Chapter 4. Digital Processing of Continuous-Time Signals.

Even though this book is concerned primarily with the processing of discrete-time signals, most signals we encounter in the real word are continuous in time, such as speech, music and images. The interface circuit performing the conversion of a continuous-time signal into a digital form is called the analog-to-digital (A/D) converter. Likewise, the reverse operation of converting a digital signal into a continuous-time signal is implemented by the interface circuit called the digital-to-analog (D/A) converter.

4.2 Sampling of Continuous-Time Signals

4.2.1 Effect of Sampling in the Frequency Domain

Let $g_a(t)$ be a continuous-time signal that is sampled uniformly at $t = nT$, generating a sequence $g[n]$ where

$$g[n] = g_a(nT), \qquad -\infty < n < \infty$$

with T being the sampling period and $F_T = \frac{1}{T}$ being the sampling frequency.

The frequency-domain representation of $g_a(t)$ is given by its continuous-time Fourier transform (CTFT):

$$G_a(j\Omega) = \int_{-\infty}^{\infty} g_a(t)\, e^{-j\Omega t}\, dt$$

On the other hand, the frequency-representation of $g[n]$ is given by the discrete-time Fourier transform:

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]\, e^{-j\omega n}.$$

Now we would like to establish the relationship between the CTFT $G_a(j\Omega)$ and the DTFT $G(e^{j\omega})$. Since the discrete-time sequence $g[n]$ is the sampled version of the continuous-time signal $g_a(t)$, we introduce the sampling operation mathematically as a multiplication of $g_a(t)$ by a periodic impulse train $p(t)$:

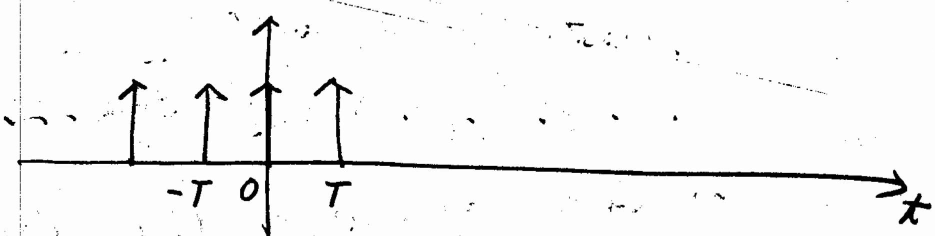
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

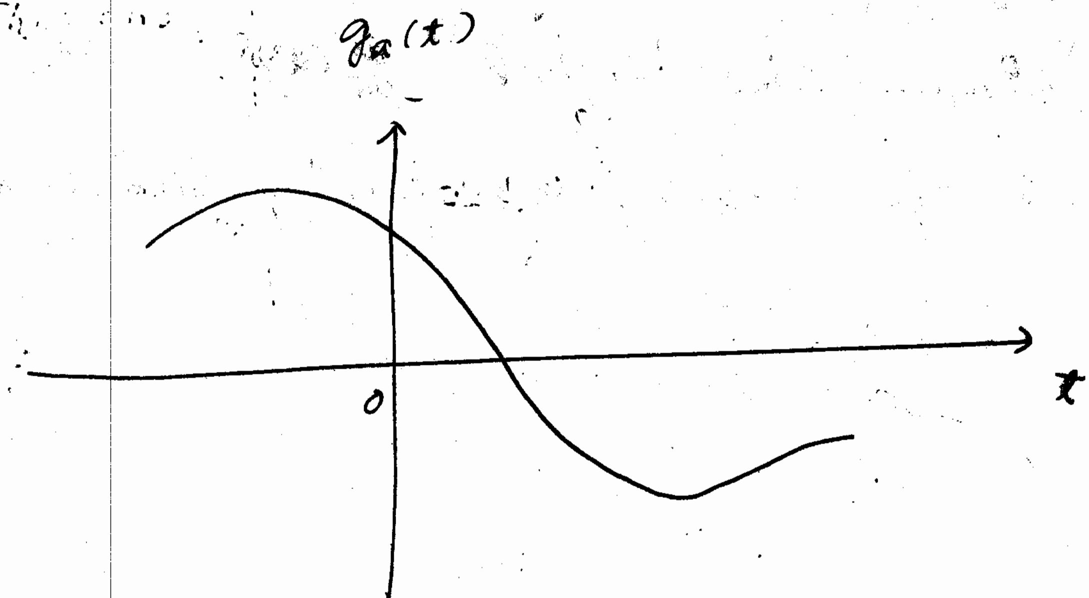
Such a multiplication yields an impulse train $g_p(t)$:

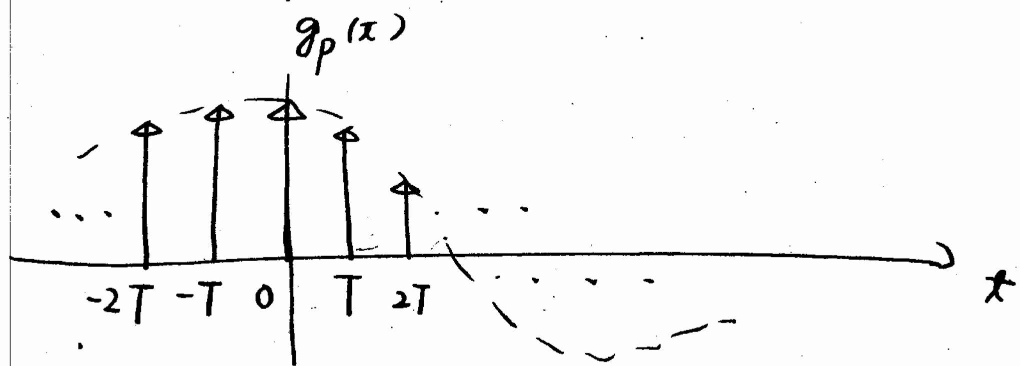
$$g_p(t) = g_a(t)\,p(t)$$

$$= \sum_{n=-\infty}^{\infty} g_a(nT)\,\delta(t-nT).$$

$$\begin{tikzcd} g_a(t) \arrow[r] & \otimes \arrow[r] & g_p(t) \\ & p(t) \arrow[u] & \end{tikzcd}$$







The CTFT $G_p(j\Omega)$ of $g_p(t)$ is given by

$$G_p(j\Omega) \doteq \sum_{n=-\infty}^{\infty} g_a(nT) \int_{-\infty}^{\infty} \delta(t-nT) e^{-j\Omega t} dt$$

$$= \sum_{n=-\infty}^{\infty} g_a(nT) e^{-j\Omega nT}$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} G_a\left(j\left(\Omega + \frac{2\pi k}{T}\right)\right)$$

Proof:

According to the Poisson's sum formula,

$$\sum_{n=-\infty}^{\infty} \phi(t+nT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \Phi\left(j\frac{2\pi k}{T}\right) e^{j\frac{2\pi k t}{T}},$$

where $\Phi(j\Omega)$ is the CTFT of $\phi(t)$.

Let $\phi(t) = g_a(t) e^{-j\Omega t}$;

$$\sum_{n=-\infty}^{\infty} g_a(t+nT) e^{-j\Omega t} e^{-j\Omega nT}$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} G_a\left(j\left(\frac{2\pi k}{T} + \Omega\right)\right) e^{j\frac{2\pi k t}{T}}$$

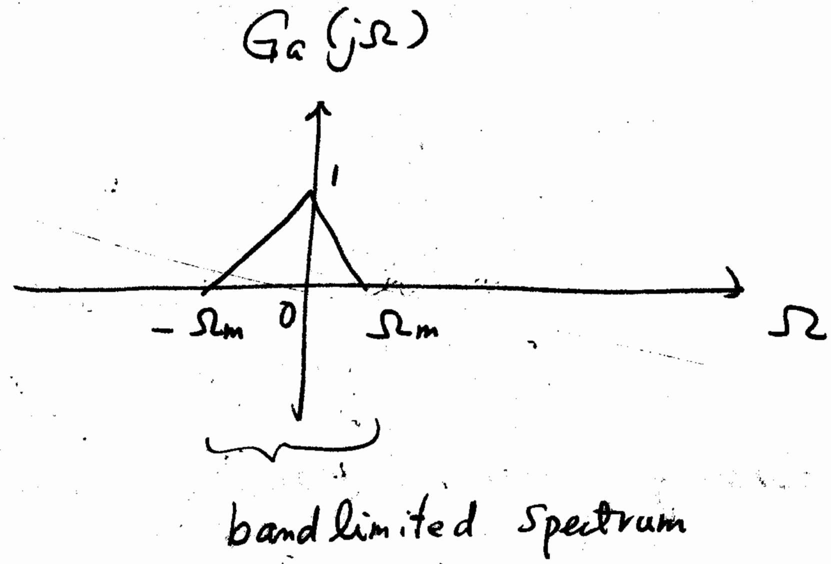
$t = 0 \Rightarrow$

$$\sum_{n=-\infty}^{\infty} g_a(nT) e^{-j\Omega nT} = \frac{1}{T} \sum_{k=-\infty}^{\infty} G_a\left(j\left(\Omega + \frac{2\pi k}{T}\right)\right)$$

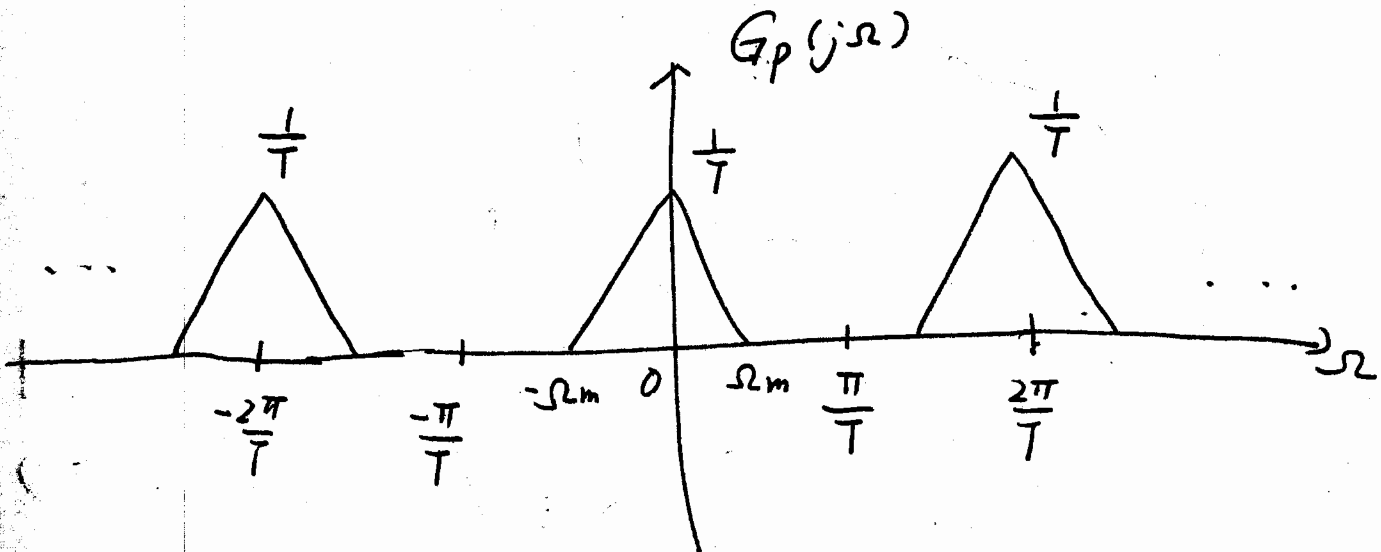
Q.E.D.

Illustration of the Frequency-domain Effects of time-domain Sampling:

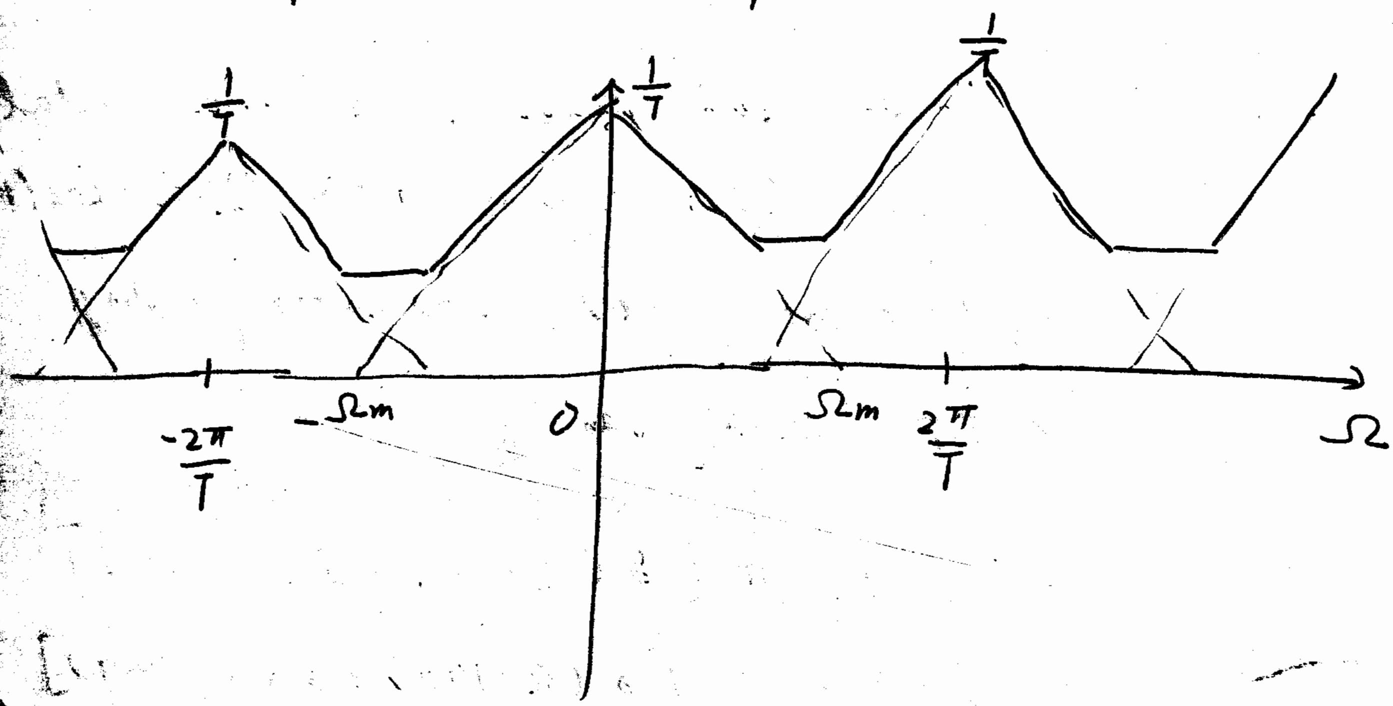
(a) Spectrum of Original continuous-time signal $g_a(t)$



(b) The spectrum of the sampled signal $g_p(t)$ when $\frac{2\pi}{T} > 2\Omega_m$



(c) The spectrum of the sampled signal $g_p(t)$ when $\frac{2\pi}{T} < 2\Omega_m$ (aliasing)



As illustrated by the figures above, when $\frac{2\pi}{T} < 2\Omega_m$, the aliasing or the spectral distortion is observed. The frequency $\frac{\pi}{T}$ is often referred to as the folding frequency or Nyquist frequency.

※ Sampling Theorem

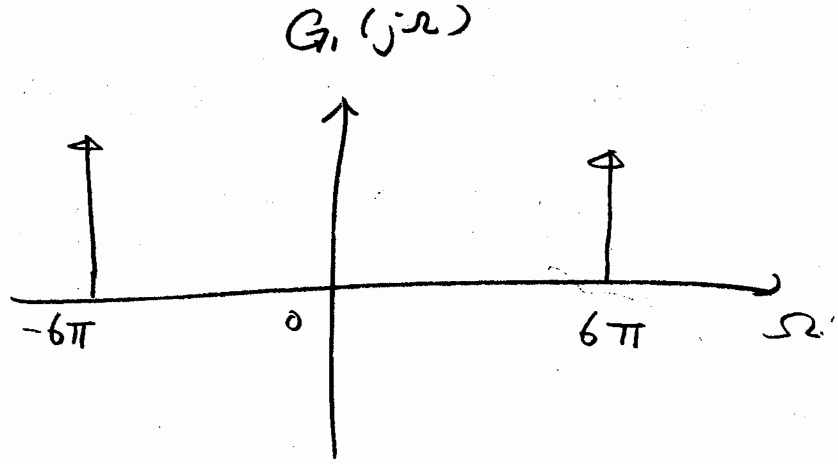
Let $g_a(t)$ be a bandlimited signal with $G_a(j\Omega) = 0$, for $|\Omega| > \Omega_m$. Then $g_a(t)$ is uniquely determined by its samples $g_a(nT)$, $-\infty < n < \infty$, if $\frac{2\pi}{T} \geq 2\Omega_m$, or $\frac{\pi}{T} \geq \Omega_m$.

Example: Consider the three pure sinusoidal signals: $g_1(t) = \cos(6\pi t)$, $g_2(t) = \cos(14\pi t)$, $g_3(t) = \cos(26\pi t)$. The corresponding CTFTs are given by

$$G_1(j\Omega) = \pi\left[\delta(\Omega - 6\pi) + \delta(\Omega + 6\pi)\right]$$

$$G_2(j\Omega) = \pi\left[\delta(\Omega - 14\pi) + \delta(\Omega + 14\pi)\right]$$

$$G_3(j\Omega) = \pi\left[\delta(\Omega - 26\pi) + \delta(\Omega + 26\pi)\right]$$



The Nyquist frequency is $\frac{\pi}{T} = 6\pi$

$\Rightarrow T \leq \frac{1}{6}$

Similarly, the Nyquist frequency for $g_2(t)$ is $\frac{\pi}{T} = 14\pi$ $(T \le \frac{1}{14})$ and that for $g_3(t)$ is $\frac{\pi}{T} = 26\pi$ $(T \le \frac{1}{26})$.

We now establish the relation between the discrete-time Fourier transform $G(e^{j\omega})$ of the sequence $g[n]$ and the continuous-time Fourier transform $G_a(j\Omega)$ of the analog signal $g_a(t)$. Compare Eqs. (4.3) and (4.6) with (4.1), and we have

$$G(e^{j\omega}) = G_p(j\Omega)\Big|_{\Omega=\frac{\omega}{T}}$$

$$\Rightarrow \quad G_p(j\Omega) = G(e^{j\omega})\Big|_{\omega=\Omega T}.$$

According to Eq. (4.10), we have

$$G(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\Omega - j\frac{2\pi k}{T}\right)\Bigg|_{\Omega=\frac{\omega}{T}}$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right)$$

4.2.2 Recovery of the Analog Signal

Now the impulse response $h_r(t)$ of the ideal lowpass filter is obtained by taking the inverse CTFT of the corresponding frequency response $H_r(j\Omega)$:

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \le \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}.$$

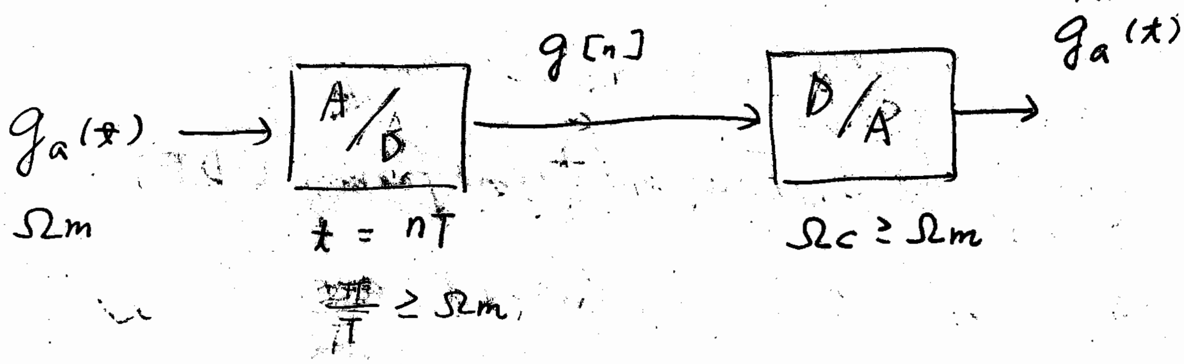
and is given by

$$h_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$= \frac{T}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega t}\, d\Omega$$

$$= \frac{\sin(\Omega_c t)}{\frac{\pi t}{T}}, \quad -\infty < t < \infty$$

The digita-to-analog converter (D/A) operates like a lowpass filter, which will have the cut-off frequency Ω_c larger than the highest frequency Ω_m for the original signal $g_a(t)$.



$$g[n] = g_a(t)\Big|_{t=nT}, \quad -\infty < n < \infty$$

$$\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g[n]\, h_r(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} g[n] \frac{\sin\left[\frac{\pi(t-nT)}{T}\right]}{\frac{\pi(t-nT)}{T}}$$

It is noted that $\hat{g}_a(t) = g_a(t)$ only when $t = nT$, $-\infty < n < \infty$. $\hat{g}_a(t)$ may not be equivalent to $g_a(t)$ away from those time instants $t = nT$, $-\infty < n < \infty$.