frac{1}{2}\left(x[n] + x^*[\langle -n \rangle_N]\right)$$

$$x_{ca}[n] = \frac{1}{2}\left(x[n] - x^*[\langle -n \rangle_N]\right), \quad 0 \le n \le N-1$$

A length-N complex sequence $x[n]$ is circular conjugate-symmetric if

$$x[n] = x^*[\langle -n \rangle_N] = x^*[\langle N-n \rangle_N].$$

A length-N real sequence $x[n]$ is circular even if

$$x[n] = x[\langle -n \rangle_N] = x[\langle N-n \rangle_N]$$

On the other hand, a length-N complex sequence is circular conjugate-antisymmetric if

$$x[n] = -x^*[\langle -n\rangle_N] = -x^*[\langle N-n\rangle_N]$$

A length-N real sequence is circular odd if

$$x[n] = -x[\langle -n\rangle_N] = -x[\langle N-n\rangle_N]$$

Example:

Consider length-4 sequence $\{u[n]\}$, $0 \le n \le 3$:

$$\{u[n]\} = \{\underset{\substack{\uparrow\\ n=0}}{1+j4},\ -2+j3,\ 4-j2,\ -5-j6\}$$

$$\{u^*[\langle -n\rangle_4]\} = \{\underset{\substack{\uparrow\\ n=0}}{1-j4},\ -5+j6,\ 4+j2,\ -2-j3\}$$

$$\therefore\ u_{cs}[n] = \frac{u[n] + u^*[\langle -n\rangle_4]}{2}$$

$$= \{\underset{\substack{\uparrow\\ n=0}}{1},\ -3.5+j4.5,\ 4,\ -3.5-j4.5\}$$

$$u_{ca}[n] = \frac{1}{2}\left(u[n] - u^*[\langle -n\rangle_4]\right)$$

$$= \{\underset{\substack{\uparrow\\ n=0}}{j4},\ 1.5-j1.5,\ -j2,\ -1.5-j1.5\}$$

Similarly, the frequency domain sequence (DFT) is circular conjugate-symmetric if

$$X[k] = X^*[\langle -k \rangle_N] = X^*[\langle N-k \rangle_N].$$

Likewise, a DFT $X[k]$ is circular conjugate-antisymmetric if

$$X[k] = -X^*[\langle -k \rangle_N] = X^*[\langle N-k \rangle_N].$$

Any DFT $X[k]$ can be expressed as

$$X[k] = X_{cs}[k] + X_{ca}[k], \quad 0 \le k \le N-1$$

where

$$X_{cs}[k] = \frac{1}{2}\left(X[k] + X^*[\langle -k \rangle_N] \right)$$

$$X_{ca}[k] = \frac{1}{2}\left(X[k] - X^*[\langle -k \rangle_N] \right),$$

$$0 \le k \le N-1$$

5.5.2 Classification Based on Geometric Symmetry

Two types of geometric symmetrices are usually defined for digital signal processing:

(1) Symmetric: A length-N symmetric sequence $x[n]$ satisfies

$$x[n] = x[N-1-n].$$

(2) Antisymmetric: A length-N antisymmetric sequence $x[n]$ satisfies

$$x[n] = -x[N-1-n].$$

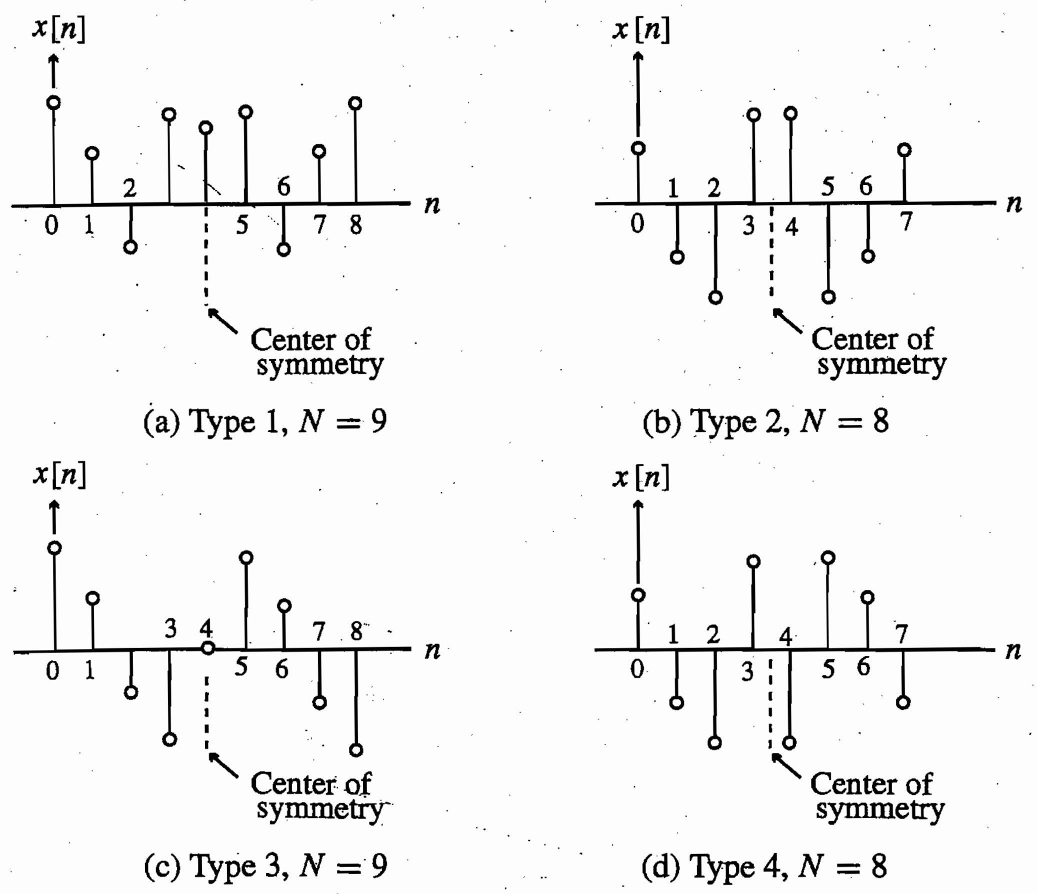


Figure 5.10: Illustration of the four types of geometric symmetry of a sequence.

Type I: Symmetric with Odd Length

$$X(e^{j\omega}) = e^{-j\frac{(N-1)\omega}{2}} \left\{ x\left[\frac{N-1}{2}\right] + 2\sum_{n=1}^{(\frac{N-1}{2})} x\left[\frac{N-1}{2}-n\right] \times \cos(\omega n) \right\}$$

$$X[k] = e^{-j\frac{(N-1)\pi k}{N}} \left\{ x\left[\frac{N-1}{2}\right] + 2\sum_{n=1}^{\frac{N-1}{2}} x\left[\frac{N-1}{2}-n\right] \times \cos\left(\frac{2\pi k n}{N}\right) \right\}$$

Type II: Symmetric Sequence with Even Length

$$X(e^{j\omega}) = e^{-j\frac{(N-1)\omega}{2}} \left\{ 2\sum_{n=1}^{N/2} x\left[\frac{N}{2}-n\right] \cos\left(\omega\left(n-\frac{1}{2}\right)\right) \right\}$$

$$X[k] = e^{-j\frac{(N-1)\pi k}{N}} \left\{ 2\sum_{n=1}^{N/2} x\left[\frac{N}{2}-n\right] \cos\left(\frac{\pi k(2n-1)}{N}\right) \right\}$$

Type III: Antisymmetric Sequence with Odd Length

$$X(e^{j\omega}) = j e^{-j\frac{(N-1)\omega}{2}} \left\{ 2\sum_{n=1}^{\frac{N-1}{2}} x\left[\frac{N-1}{2}-n\right] \sin(\omega n) \right\}$$

$$X[k] = j e^{-j\frac{(N-1)\pi k}{N}} \left\{ 2\sum_{n=1}^{\frac{N-1}{2}} x\left[\frac{N-1}{2}-n\right] \sin\left(\frac{2\pi k n}{N}\right) \right\}$$

Type IV: Antisymmetric Sequence with Even Length

$$X(e^{j\omega}) = j e^{-j\frac{(N-1)\omega}{2}} \left\{ 2 \sum_{n=1}^{N/2} x\left[\frac{N}{2}-n\right] \sin\left(\omega\left(n-\frac{1}{2}\right)\right)\right.$$

$$X[k] = j e^{-j\frac{(N-1)\pi k}{N}} \left\{ 2 \sum_{n=1}^{N/2} x\left[\frac{N}{2}-n\right] \sin\left(\frac{\pi k(2n-1)}{N}\right) \right\}$$

5.6 DFT Symmetry Relations

The DFT can be decomposed into the two parts such that

$$X[k] = X_{re}[k] + j X_{im}[k]$$

where

$$X_{re}[k] = \frac{1}{2}\left(X[k] + X^*[k]\right)$$

$$X_{im}[k] = \frac{1}{2}\left(X[k] - X^*[k]\right).$$

The complex-valued $x[n] = x_{re}[n] + j x_{im}[n]$ can be used to construct the DFT as

$$X[k] = \sum_{n=0}^{N-1} \left(x_{re}[n] + j x_{im}[n]\right)\left[\cos\left(\frac{2\pi k}{N}\right) - j \sin\left(\frac{2\pi k}{N}\right)\right]$$

$$= \sum_{n=0}^{N-1} \left(x_{re}[n] \cos\left(\frac{2\pi k}{N}\right) + x_{im}[n] \sin\left(\frac{2\pi k}{N}\right)\right)$$

$$+ j \sum_{n=0}^{N-1} \left(x_{im}[n] \cos\left(\frac{2\pi k}{N}\right) - x_{re}[n] \sin\left(\frac{2\pi k}{N}\right) \right).$$

Thus,

$$X_{re}[k] = \sum_{n=0}^{N-1} \left[x_{re}[n] \cos\left(\frac{2\pi k}{N}\right) + x_{im}[n] \sin\left(\frac{2\pi k}{N}\right) \right]$$

$$= \sum_{n=0}^{N-1} x_{cs}[n] \, e^{-j\frac{2\pi k}{N}}$$

$$X_{im}[k] = \sum_{n=0}^{N-1} \left[x_{im}[n] \cos\left(\frac{2\pi k}{N}\right) - x_{re}[n] \sin\left(\frac{2\pi k}{N}\right) \right]$$

$$= \sum_{n=0}^{N-1} x_{ca}[n] \, e^{-j\frac{2\pi k}{N}}$$

$$x_{cs}[n] \xleftrightarrow{DFT} X_{re}[k]$$

$$x_{ca}[n] \xleftrightarrow{DFT} X_{im}[k]$$

$$X^*[k] = \sum_{n=0}^{N-1} x^*[n] \, e^{j\frac{2\pi k n}{N}}$$

$$\Rightarrow X^*[\langle -k \rangle_N] = \sum_{n=0}^{N-1} x^*[n] \, e^{j\frac{2\pi \langle -k \rangle_N n}{N}}$$

For $k=0$, we have

$$X^*[0] = \sum_{n=0}^{N-1} x^*[n]$$

For $1 \le k \le N-1$, we have

$$X^*[\langle -k \rangle_N] = X^*[N-k] = \sum_{n=0}^{N-1} x^*[n] \cdot e^{j\frac{2\pi(N-k)n}{N}}$$

$$= \sum_{n=0}^{N-1} x^*[n] e^{-j\frac{2\pi k n}{N}}$$

Combine for $0 \le k \le N-1$, we have

$$X^*[\langle -k \rangle_N] = \sum_{n=0}^{N-1} x^*[n] e^{-j\frac{2\pi k n}{N}}$$

Similarly, we can derive

$$x^*[\langle -n \rangle_N] \xleftrightarrow{DFT} X^*[k]$$

$$x_{re}[n] \xleftrightarrow{DFT} X_{cs}[k]$$

$$j x_{im}[n] \xleftrightarrow{DFT} X_{ca}[k]$$

and $\quad x^*[n] \xleftrightarrow{DFT} X^*[\langle -k \rangle_N]$

※ Finite-Length Real Sequences

For a real sequence $x[n]$, $x_{im}[n] = 0$,

we have

$$X[k] = \sum_{n=0}^{N-1} x_{re}[n] \cos\left(\frac{2\pi k}{N}\right) - j \sum_{n=0}^{N-1} x_{re}[n] \sin\left(\frac{2\pi k}{N}\right.$$

Thus,

$$X_{re}[k] = \sum_{n=0}^{N-1} x_{re}[n] \cos\left(\frac{2\pi k}{N}\right)$$

$$X_{im}[k] = -\sum_{n=0}^{N-1} x_{re}[n] \sin\left(\frac{2\pi k}{N}\right)$$

According to Eq. (5.90), we have

$$x_{ev}[n] \overset{DFT}{\longleftrightarrow} X_{re}[k]$$

and $\quad x_{od}[n] \overset{DFT}{\longleftrightarrow} j\, X_{im}[k]$

where $\quad x[n] = x_{ev}[n] + x_{od}[n]$

$$X[\langle -k\rangle_N] = \sum_{n=0}^{N-1} x_{re}[n] \cos\left[\frac{2\pi(\langle -k\rangle_N)}{N}\right]$$

$$-j \sum_{n=0}^{N-1} x_{re}[n] \sin\left[\frac{2\pi(\langle -k\rangle_N)}{N}\right]$$

$$= \sum_{n=0}^{N-1} x_{re}[n] \cos\left[\frac{2\pi(N-k)}{N}\right]$$

$$-j \sum_{n=0}^{N-1} x_{re}[n] \sin\left[\frac{2\pi(N-k)}{N}\right]$$

$$= \sum_{n=0}^{N-1} x_{re}[n] \cos\left(\frac{2\pi k}{N}\right)$$

$$+ j \sum_{n=0}^{N-1} x_{re}[n] \sin\left(\frac{2\pi k}{N}\right) = X^*[k].$$

Consequently, $X[k] = X^*[\langle -k \rangle_N]$.

We arrive the following symmetry relations:

$$X_{re}[k] = X_{re}[\langle -k \rangle_N]$$

$$X_{im}[k] = -X_{im}[\langle -k \rangle_N]$$

$$|X[k]| = |X[\langle -k \rangle_N]|$$

$$\measuredangle X[k] = \measuredangle X[\langle -k \rangle_N]$$

5.7 Discrete Fourier Transform Theorems

Assume that two DFT pairs are given:

$$g[n] \xleftrightarrow{DFT} G[k]$$
$$h[n] \xleftrightarrow{DFT} H[k].$$

We have the following DFT theorems.

(i) Linearity Theorem:

Consider a sequence $x[n] = \alpha g[n] + \beta h[n]$.

Then

$$\alpha g[n] + \beta h[n] \xleftrightarrow{DFT} \alpha G[k] + \beta H[k].$$

(ii) Circular Time-shifting Theorem

$$g[\langle n-n_o\rangle_N] \xleftrightarrow{DFT} W_N^{kn_o} G[k]$$

(iii) Circular Frequency-shifting Theorem

$$W_N^{-k_o n} g[n] \xleftrightarrow{DFT} G[\langle k-k_o\rangle_N]$$

(iv) Duality Theorem

$$G[n] \xleftrightarrow{DFT} N g[\langle -k\rangle_N]$$

(v) Circular Convolution Theorem

$$g[n] \circledN h[n] \xleftrightarrow{DFT} G[k] H[k]$$

Example: Implementation of the Linear Convolution of Two Finite-Length Sequences Using the Circular Convolution

Consider two length-4 sequences:

$$g[n] \begin{cases} \neq 0, & 0 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

$$h[n] \begin{cases} \neq 0, & 0 \le n \le 3 \\ = 0, & \text{otherwise} \end{cases}$$

We need to compute the $y[n] = g[n] \circledast h[n]$ using the circular convolution. First, we need to zero-pad both sequences to have two length-7 sequences since the resulting sequence $y[n]$ should have length 7. Hence,

$$g_e[n] = \begin{cases} g[n], & 0 \le n \le 3 \\ 0, & 4 \le n \le 6. \end{cases}$$

$$h_e[n] = \begin{cases} h[n], & 0 \le n \le 3 \\ 0, & 4 \le n \le 6 \end{cases}$$

Thus, the linear convolution can be determined using the 7-point circular convolution

$$y[n] = \sum_{m=0}^{6} g_e[m]\, h_e[\langle n-m \rangle_7], \quad 0 \le n \le 6$$

$$= g_e[n] \, ⑥ \, h_e[n].$$

(vi) Modulation Theorem

$$g[n]h[n] \xleftrightarrow{DFT} \frac{1}{N} \sum_{\ell=0}^{N-1} G[\ell] H[\langle k-\ell \rangle_N]$$

(vii) Parseval's Theorem

The total energy of a length-N sequence $g[n]$ can be computed by summing the square of the absolute values of the DFT samples $G[k]$ such that

$$\sum_{n=0}^{N-1} |g[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |G[k]|^2$$

A more general Parseval's relation can be given by

$$\sum_{n=0}^{N-1} g[n]\, h^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} G[k]\, H^*[k]$$

Table 5.1: Symmetry properties of the DFT of a complex sequence.

Length-N Sequence	**N-point DFT**
$x[n] = x_{re}[n] + jx_{im}[n]$	$X[k] = X_{re}[k] + jX_{im}[k]$
$x^*[n]$	$X^*[\langle -k \rangle_N]$
$x^*[\langle -n \rangle_N]$	$X^*[k]$
$x_{re}[n]$	$X_{cs}[k] = \frac{1}{2}\{X[k] + X^*[\langle -k \rangle_N]\}$
jx_{im}	$X_{ca}[k] = \frac{1}{2}\{X[k] - X^*[\langle -k \rangle_N]\}$
$x_{cs}[n]$	$X_{re}[k]$
$x_{ca}[n]$	$jX_{im}[k]$

Note: $x_{cs}[n]$ and $x_{ca}[n]$ are the circular conjugate-symmetric and circular conjugate-antisymmetric parts of $x[n]$, respectively. Likewise, $X_{cs}[k]$ and $X_{ca}[k]$ are the circular conjugate-symmetric and circular conjugate-antisymmetric parts of $X[k]$, respectively.

Table 5.2: Symmetry properties of the DFT of a length-N real sequence.

Length-N Sequence	**N-point DFT**
$x[n] = x_{ev}[n] + x_{od}[n]$	$X[k] = X_{re}[k] + jX_{im}[k]$
$x_{ev}[n]$	$X_{re}[k]$
$x_{od}[n]$	$jX_{im}[k]$
Symmetry relations	$X[k] = X^*[\langle -k \rangle_N]$ $X_{re}[k] = X_{re}[\langle -k \rangle_N]$ $X_{im}[k] = -X_{im}[\langle -k \rangle_N]$ $\lvert X[k] \rvert = \lvert X[\langle -k \rangle_N] \rvert$ $\arg X[k] = -\arg X[\langle -k \rangle_N]$

Note: $x_{ev}[n]$ and $x_{od}[n]$ are the circular even and circular odd parts of $x[n]$, respectively.

Table 5.3: DFT Theorems.

Theorem	Length-N Sequence	N-point DFT
	$g[n]$ $h[n]$	$G[k]$ $H[k]$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G[k] + \beta H[k]$
Circular time-shifting	$g[\langle n - n_o \rangle_N]$	$W_N^{-kn_o} G[k]$
Circular frequency-shifting	$W_N^{-k_o} g[n]$	$G[\langle k - k_o \rangle_N]$
Duality	$G[n]$	$N g[\langle -k \rangle_N]$
N-point circular convolution	$\sum_{m=0}^{N-1} g[m] h[\langle n - m \rangle_N]$	$G[k]H[k]$
Modulation	$g[n]h[n]$	$\frac{1}{N} \sum_{m=0}^{N-1} G[m] H[\langle k - m \rangle_N]$
Parseval's relation	$\sum_{n=0}^{N-1} \lvert g[n] \rvert^2 = \frac{1}{N} \sum_{k=0}^{N-1} \lvert G[k] \rvert^2$	

5.9 Computation of the DFT of Real Sequences

5.9.1 Let $g[n]$ and $h[n]$ be the real sequences of length N each, with $G[k]$ and $H[k]$ are their corresponding N-point DFTs. We can apply the N-point complex sequence's DFT to carry out the two DFTs $G[k]$, $H[k]$ together.

Define $x[n] = g[n] + j\, h[n]$.

$$G[k] = \frac{1}{2}\left\{ X[k] + X^*[\langle -k \rangle_N] \right\}$$

$$H[k] = \frac{1}{2j}\left\{ X[k] - X^*[\langle -k \rangle_N] \right\}.$$

Example:

Compute the DFTs for $g[n] = \{\underset{\substack{\uparrow\\ n=0}}{1}, 2, 0, 1\}$ and $h[n] = \{\underset{\substack{\uparrow\\ n=0}}{2}, 2, 1, 1\}$ using one 4-point DFT.

Solution:

$$x[n] = g[n] + j\, h[n]$$
$$= \{\underset{\substack{\uparrow\\ n=0}}{1+j2}, 2+j2, j, 1+j\}$$

The DFT $X[k]$ =

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -j & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1+j2 \\ 2+j2 \\ j \\ 1+j \end{bmatrix}$$

$$= \begin{bmatrix} 4+j6 \\ 2 \\ -2 \\ j2 \end{bmatrix}$$

$$\{X^*[k]\} = \{\underset{\substack{\uparrow\\ k=0}}{4-j6},\ 2,\ -2,\ -j2\}$$

$$\{X^*[\langle -k\rangle_4]\} = \{\underset{\substack{\uparrow\\ k=0}}{4-j6},\ -j2,\ -2,\ 2\}$$

$$\therefore\ \{G[k]\} = \frac{1}{2}\{X[k] + X^*[\langle -k\rangle_4]\}$$

$$= \{\underset{\substack{\uparrow\\ k=0}}{4},\ 1-j,\ -2,\ 1+j\}$$

$$\{H[k]\} = \frac{1}{2j}\{X[k] - X^*[\langle -k\rangle_4]\}$$

$$= \{\underset{\substack{\uparrow\\ k=0}}{6},\ 1-j,\ 0,\ 1+j\}$$

5.9.2 2N-point DFT of a Real Sequence Using a Single N-point DFT

Let $v[n]$ be a real sequence of length $2N$ with $V[k]$ as the corresponding DFT.

We try to shorten the DFT window size to N to carry out $V[k]$.

Define two real sequences

$$g[n] = v[2n]$$

$$h[n] = v[2n+1], \qquad 0 \le n \le N-1$$

Thus,

$$V[k] = \sum_{n=0}^{2N-1} v[n]\, W_{2N}^{nk}$$

$$= \sum_{n=0}^{N-1} v[2n]\, W_{2N}^{2nk}$$

$$+ \sum_{n=0}^{N-1} v[2n+1]\, W_{2N}^{(2n+1)k}$$

$$W_{2N}^{2nk} = e^{-j\frac{4\pi nk}{2N}} = e^{-j\frac{2\pi nk}{N}} = W_N^{nk}$$

$$= \sum_{n=0}^{N-1} g[n]\, W_N^{nk} + \sum_{n=0}^{N-1} h[n]\, W_N^{nk}\, W_{2N}^{k}$$

$$= \sum_{n=0}^{N-1} g[n]\, W_N^{nk} + W_{2N}^{k} \sum_{n=0}^{N-1} h[n]\, W_N^{nk}$$

Therefore,

$$V[k] = G[\langle k \rangle_N] + W_{2N}^{k}\, H[\langle k \rangle_N], \qquad 0 \le k \le 2N-1$$

Example:

Determine the 8-point DFT $V[k]$ of the real sequence $v[n]$ given as

$$v[n] = \{\underset{\substack{\uparrow \\ n=0}}{1}, 2, 2, 2, 0, 1, 1, 1\}$$

$$g[n] = v[2n] = \{\underset{\substack{\uparrow \\ n=0}}{1}, 2, 0, 1\}$$

$$h[n] = v[2n+1] = \{\underset{\substack{\uparrow \\ n=0}}{2}, 2, 1, 1\}$$

$$\{G[k]\} = \text{DFT}\{g[n]\}$$

$$= \{\underset{\substack{\uparrow \\ k=0}}{4}, 1-j, -2, 1+j\}$$

$$\{H[k]\} = \text{DFT}\{h[n]\}$$

$$= \{\underset{\substack{\uparrow \\ k=0}}{6}, 1-j, 0, 1+j\}$$

$$V[k] = G[\langle k \rangle_4] + W_8^k H[\langle k \rangle_4]$$

$$= \{ \underset{\substack{\uparrow \\ k=0}}{10},\ 1-j2.41,\ -2,\ 1-j0.41,\ -2,\ 1+j0.4 \\ -2,\ 1+j2.41 \}$$

5.10 Linear Convolution Using DFT

5.10.1 Since the DFT can be efficiently implemented as the fast Fourier transform (FFT), we can take advantage of such computational efficien and carry out the long linear convolution using DFT.

Let $g[n]$ and $h[n]$ be finite-length sequences of lengths N and M, respectively. Our objective is to implement their linear convolution

$$y_L[n] = g[n] \circledast h[n],$$

using a circular convolution (DFT followed by a IDFT).

Accordingly, the length of the sequence $y_L[n]$ is $L = M+N-1$. Since the circular convolution has to involve two "equal-length" sequences and the resulting sequence will also possess the same length, we have to "zero-pad" the two sequences $g[n]$, $h[n]$ to form the two new sequences of length $(L+M-1)$ in preparation of the circular convolution such that

$$g_e[n] = \begin{cases} g[n], & 0 \le n \le N-1 \\ 0, & N \le n \le L-1, \end{cases}$$

$$h_e[n] = \begin{cases} h[n], & 0 \le n \le M-1, \\ 0, & M \le n \le L-1. \end{cases}$$

Obviously,

$$y_L[n] = y_C[n] = g_e[n] \textcircled{L} h_e[n].$$

5.10.2

There are applications where we need to perform a linear convolution of a finite-length sequence with a sequence of infinite length.

For example, the application may involve the processing of a speech signal by an FIR filt

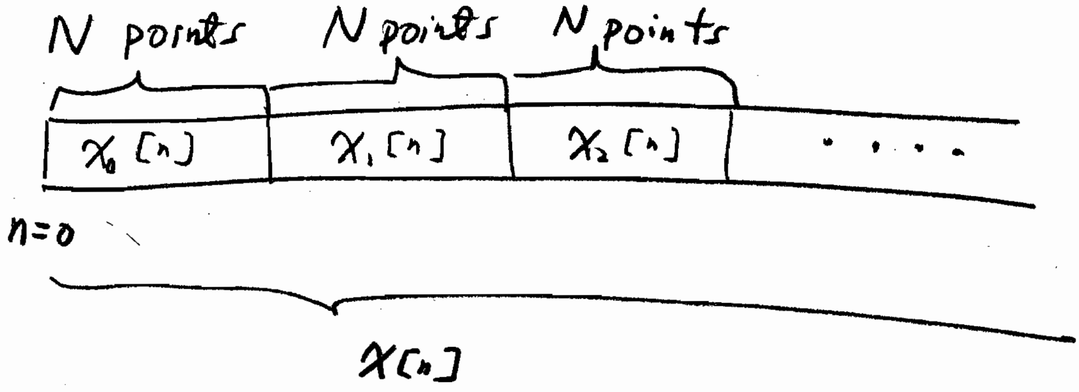
Thus, we cannot wait until the complete signal of infinite length to be acquired for carrying out this long convolution. Alternative we apply the "segment-and-convolution" technique such as "Overlap-Add" and "Overlap-Save" methods to produce real-time partial output (convolved) sequences.

Overlap-Add Method

In this method, we first segment $x[n]$ (the long sequence) into successive finite-length subsequences $x_m[n]$ of length N (the length of the short sequence $h[n]$) such that

$$x[n] = \sum_{m=0}^{\infty} x_m[n - mN].$$

where $x_m[n] = \begin{cases} x[n+mN], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$



Therefore,

$$y[n] = x[n] \circledast h[n]$$

$$= \sum_{l=0}^{M-1} h[l]\, x[n-l]$$

$$= \sum_{l=0}^{M-1} h[l] \left(\sum_{m=0}^{\infty} x_m[n-l-mN] \right)$$

$$= \sum_{m=0}^{\infty} \left(\sum_{l=0}^{M-1} h[l]\, x_m[n-l-mN] \right)$$

Since we can denote

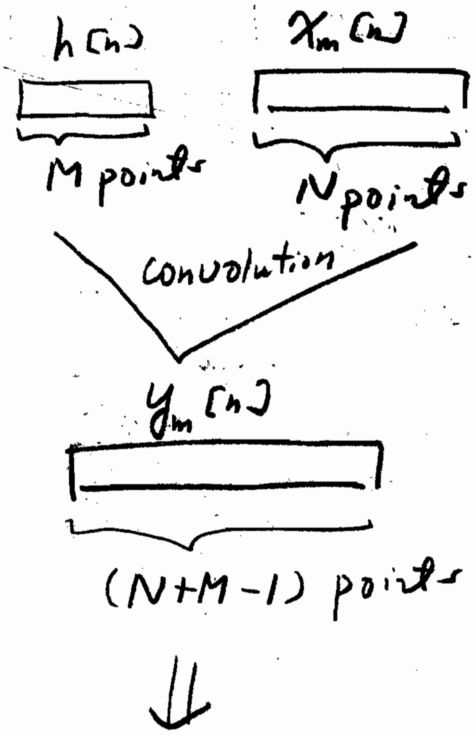
$$y_m[n] = \sum_{l=0}^{M-1} h[l]\, x_m[n-l]$$

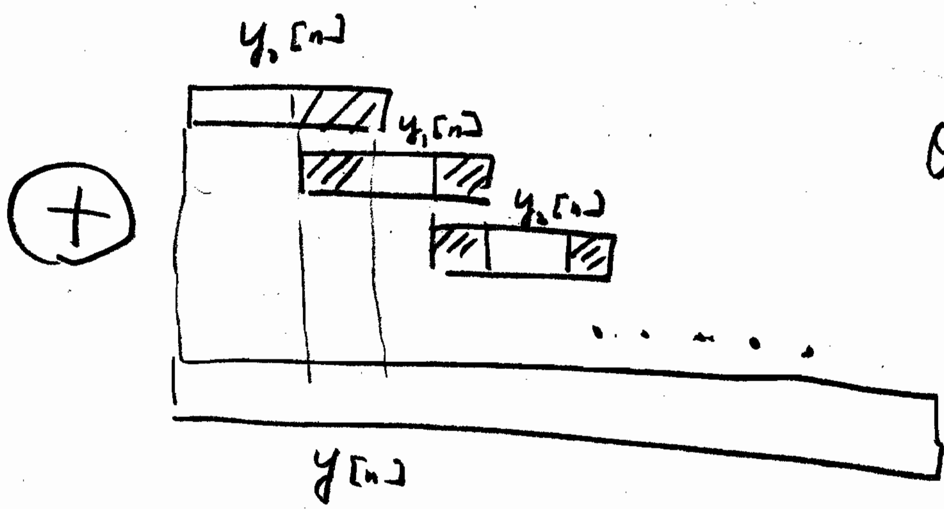
$$= h[n] \circledast x_m[n],$$

$$\sum_{l=0}^{M-1} h[l]\, x_m[n-l-mN] = y_m[n-mN].$$

$$\therefore \quad y[n] = \sum_{m=0}^{\infty} y_m[n-mN].$$

Note that $y_m[n]$ is the convolutional sequence of $h[n]$ and $x_m[n]$; hence it has the sequence length $(N+M-1)$. There are the overlaps between the successive outputs $y_m[n-mN]$ definitely, illustrated as below:





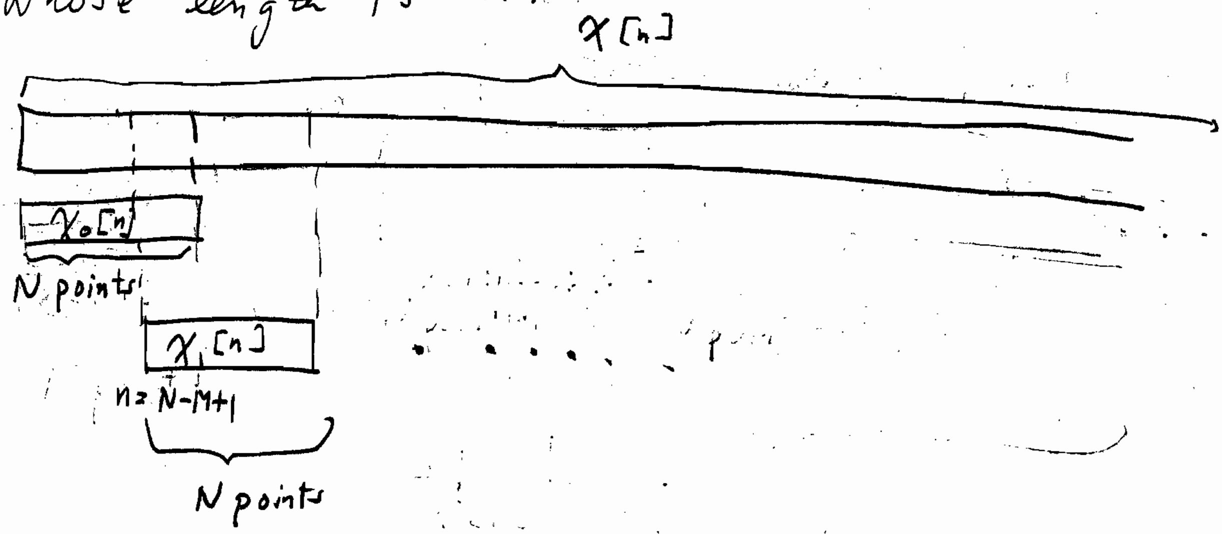
Overlap-Add

Overlap-Save Method

In this method, we segment $x[n]$ into subsequences $x_m[n]$ differently. The chosen subsequence length N should be larger than M ($M \le N$). The mth section of the long sequence $x[n]$ is defined as

$$x_m[n] = x[n + m(N-M+1)], \quad 0 \le n \le N-1,$$

whose length is N.



Then zero-pad $h[n]$ (length M) to form $h_e[n]$ such that

$$h_e[n] = \begin{cases} h[n], & 0 \le n \le M-1 \\ 0, & M \le n \le N-1 \end{cases},$$

whose length is N

The N-point circular convolution can thus be carried out as

$$w_m[n] = h_e[n] \; \textcircled{N} \; x_m[n].$$

Once $w_m[n]$ is obtained, $\forall m$, we can save portion of $w_m[n]$ to construct the m^{th} section of convolutional result $y_m[n]$ that is

$$y_m[n] = \begin{cases} 0, & 0 \le n \le M-2 \\ w_m[n], & M-1 \le n \le N-1 \end{cases}$$

and the complete convolution can be expressed as

$$y_L[n+m(N-M+1)] = y_m[n], \quad M-1 \le n \le N-1$$