Chapter 3 Discrete-time Fourier Transform

3.2 Discrete-time Fourier Transform

The discrete-time Fourier transform (DTFT) of a discrete-time sequence $x[n]$ is a representation of the sequence in terms of the complex exponential sequence $\{e^{j\omega n}\}$, where ω is the real frequency variable. In text, the discrete-time Fourier transform is just briefed as Fourier Transform (FT).

3.2.1 Definition

The discrete-time Fourier transform (FT) of a sequence $x[n]$ is defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}.$$

Example: The FT $\Delta(e^{j\omega})$ of the unit sample sequence $\delta[n]$ is given by

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\omega n} = 1$$

Example: Consider the causal sequence

$$x[n] = \alpha^n \mu[n], \quad |\alpha| < 1$$

The FT $X(e^{j\omega})$ can be derived as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n$$

$$= \frac{1}{1 - \alpha e^{-j\omega}},$$

as $|\alpha e^{-j\omega}| = |\alpha| < 1$

As can be seen from the definition, the discrete-time Fourier transform $X(e^{j\omega})$ of any sequence $x[n]$ is a continuous function of ω and periodic with a period 2π.

$$X(e^{j(\omega+2\pi k)}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega+2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \underbrace{e^{-j2\pi kn}}_{1}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = X(e^{j\omega})$$

for all integers k.

$$\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \int_{-\pi}^{\pi} \left(\sum_{m=-\infty}^{\infty} x[m]\, e^{-j\omega m} \right) e^{j\omega n}\, d\omega$$

$$= \sum_{m=-\infty}^{\infty} x[m] \underbrace{\int_{-\pi}^{\pi} e^{j\omega(n-m)}\, d\omega}_{2\pi\,\delta[n-m]}$$

from Eq. (3.18)

$$= \sum_{m=-\infty}^{\infty} 2\pi\, x[m]\, \delta[n-m] = 2\pi\, x[n]$$

Thus,

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega,$$

which is called the inverse discrete-time Fourier transform. We also denote them as

$$\mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$\mathcal{F}^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

or

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

3.2.2. Basic FT properties

In general, $X(e^{j\omega})$ is a complex function of real-valued variable ω,

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + j\, X_{im}(e^{j\omega})$$

where

$$X_{re}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{j\omega})\}$$

and $$X_{im}(e^{j\omega}) = \frac{1}{2j}\{X(e^{j\omega}) - X^*(e^{j\omega})\}$$

In the polar form,

$$X(e^{j\omega}) = |X(e^{j\omega})| e^{j\theta(\omega)},$$

where $\theta(\omega) = \arg\{X(e^{j\omega})\}$.

Then, we can obtain

$$X_{re}(e^{j\omega}) = |X(e^{j\omega})| \cos(\theta(\omega))$$

$$X_{im}(e^{j\omega}) = |X(e^{j\omega})| \sin(\theta(\omega))$$

$$|X(e^{j\omega})|^2 = X(e^{j\omega}) X^*(e^{j\omega})$$

$$= X_{re}^2(e^{j\omega}) + X_{im}^2(e^{j\omega}),$$

$$\tan(\theta(\omega)) = \frac{X_{im}(e^{j\omega})}{X_{re}(e^{j\omega})}$$

We call $X(e^{j\omega})$ the Fourier spectrum and $|X(e^{j\omega})|$, $\theta(\omega)$ the magnitude spectrum and the phase spectrum, respectively. We restrict the range of $\theta(\omega)$ in the primary range,

$$-\pi \leq \theta(\omega) < \pi.$$

3.2.3 Symmetry Relations

$$\mathcal{F}\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[\underbrace{-n}_{m}]\, e^{-j\omega n} = \sum_{m=\infty}^{-\infty} x[m]\, e^{j\omega m}$$

$m = -n$

$$= X(e^{-j\omega})$$

$$\therefore\ x[-n] \xleftrightarrow{\mathcal{F}} X(e^{-j\omega})$$

$$\mathcal{F}\{x^*[n]\} = \sum_{n=-\infty}^{\infty} x^*[n]\, e^{-j\omega n}$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n]\, e^{j\omega n}\right)^*$$

$$= X^*(e^{-j\omega})$$

$$\therefore\ x^*[n] \xleftrightarrow{\mathcal{F}} X^*(e^{-j\omega}).$$

$$\Rightarrow\ x^*[-n] \xleftrightarrow{\mathcal{F}} X^*(e^{j\omega}).$$

$$X_{re}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(x_{re}[n]\cos(\omega n) - x_{im}[n]\sin(\omega n)\right)$$

$$X_{im}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(x_{im}[n]\cos(\omega n) + x_{re}[n]\sin(\omega n)\right)$$

where $x[n] = x_{re}[n] + j\, x_{im}[n]$

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + X_{im}(e^{j\omega})$$

A complex-valued FT $X(e^{j\omega})$ can be expressed as

$$X(e^{j\omega}) = \underbrace{X_{cs}(e^{j\omega})}_{\substack{\text{conjugate-} \\ \text{symmetric} \\ \text{part}}} + \underbrace{X_{ca}(e^{j\omega})}_{\substack{\text{conjugate-} \\ \text{antisymmetric} \\ \text{part}}}$$

where $X_{cs}(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega}) + X^*(e^{-j\omega})\right]$

$$X_{ca}(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega}) - X^*(e^{-j\omega})\right]$$

$$X_{cs}(e^{j\omega}) = \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} x_{re}[n]\,e^{-j\omega n} + \sum_{n=-\infty}^{\infty} x_{im}[n]\,e^{-j\omega n}\right)$$

$$+ \frac{1}{2}\left(\sum_{m=-\infty}^{\infty} x_{re}[n]\,e^{-j\omega n} - j\sum_{n=-\infty}^{\infty} x_{im}[n]\,e^{-j\omega n}\right)$$

$$= \sum_{n=-\infty}^{\infty} x_{re}[n]\,e^{-j\omega n} = \mathcal{F}\{x_{re}[n]\}$$

$$\therefore \quad x_{re}[n] \overset{\mathcal{F}}{\longleftrightarrow} X_{cs}(e^{j\omega})$$

Similarly,

$$j\,x_{im}[n] \overset{\mathcal{F}}{\longleftrightarrow} X_{ca}(e^{j\omega}).$$

According to ...

$$\mathcal{F}\{x_{cs}[n]\} = \frac{1}{2}\left(\mathcal{F}\{x[n]\} + \mathcal{F}\{x^*[-n]\}\right)$$

$$= \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{j\omega})\}$$

$$= X_{re}(e^{j\omega})$$

$$\therefore \quad x_{cs}[n] \overset{\mathcal{F}}{\longleftrightarrow} X_{re}(e^{j\omega})$$

Similarly,

$$x_{ca}[n] \overset{\mathcal{F}}{\longleftrightarrow} j X_{im}(e^{j\omega})$$

Table 3.1: Symmetry relations of the discrete-time Fourier transform of a complex sequence.

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_{cs}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\mathrm{Im}\{x[n]\}$	$X_{ca}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{cs}[n]$	$X_{re}(e^{j\omega})$
$x_{ca}[n]$	$jX_{im}(e^{j\omega})$

Note: $X_{cs}(e^{j\omega})$ and $X_{ca}(e^{j\omega})$ are the conjugate-symmetric and conjugate-antisymmetric parts of $X(e^{j\omega})$, respectively. Likewise, $x_{cs}[n]$ and $x_{ca}[n]$ are the conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$, respectively.

3.2.4 Convergence Condition

Similar to the z-transform, if $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$,

(absolutely summable), then

$$|X(e^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right| \le \sum_{n=-\infty}^{\infty} |x[n]| |e^{-j\omega n}|$$

$$\le \sum_{n=-\infty}^{\infty} |x[n]| < \infty,$$

for all ω values guaranteeing the existence of $X(e^{j\omega})$.

Eq. (3.43) is a sufficient condition for the existence of the FT $X(e^{j\omega})$ of the sequence $x[n]$. That is,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

$$\Rightarrow \lim_{K\to\infty} X_K(e^{j\omega}) = \lim_{K\to\infty} \sum_{n=-K}^{K} x[n] e^{-j\omega n}$$

$$= X(e^{j\omega})$$

(uniform convergence).

Table 3.2: Symmetry relations of the discrete-time Fourier transform of a real sequence.

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$
$x_{ev}[n]$	$X_{re}(e^{j\omega})$
$x_{od}[n]$	$jX_{im}(e^{j\omega})$
Symmetry relations	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_{re}(e^{j\omega}) = X_{re}(e^{-j\omega})$ $X_{im}(e^{j\omega}) = -X_{im}(e^{-j\omega})$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ $\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Note: $x_{ev}[n]$ and $x_{od}[n]$ denote the even and odd parts of $x[n]$, respectively.

Table 3.3: Commonly used discrete-time Fourier transform pairs.

Sequence	Discrete-Time Fourier Transform
$\delta[n]$	1
$1, \; (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$\mu[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$e^{j\omega_o n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$
$\alpha^n \mu[n], \quad (\lvert\alpha\rvert < 1)$	$\frac{1}{1 - \alpha e^{-j\omega}}$
$(n+1)\alpha^n \mu[n], \; (\lvert\alpha\rvert < 1)$	$\frac{1}{(1-\alpha e^{-j\omega})^2}$
$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \; (-\infty < n < \infty)$	$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$

Example: Consider the FT

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

The inverse DTFT of $H_{LP}(e^{j\omega})$ is given by

$$h_{LP}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn} \right)$$

$$= \frac{\sin(\omega_c n)}{\pi n}, \quad -\infty < n < \infty, \quad n \neq 0$$

For $n = 0$,

$$h_{LP}[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\omega}) d\omega$$

$$= \frac{\omega_c}{\pi}$$

Since $\lim_{n \to 0} \frac{\sin(\omega_c n)}{\pi n} = \frac{\omega_c}{\pi}$,

$$\therefore \quad h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n}, \quad -\infty < n < \infty$$

Example:

Consider the complex exponential sequence

$$x[n] = e^{j\omega_0 n}, \quad \omega_0 \text{ is real.}$$

The corresponding FT is given by

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_0 + 2\pi k).$$

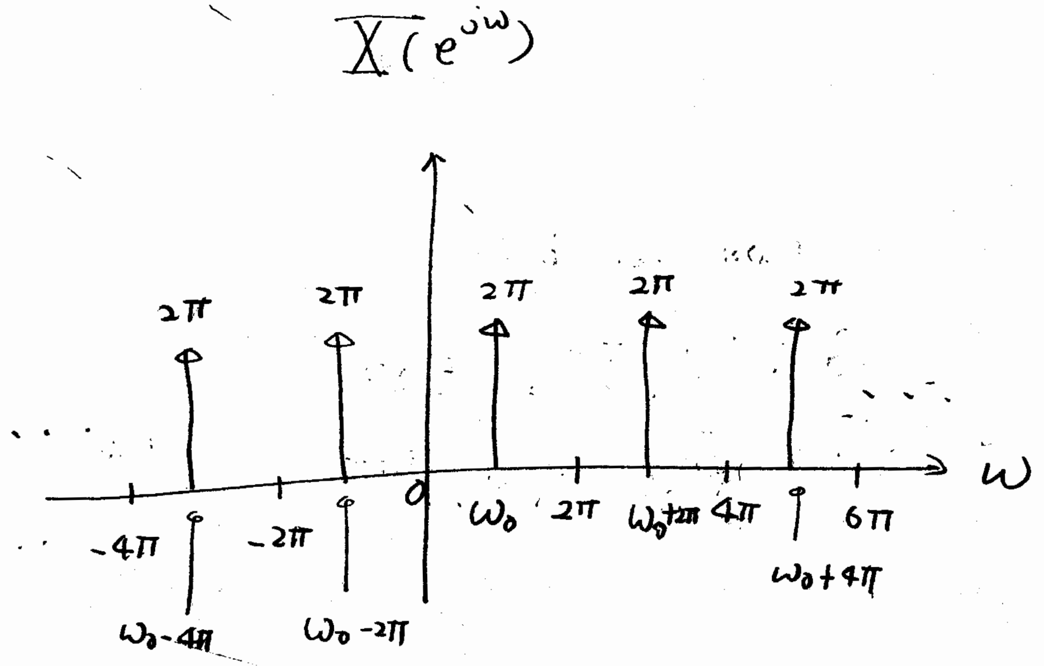
It is very diffiult to derive $X(e^{j\omega})$ from $x[n]$. Rather, we test if the above-expressed $X(e^{j\omega})$ would induce the inverse DTFT $x[n] = e^{j\omega_0 n}$.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_0 + 2\pi k)\, e^{j\omega n}\, d\omega$$

$$= e^{j(\omega_0 - 2\pi m)n} \quad \text{for } -\pi \le \omega_0 - 2\pi m < \pi$$

$$= e^{j\omega_0 n}$$

Thus, $$e^{j\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_0 + 2\pi k)$$



3.3 Discrete-time Fourier Transform Theorems

We assume $g[n] \xleftrightarrow{\mathcal{F}} G(e^{j\omega})$

$h[n] \xleftrightarrow{\mathcal{F}} H(e^{j\omega})$

Table 3.4: Discrete-time Fourier transform theorems.

Theorem	Sequence	Discrete-Time Fourier Transform
	$g[n]$	$G(e^{j\omega})$
	$h[n]$	$H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time-reversal	$g[-n]$	$G(e^{-j\omega})$
Time-shifting	$g[n - n_o]$	$e^{-j\omega n_o} G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_o n} g[n]$	$G\left(e^{j(\omega-\omega_o)}\right)$
Differentiation-in-frequency	$ng[n]$	$j\frac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] \circledast h[n]$	$G(e^{j\omega})H(e^{j\omega})$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\,d\theta$
Parseval's Relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})\,d\omega$	

Example:

Determine the FT $Y(e^{j\omega})$ of the sequence

$$y[n] = \begin{cases} \alpha^n, & 0 \le n \le M-1, \\ 0, & \text{otherwise} \end{cases}, \quad |\alpha| < 1$$

Solution:

We first rewrite $y[n]$ as

$$y[n] = \alpha^n \mu[n] - \alpha^n \mu[n-M] = \alpha^n \mu[n] - \alpha^M \alpha^{n-M} \mu[n-M]$$

According to the time-shifting property,

$$Y(e^{j\omega}) = \mathcal{F}\{\alpha^n \mu[n]\} - \alpha^M \mathcal{F}\{\alpha^{n-M} \mu[n-M]\}$$

$$= \frac{1}{1-\alpha e^{-j\omega}} - \alpha^M \frac{e^{-j\omega M}}{1-\alpha e^{-j\omega}}$$

$$= \frac{1-\alpha^M e^{-j\omega M}}{1-\alpha e^{-j\omega}}$$

Example:

Determine the FT $V(e^{j\omega})$ of the sequence $v[n]$ given by

$$d_0 v[n] + d_1 v[n-1] = P_0 \delta[n] + P_1 \delta[n-1],$$

$$|d_1/d_0| < 1.$$

Taking the FT for each term, we have

$$d_0 \mathcal{F}\{v[n]\} + d_1 \mathcal{F}\{v[n-1]\} = P_0 \mathcal{F}\{\delta[n]\} + P_1 \mathcal{F}\{\delta[n-1]\}$$

$$\Rightarrow d_0 V(e^{j\omega}) + d_1 e^{-j\omega} V(e^{j\omega}) = P_0 + P_1 e^{-j\omega}$$

$$\therefore V(e^{j\omega}) = \frac{P_0 + P_1 e^{-j\omega}}{d_0 + d_1 e^{-j\omega}}$$

Example: Consider the sequence

$$y[n] = (-1)^n \alpha^n \mu[n], \quad |\alpha| < 1$$

The sequence $y[n]$ can be expressed as $y[n] = e^{j\pi n} x[n]$, where $x[n]$ is the complex exponential sequence of $\alpha^n \mu[n]$ with FT $\frac{1}{1-\alpha e^{-j\omega}}$. The FT of $y[n]$ is given by

$$Y(e^{j\omega}) = X(e^{j(\omega-\pi)}) = \frac{1}{1-\alpha e^{-j(\omega-\pi)}}$$

$$= \frac{1}{1+\alpha e^{-j\omega}}$$

Example:

Determine the Fourier transform of the sequence

$$y[n] = (n+1)\alpha^n \mu[n], \quad |\alpha| < 1$$

Let $x[n] = \alpha^n \mu[n]$, $|\alpha| < 1$. We can therefore write

$$y[n] = n\,x[n] + x[n]$$

Since
$$X(e^{j\omega}) = \mathcal{F}\{x[n]\} = \frac{1}{1-\alpha e^{-j\omega}},$$

$$\mathcal{F}\{n\,x[n]\} = j\frac{dX(e^{j\omega})}{d\omega}$$

$$= j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$= \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

Therefore,

$$Y(e^{jw}) = \mathcal{F}\{y[n]\}$$

$$= \frac{\alpha e^{-jw}}{(1-\alpha e^{-jw})^2} + \frac{1}{1-\alpha e^{-jw}} = \frac{1}{(1-\alpha e^{-jw})^2}$$

3.8 The Frequency Response of an LTI Discrete-Time System

3.8.1
$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

specifies the output $y[n]$ of an LTI system with an impulse response $h[n]$ in response to $x[n]$. If the input $x[n]$ is

$$x[n] = e^{jwn}, \quad -\infty < n < \infty$$

then
$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, e^{jw(n-k)} = \left(\sum_{k=-\infty}^{\infty} h[k] e^{-jwk}\right) e^{jwn},$$

or
$$y[n] = H(e^{jw})\, e^{jwn}$$

where $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$.

The FT of $h[n]$, $H(e^{j\omega})$, is called the frequency response of such an LTI discrete-time system. $H(e^{j\omega})$ is a complex function of ω with a period 2π and can be expressed as

$$H(e^{j\omega}) = \underbrace{|H(e^{j\omega})|}_{\text{magnitude response}} \underbrace{e^{j\theta(\omega)}}_{\text{phase response}}.$$

3.8.2.

If $Y(e^{j\omega})$ and $X(e^{j\omega})$ denote the FTs of the output and input sequences, $y[n]$ and $x[n]$ respectively, then the frequency-domain representation of the LTI discrete-time system can be written as

$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega}),$$

Or the frequency response of this LTI system is

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

Example: The input sequence for an LTI system is $x[n] = \alpha^n \mu[n]$, $|\alpha| < 1$. The system is characterized as the impulse response $h[n] = \beta^n \mu[n]$, $|\beta| < 1$. Calculate $y[n]$ from the FTs.

Solution:

$$X(e^{j\omega}) = \mathcal{F}\{x[n]\}$$

$$= \frac{1}{1 - \alpha e^{-j\omega}}$$

$$H(e^{j\omega}) = \mathcal{F}\{h[n]\}$$

$$= \frac{1}{1 - \beta e^{-j\omega}}$$

$$Y(e^{j\omega}) = \mathcal{F}\{y[n]\} = \mathcal{F}\{x[n] \circledast h[n]\}$$

$$= H(e^{j\omega})\, X(e^{j\omega})$$

$$= \frac{1}{(1-\alpha e^{-j\omega})\,(1-\beta e^{-j\omega})}$$

$$= \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}$$

$$= \frac{(A+B) - (A\beta + B\alpha)\, e^{-j\omega}}{(1-\alpha e^{-j\omega})\,(1-\beta e^{-j\omega})}$$

$$\Rightarrow \quad A+B = 1, \quad A\beta + B\alpha = 0$$

$$\Rightarrow \quad A = \frac{\alpha}{\alpha-\beta}, \quad B = -\frac{\beta}{\alpha-\beta}$$

$$\Rightarrow \quad Y(e^{j\omega}) = \frac{\frac{\alpha}{\alpha-\beta}}{1-\alpha e^{-j\omega}} - \frac{\frac{\beta}{\alpha-\beta}}{1-\beta e^{-j\omega}}$$

$$\Rightarrow \quad y[n] = \mathcal{F}^{-1}\{Y(e^{j\omega})\}$$

$$= \frac{\alpha}{\alpha-\beta}\,\alpha^n \mu[n] - \frac{\beta}{\alpha-\beta}\,\beta^n \mu[n]$$

$$= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha-\beta}\,\mu[n]$$

3.8.3 Frequency Response of LTI Discrete-Time Systems

* Frequency Response of LTI FIR systems

The LTI FIR system can be characterized as

$$y[n] = \sum_{k=N_1}^{N_2} h[k]\, x[n-k], \quad N_1 < N_2.$$

Applying the DTFT for each term, we have

$$Y(e^{j\omega}) = \sum_{k=N_1}^{N_2} h[k]\, e^{-j\omega k} X(e^{j\omega})$$

$$= H(e^{j\omega}) X(e^{j\omega})$$

Thus, the frequency response $H(e^{j\omega}) = \mathcal{F}\{h[n]\}$ is given by

$$H(e^{j\omega}) = \sum_{k=N_1}^{N_2} h[k]\, e^{-j\omega k}.$$

* Frequency Response of LTI IIR systems

The LTI IIR system can be characterized as

$$\sum_{k=0}^{N} d_k \, y[n-k] = \sum_{k=0}^{M} p_k \, x[n-k].$$

Applying the DTFT for each term, we obtain

$$\sum_{k=0}^{N} d_k e^{-j\omega k} Y(e^{j\omega}) = \sum_{k=0}^{M} p_k e^{-j\omega k} X(e^{j\omega})$$

$$\Rightarrow \left(\sum_{k=0}^{N} d_k e^{-j\omega k}\right) Y(e^{j\omega}) = \left(\sum_{k=0}^{M} p_k e^{-j\omega k}\right) X(e^{j\omega})$$

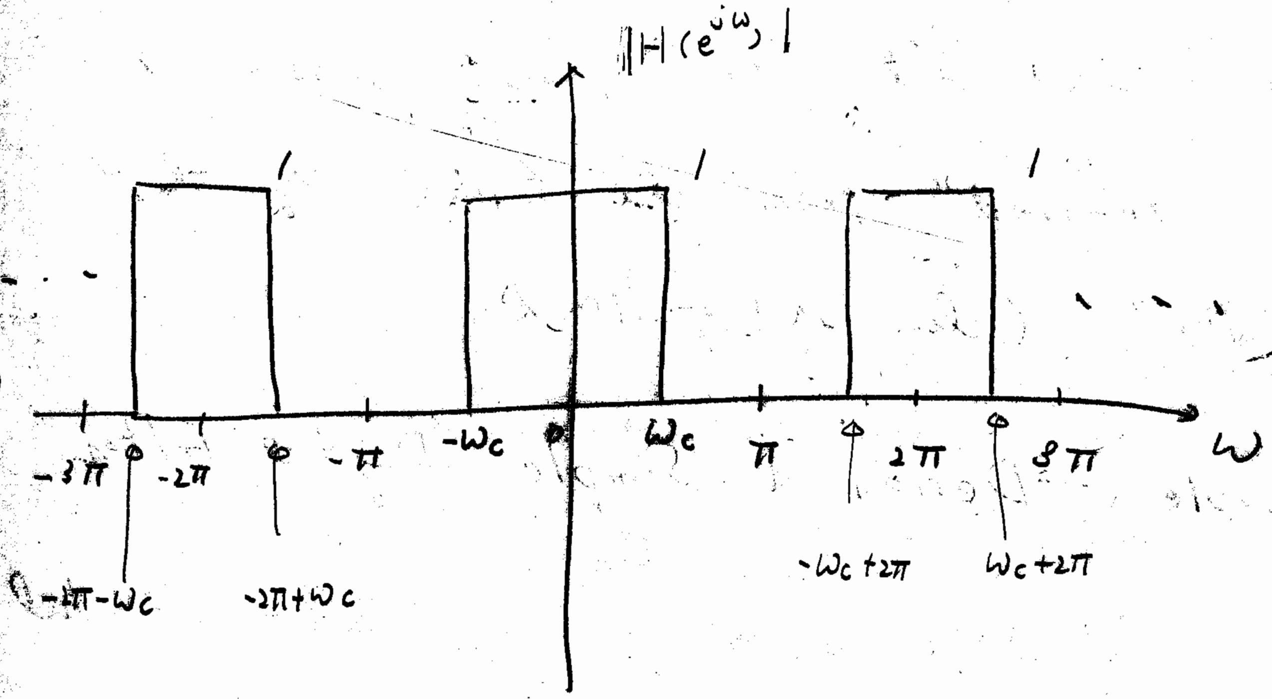
$$\Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} p_k e^{-j\omega k}}{\sum_{k=0}^{N} d_k e^{-j\omega k}}$$

3.8.7 Concept of Filtering

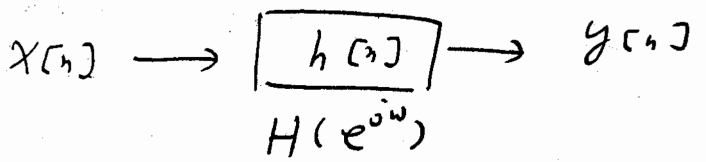
The DTFT can be utilized as a tool to establish the concept of "Filtering", especially the magnitude spectrum of the DTFT.

For illustration, a magnitude spectrum can be applied to specify a "Filter" in the frequency domain as given by

$$|H(e^{j\omega})| \cong \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| < \pi. \end{cases}$$



Frequency-selective characteristics of a "Filter" specified by $|H(e^{j\omega})|$



Apply an input $x[n] = A\cos(\omega_1 n) + B\cos(\omega_2 n)$

$0 < \omega_1 < \omega_c < \omega_2 < \pi$.

$$y[n] = A\,|H(e^{j\omega_1})|\cos(\omega_1 n + \theta(\omega_1)) + B\,|H(e^{j\omega_2})|\cos(\omega_2 n + \theta(\omega_2))$$

$$= A\,|H(e^{j\omega_1})|\cos(\omega_1 n + \theta(\omega_1)).$$

It is noted that, the componant $B\cos(\omega_2 n)$ is "removed" from the signal by the "filter". (low-pass filter)

Example: Design a Simple Digital Filter

We would like to design a length-3 digital filter (highpass) to pass the high-frequency componant (at the angular freqency 0.1 rad/samples) and block the low-frequency componant (at 0.4 rad/samples). We assume that the impulse response of such a filter is

$h[0] = h[2] = \alpha_0$, $h[1] = \alpha_1$

Then, we have to determine α_0, α_1 to serve our purpose.

$$y[n] = h[0]\, x[n] + h[1]\, x[n-1] + h[2]\, x[n-2]$$

$$= \alpha_0\, x[n] + \alpha_1\, x[n-1] + \alpha_0\, x[n-2],$$

$$H(e^{j\omega}) = h[0] + h[1]\, e^{-j\omega} + h[2]\, e^{-j2\omega}$$

$$= \alpha_0 (1 + e^{-j2\omega}) + \alpha_1 e^{-j\omega} = 2\alpha_0 \left(\frac{e^{j\omega} + e^{-j\omega}}{2} \right) e^{-j\omega}$$

$$+ \alpha_1 e^{-j\omega}$$

$$= (2\alpha_0 \cos(\omega) + \alpha_1)\, e^{-j\omega}.$$

The magnitude and phase spectra of this filter are

$$|H(e^{j\omega})| = |2\alpha_0 \cos(\omega) + \alpha_1|$$

$$\theta(\omega) = -\omega + \beta$$

where $\beta = 0$ when $2\alpha_0 \cos(\omega) + \alpha_1 > 0$

$\beta = \pi$ when $2\alpha_0 \cos(\omega) + \alpha_1 < 0$

Two conditions have to be satisfied:

$$\begin{cases} 2\alpha_0 \cos(0.1) + \alpha_1 = 0 \\ 2\alpha_0 \cos(0.4) + \alpha_1 = 1 \end{cases}$$

$\Rightarrow$ $\alpha_0 = -6.76$, $\alpha_1 = 13.46$

$\Rightarrow$ The input-output relation of the desired FIR filter is

$$y[n] = -6.76\,(x[n] + x[n-2]) + 13.46\,x[n-1]$$

$$H(e^{j\omega}) = \mathcal{F}\{h[n]\} = X(z)\Big|_{z=e^{j\omega}} = \mathcal{Z}\{h[n]\}$$

if the ROC of $X(z)$ includes the unit circle $|z| = 1$ or $z = e^{j\omega}$.