Chapter 2

2.1 Discrete-time Signals

2.1.1 Time-domain Representation

An example of a discrete-time signal with real-valued samples is given by

$$\underset{\uparrow\ \text{time index}}{\{x[n]\}} = \{\ldots, 0.95, -0.2, \underset{\uparrow\ n=0}{2.17}, 1, 1, 0.2, -3.67, \ldots\}$$

For the above signal, it reads

$\ldots\ x[-1] = -0.2,\quad x[0] = 2.17,\quad x[1] = 1,\quad x[2] = 1 \ldots$

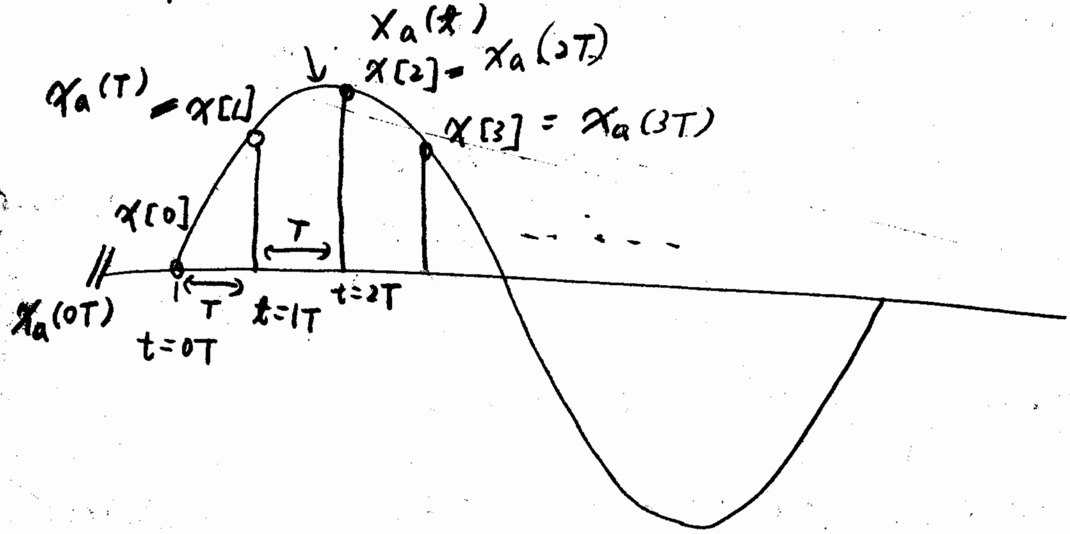
and so on.

Usually, this kind of discrete-time signal (sequence) $\{x[n]\}$ is generated by periodically sampling a continuous-time signal $x_a(t)$ at uniform time intervals:

$$x[n] = x_a(t)\big|_{t=nT} = x_a(nT), \quad n = \ldots, -2, -1, 0, 1, 2, \ldots,$$

where T is called the sampling interval or sampling period. and $F_T \triangleq \frac{1}{T}$ is called the sampling frequency. The unit of sampling frequency is cycles per second or Hertz (Hz) if the sampling period is in seconds.

Example:



discrete-time sequence generated by sampling a continuous-time signal $x_a(t)$.

The quantity $x[n]$ is called the n^{th} sample of the sequence (signal).

Length of a discrete-time signal

The discrete-time signal may be a finite-length or an infinite-length sequence. A finite length signal can be defined as

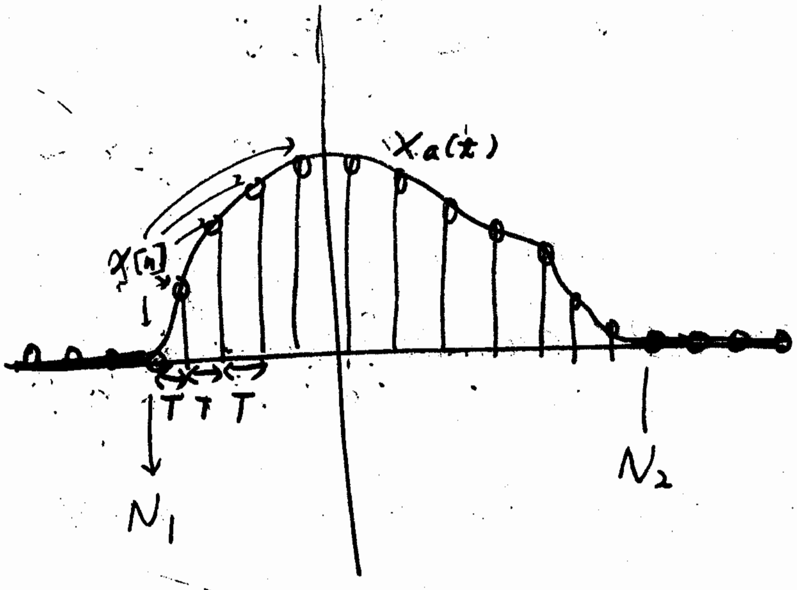
$$x[n] = 0, \quad \text{for } n < N_1, \ n > N_2$$

and the corresponding length or duration N is

$$N = N_2 - N_1 + 1.$$

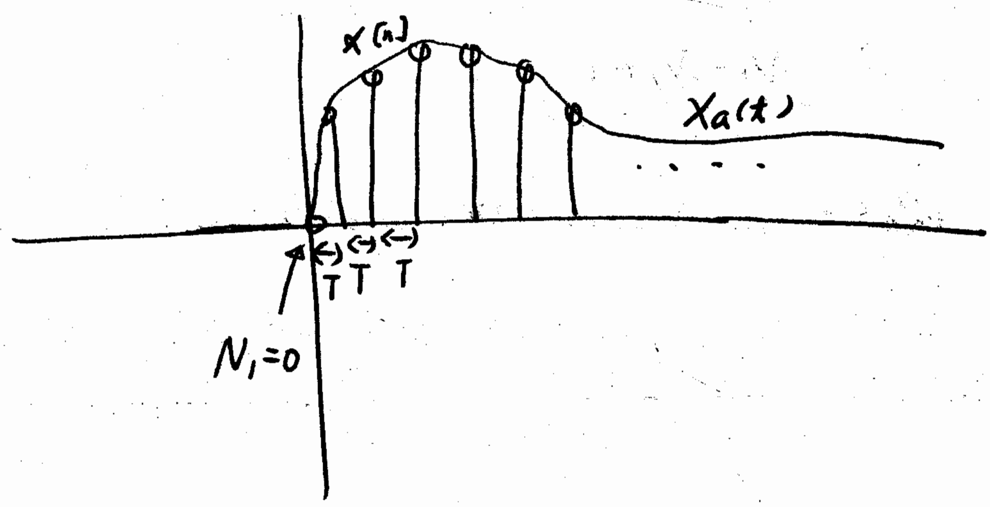
If the signal has the nontrivial values for very large positive integer N_2 and also for very large negative integer N_1 ($N_1 \to -\infty$) ($N_2 \to \infty$), then $x[n]$ is called the two-sided sequence. For a finite N_1, $x[n] = 0$, $n < N_1$, $x[n]$ is called the right-sided or causal sequence ($N_1 \geq 0$). On the other hand, for a finite N_2 $x[n] = 0$, $n > N_2$, $x[n]$ is called left-sided or anticausal sequence ($N_2 \leq 0$).

Finite-length sequence

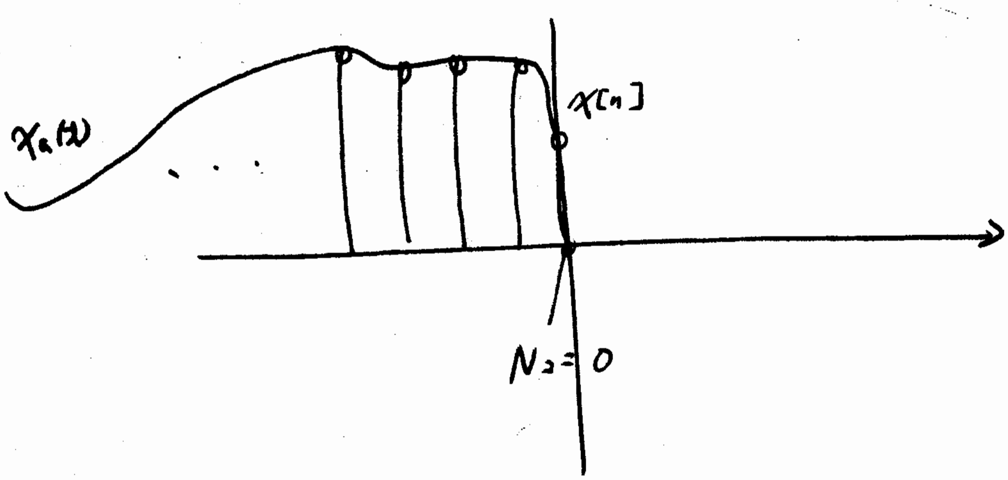


circles represent $x(n)$ in each figure

Causal sequence



anticausal sequence



2.1.2 Operations on Sequences

① Product:

$w_1[n] = x[n] \cdot y[n]$, $-\infty < n < \infty$

$w_1[n]$ results from the sample-wise multiplication of $x[n]$ and $y[n]$

② Scalar multiplication:

$w_2[n] = A\,x[n]$, where A is a scalar constant

③ addition:

$w_3[n] = x[n] + y[n]$, $-\infty < n < \infty$

$w_3[n]$ results from the sample-wise summation of $x[n]$ & $y[n]$.

④ time-shifting:

$w_4[n] = x[n-N]$

$w_4[n]$ results from the N-sample delaying of $x[n]$ if $N > 0$

$w_4[n]$ results from the N-sample advance of $x[n]$ if $N < 0$

⑥ time-reversal:

$w_6[n] = x[-n]$

$w_6[n]$ results from the flip-around-the-vertical-axis version of $x[n]$.

Example: Given two finite-length sequences

$$c[n] = \{\underset{\uparrow}{3.2},\ 41,\ 36,\ -9.5,\ 0\}$$

$$d[n] = \{\underset{\uparrow}{1.7},\ -0.5,\ 0,\ 0.8,\ 1\}$$

① $w_1[n] = c[n] \cdot d[n] = \{\underset{\uparrow}{5.44},\ -20.5,\ 0,\ -7.6,\ 0\}$

② $w_2[n] : c[n] + d[n] = \{\underset{\uparrow}{4.9},\ 40.5,\ 36,\ -8.7,\ 1\}$

③ $w_3[n] = \frac{7}{2} c[n] = \{\underset{\uparrow}{11.2},\ 143.5,\ 126,\ -33.25,\ 0\}$

④ $w_4[n] = d[n-3] = \{\underset{\uparrow}{0},\ 0,\ 0,\ 1.7,\ -0.5,\ 0,\ 0.8,\ 1$

$w_4[n] = c[n+2] = \{3.2,\ 41,\ \underset{\uparrow}{36},\ -9.5,\ 0\}$

$w_6[n] = c[-n]$

$= \{0,\ -9.5,\ 36,\ 41,\ \underset{\uparrow}{3.2}\}$

Example :

$\{c[n]\} = \{3.2, 41, 36, -9.5, 0\}$

$\{g[n]\} = \{-21, 1.5, 3\}$

First, we have make $\{c[n]\}$, $\{g[n]\}$ equal-length.

$\{g[n]\} = \{-21, 1.5, 3, 0, 0\}$

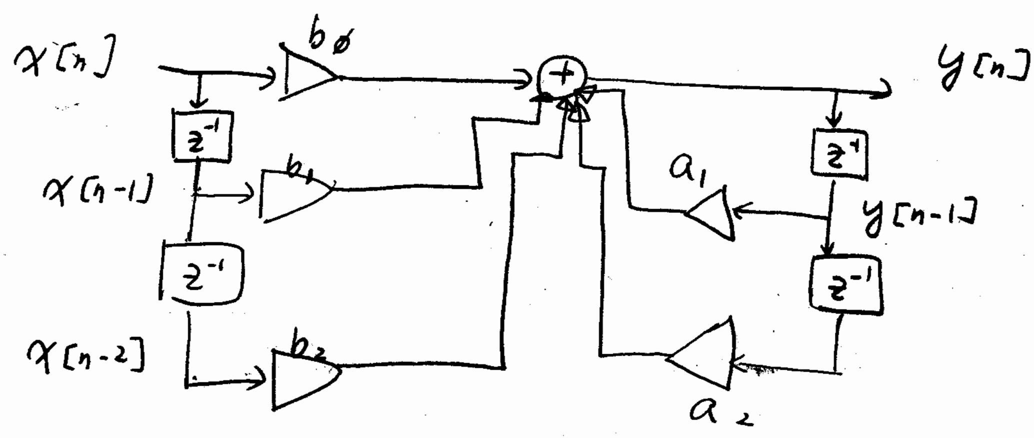
$$\therefore \{w_4[n]\} = \{c[n] \cdot g[n]\}$$
$$= \{-67.2, 61.5, 108, 0, 0\}$$

$$\{w_5[n]\} = \{c[n] + g[n]\}$$
$$= \{-17.8, 42.5, 39, -9.5, 0\}$$

Example : Any difference equation characterizing a discrete-time system can be a combination of elementary operations such as

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + a_1 y[n-1] + a_2 y[n-2]$$

Such a system can be depicted as



Sampling rate Alternation

$x[n]$ is a sequence with a sampling rate of F_T Hz and it is used to generate another sequence $y[n]$ with a desired sampling rate of F_T' Hz; then the sampling rate alternation ratio is given by

$$R \triangleq \frac{F_T'}{F_T} \begin{matrix} \leftarrow \text{new sampling frequency} \\ \leftarrow \text{original sampling frequency} \end{matrix}$$

If $R > 1$, the process is called interpolation and results in a sequence with a higher sampling rate. If $R < 1$, the process is called decimation with a lower sampling rate.

If $R = L$ (integer), it is called up-sampling and $(L-1)$ equidistant zeros are inserted by such an up-sampler between each two consecutive samples of the original input sequence $x[n]$,

$$x_u[n] = \begin{cases} x\left[\frac{n}{L}\right], & n = 0, \pm L, \pm 2L, \cdots \\ 0, & \text{otherwise} \end{cases}$$

Example: $L = 3$

$$x[n] = \{1, 3, \underset{\substack{\uparrow \\ n=0}}{2}, 7, -1\}$$

$$x_u[n] = \{1, 0, 0, 3, 0, 0, \underset{\substack{\uparrow \\ n=0}}{2}, 0, 0, 7, 0, 0, -1\}$$

The upsampler can be depicted as

$$x[n] \rightarrow \boxed{\uparrow L} \rightarrow x_u[n]$$

On the other hand, If $R = \frac{1}{M}$ (M is integer), it is called downsampling and every M^{th} sample of original input sequence $x[n]$ remains but all $(M-1)$

samples are removed in between,

$$y[n] = x[nM].$$

Example: $x[n] = \{1, 3, \underset{\substack{\uparrow \\ n=0}}{2}, 8, 7, -5\}$

$M = 3$

$y[n] = \{\underset{\substack{\uparrow \\ n=0}}{2}, -5\}$

The downsampler can be depicted as

$$x[n] \rightarrow \boxed{\downarrow M} \rightarrow y[n]$$

2.1.3 Classification of Sequences

× Symmetry:

A sequence $x[n]$ is called a conjugate symmetric sequence if $x[n] = x^*[-n]$.

A sequence $x[n]$ is called a conjugate antisymmetric sequence if $x[n] = -x^*[-n]$

Any complex sequence $x[n]$ can be expressed (decomposed) as a sum of its conjugate-symmetric part $x_{cs}[n]$ and its conjugate-antisymmetric part $x_{ca}[n]$, such that

$$x[n] = x_{cs}[n] + x_{ca}[n]$$

where
$$x_{cs}[n] = \frac{1}{2}\left(x[n] + x^*[-n]\right)$$
$$x_{ca}[n] = \frac{1}{2}\left(x[n] - x^*[-n]\right)$$

Example:

A sequence $\{g[n]\}$ is defined for $-3 \le n \le 3$, with length 7, such that

$$\{g[n]\} = \{0,\ 1+j4,\ -2+j3,\ \underset{\substack{\uparrow \\ n=0}}{4-j2},\ -5-j6,\ -2j,\ 3\}$$

where $j = \sqrt{-1}$.

The conjugate-symmetric part $g_{cs}[n]$ can be constructed as

$$\{g^*[n]\} = \{0,\ 1-j4,\ -2-j3,\ 4+j2,\ -5+j6,\ j2,\ 3\}$$

$$\{g^*[-n]\} = \{3,\ j2,\ -5+j6,\ 4+j2,\ -2-j3,\ 1-j4,\ 0\}$$

$$\{g_{cs}[n]\} = \{1.5,\ 0.5+j3,\ -3.5+j4.5,\ 4,\ -3.5-j4.5,\ 0.5-j3,\ 1.5\}$$

$$= \frac{1}{2}\left(g[n] + g^*[-n]\right)$$

$$\{g_{ca}[n]\} = \{-1.5,\ 0.5+j,\ 1.5-j1.5,\ -j2,\ -1.5-j1.5,\ -0.5+j,\ 1.5\}$$

$$= \frac{1}{2}\left(g[n] - g^*[-n]\right)$$

Similarly, any real sequence $x[n]$ can be expressed (decomposed) as a sum of its even part $x_{ev}[n]$ and its odd part $x_{od}[n]$ such that

$$x[n] = x_{ev}[n] + x_{od}[n]$$

where

$$x_{ev}[n] = \frac{1}{2}\left(x[n] + x[-n]\right)$$

$$x_{od}[n] = \frac{1}{2}\left(x[n] - x[-n]\right)$$

Example :

A real sequence $\{g[n]\}$ is defined for $-2 \le n \le 3$, with length 6, such that

$$\{g[n]\} = \{1, -3.5, \underset{\uparrow}{2}, 3, -2, 4\}$$

$$\{g[-n]\} = \{4, -2, 3, \underset{\uparrow}{2}, -3.5, 1\}$$

To make $\{g[n]\}$, $\{g[-n]\}$ equal-length first,

$$\{g[n]\} = \{0, 1, -3.5, \underset{\uparrow}{2}, 3, -2, 4\}$$

$$\{g[-n]\} = \{4, -2, 3, \underset{\uparrow}{2}, -3.5, 1, 0\}$$

$$\therefore \quad g_{ev}[n] = \frac{1}{2}(g[n] + g[-n])$$

$$= \{2, -0.5, -0.25, \underset{\uparrow}{2}, -0.25, -0.5, 2\}$$

$$g_{od}[n] = \{-2, 1.5, -3.25, \underset{\uparrow}{0}, 3.25, -1.5, 2\}$$

$$g[n] = g_{ev}[n] + g_{od}[n]$$

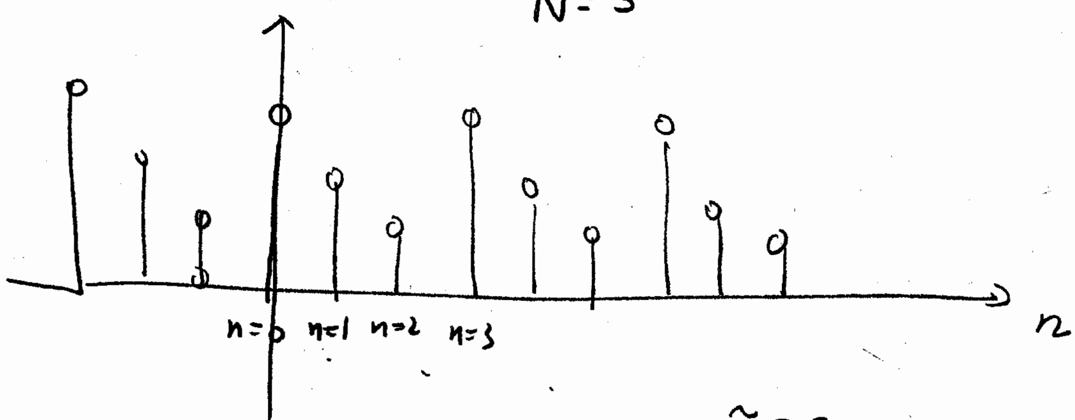
Periodic and Aperiodic Signals

A sequence $\tilde{x}[n]$ satisfying

$$\tilde{x}[n] = \tilde{x}[n + kN] \quad \text{for all } n$$

is called a periodic sequence with a period N, where N is a positive integer and k is any integer.

$N = 3$



$\tilde{x}[n]$ is periodic with period $N = 3$

If $\tilde{y}[n]$ is periodic such that

$$\tilde{y}[n] = \sum_{k=1}^{M} a_k \tilde{x}_k[n],$$

where $\tilde{x}_k[n]$ is periodic for all k,

then the fundamental period N is given by

$$N = \frac{\prod_{k=1}^{M} N_k}{GCD\,(N_1, N_2, \cdots, N_M)}$$

where $\prod_{k=1}^{M} N_k = (N_1 \cdot N_2 \cdots N_M)$

and GCD is the greatest common divisor of $N_1, N_2, \cdots, N_M$.

Example: What is the fundamental period of $\tilde{x}[n] = \cos(0.7\pi n)$, $\tilde{x}[n] = \cos(11\pi n)$?

Solution:

(i) $\cos(0.7\pi n) = \cos(0.7\pi(n+N))$

$= \cos(0.7\pi n + 0.7\pi N)$

$\therefore 0.7\pi N = 2\pi k, \quad k \in \mathbb{Z}^+$

$N = \frac{2}{0.7} k = \frac{20}{7} k, \quad k = 7, 14, 21, \ldots$

$\Rightarrow$ at least $N = 20$ for the smallest $k = 7$

(ii) $\cos(11\pi n) = \cos(11\pi n + 11\pi N)$

$\therefore 11\pi N = 2\pi k, \quad k \in \mathbb{Z}^+$

$N = \frac{2}{11} k, \quad k = 11, 22, 33, \ldots$

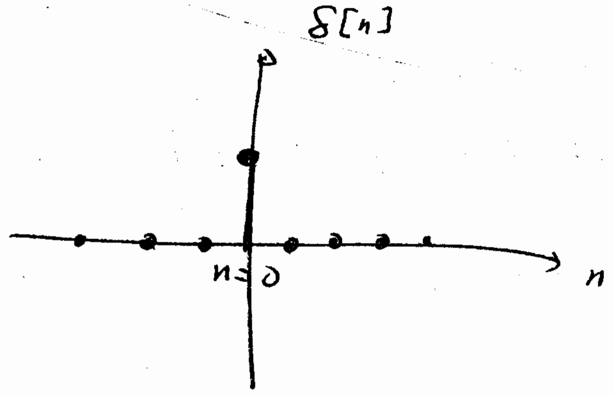
$\Rightarrow$ At least $N = 2$ (fundamental period)

for the smallest $k = 11$.

2.2 Typical Sequences and Sequence Representation

- Unit Sample Sequence

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

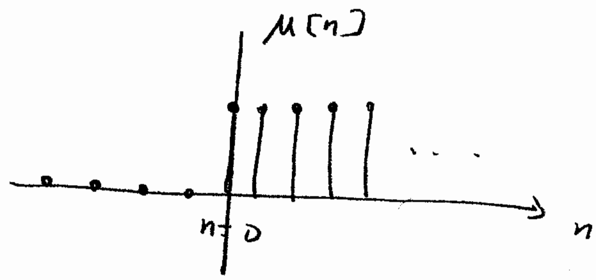


- Unit Step Sequence

$$\mu[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$\mu[n] = \sum_{m=0}^{\infty} \delta[n-m]$$

$$\delta[n] = \mu[n] - \mu[n-1]$$



X. Sinusoidal and Exponential Sequences

$$x[n] = A\cos(\omega_0 n + \phi), \quad -\infty < n < \infty$$

where A, ω_0 and ϕ are all real-valued.

An exponential sequence can be expressed as

$$x[n] = A\alpha^n, \quad -\infty < n < \infty$$

where $\alpha = e^{(\sigma_0 + j\omega_0)}$, $A = |A| e^{j\phi}$

$$\therefore \quad x[n] = |A| e^{\sigma_0 n} e^{j(\omega_0 n + \phi)}$$

$$= x_{re}[n] + x_{in}[n]$$

and $x_{re}[n] = |A| e^{\sigma_0 n} \cos(\omega_0 n + \phi)$

$x_{in}[n] = |A| e^{\sigma_0 n} \sin(\omega_0 n + \phi)$

Read 2.5 By yourself (EE 3120 material!)