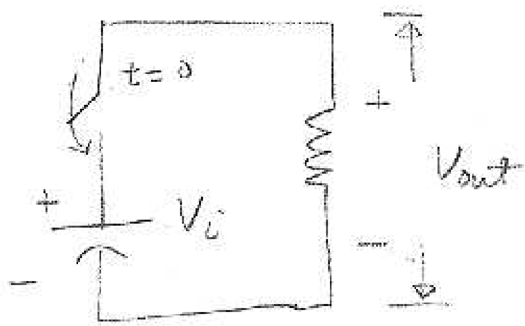
Chapter 6 Continuous-Time Signal Analysis

6.1

A signal is called a deterministic signal if it can be described without any uncertainty.

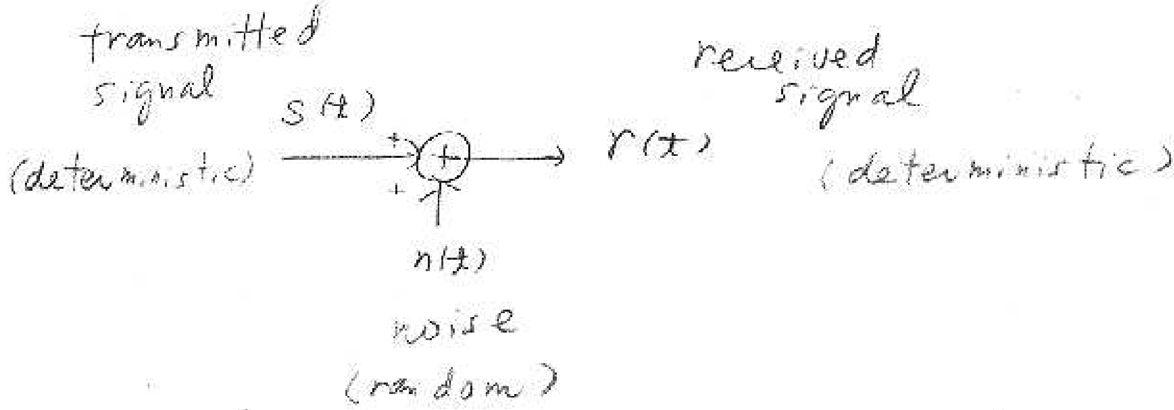
Example:



The voltage across the resister is $V_{out} = V_i\,g(t)$. V_{out} is a deterministic signal.

On the contrary, a signal is called a random signal if it cannot be described with certainty before it actually occurs.

Example: The noise in the communication system.



6.1.1 Periodic and Aperiodic Signals

A function is said to be periodic with period P if $f(t) = f(t+P)$ for all t. If $f(t)$ is periodic, then

$$f(t) = f(t+P) = f(t+2P) = \cdots = f(t+nP)$$

for all t and for every integer n.

If a function is not periodic, it is nonperiodic or aperiodic function

If $f_i(t)$ are period with period P_i, $i=1,2$, then $f_1(t)+f_2(t)$ is periodic with period $P=n_1P_1=n_2P_2$, n_1, n_2 are two positive integers. $f_1(t)+f_2(t)$ is aperiodic if no such n_1, n_2 pair can be found.

Example. Which of the following are periodic functions? Find their fundamental frequencies and periods.

(a) $\sin(3t)+\sin(\pi t)$

(b) $\sin(0.8t)+\pi\cos(2t)-21\sin(1.2t)$

Answer:

(a)
$$\sin(3(t+P_1)) = \sin(3t)$$
$$3P_1 = 2\pi \Rightarrow P_1 = \frac{2\pi}{3}$$
$$\sin(\pi(t+P_2)) = \sin(\pi t)$$
$$\pi P_2 = 2\pi \Rightarrow P_2 = 2$$
$$n_1 P_1 = n_2 P_2$$
$$\Rightarrow \frac{n_1}{n_2} = \frac{P_2}{P_1} = \frac{3}{\pi}$$

no such integer n_1, n_2 pair exists

$\therefore$ $\sin(3t) + \sin(\pi t)$ is aperiodic

(b) $\sin(0.8(t + P_1)) = \sin(0.8t)$

$0.8 P_1 = 2\pi \Rightarrow P_1 = \frac{5}{2}\pi$

$\pi \cos(2(t + P_2)) = \pi \cos(2t)$

$2P_2 = 2\pi \Rightarrow P_2 = \pi$

$-21 \sin(1.2(t + P_3)) = -21 \sin(1.2t)$

$1.2 P_3 = 2\pi \Rightarrow P_3 = \frac{5\pi}{3}$

$P_1 \Rightarrow n_1 P_1 = n_2 P_2 = n_3 P_3$

$= \frac{5}{2}\pi n_1 = \pi n_2 = \frac{5}{3}\pi n_3$

$$\left.\begin{array}{l} n_1 = 2 \\ n_2 = 5 \\ n_3 = 3 \end{array}\right\} \Rightarrow P = 5\pi \text{ radians}$$

fundamental frequency $= \frac{2\pi}{P} = \frac{2\pi}{5\pi} = 0.4$ Hz

6.1.2 Energy and Power Signals

Energy provided by the signal $f(t)$ over the time interval $[t_0, t_1]$ is

$$\int_{t_0}^{t_1} |f(t)|^2 \, dt = \int_{t_0}^{t} f(t) f^*(t) \, dt$$

※ Energy signal : A signal $f(t)$ is called an energy signal if its total energy in $(-\infty, \infty)$ is finite, that is

$$E_\infty := \int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty$$

The average power of a signal over $[t_1, t_2]$ is defined as

$$P_{av} := \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |f(t)|^2 \, dt$$

※ Power signal : A signal is called a power signal if the average power over $(-\infty, \infty)$

$$P_\infty := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 \, dt$$

is nonzero and finite.

Example: The signal e^{-2t} is energy signal or power signal?

Answer:

$$E_\infty = \int_{-\infty}^{\infty} |e^{-2t}|^2 dt$$

$$= \int_{-\infty}^{\infty} e^{-4t} dt$$

$$= -\frac{e^{-4t}}{4}\Big|_{-\infty}^{\infty} = 0 + \frac{e^{-4t}}{4}\Big|_{t\to-\infty}$$

$$= \infty$$

Hence it is not energy signal

$$P_\infty := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-4t} dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \left[-\frac{e^{-4T}}{4} + \frac{e^{4T}}{4} \right]$$

$$= \lim_{T\to\infty} \left(-\frac{e^{-4T}}{8T} \right) + \lim_{T\to\infty} \left(\frac{e^{4T}}{8T} \right)$$

$$= 0 + \lim_{T\to\infty} \frac{4e^{4T}}{8} = \infty$$

it is not a power signal either.

6.1.3 Orthogonality of Complex Exponentials

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

Consider the set of complex exponentials

$$\phi_m(t) := e^{jm\omega_0 t}, \quad \text{where } m = 0, \pm 1, \pm 2, \cdots$$

For each integer m, the function $e^{jm\omega_0 t}$ is a periodic function with fundamental frequency $m\omega_0$ and fundamental period $p_m := \frac{2\pi}{m\omega_0}$. If we define $P := \frac{2\pi}{\omega_0}$ then $P = m p_m$ and any time interval of length P contains m complete cycles of $e^{jm\omega_0 t}$.

Now we show

$$\int_{t_0}^{t_0+P} e^{jm\omega_0 t}\, dt = \int_{t_0}^{t_0+P} [\cos(m\omega_0 t) + j\sin(m\omega_0 t)]\, dt$$

$$= \begin{cases} P, & \text{if } m = 0 \\ 0, & \text{if } m \neq 0 \end{cases}$$

It implies

$$\int_{<P>} \phi_m(t)\,\phi_n^*(t)\,dt := \int_{t_0}^{t_0+P} e^{j(m-n)\omega_0 t}\,dt$$

$$= \begin{cases} P, & \text{if } m=n \\ 0, & \text{if } m\neq n \end{cases}$$

This is called the orthogonality property of the set $\phi_m(t)$

Consider a signal consisting of a linear combination of $\phi_m(t)$ such that

$$f(t) = \sum_{m=-\infty}^{\infty} C_m e^{jm\omega_0 t}$$ where C_m are complex constants. It is clear that ω_0 is the GCD of all $m\omega_0$. Thus, for every set of C_m, $f(t)$ is periodic with fundamental period ω_0. Conversely, every periodic function $f(t)$ with fundamental period ω_0 can be represented by such a linear combination for some set of C_m.

6.2 Fourier Series of Periodic Functions

Consider a periodic function $f(t)$ with fundamental period P and fundamental frequency $\omega_0 = \frac{2\pi}{P}$. We show that $f(t)$ can be expressed as

$$f(t) = \sum_{m=-\infty}^{\infty} C_m e^{jm\omega_0 t} \quad \ldots\ldots \quad (*)$$

with

$$C_m = \frac{1}{P}\int_{t_0}^{t_0+P} f(t)\, e^{-jm\omega_0 t}\, dt$$

$$=: \frac{1}{P}\int_{<P>} f(t)\, e^{-jm\omega_0 t}\, dt$$

for $m = 0, \pm1, \pm2, \pm3, \ldots$ and for any arbitrary t_0.

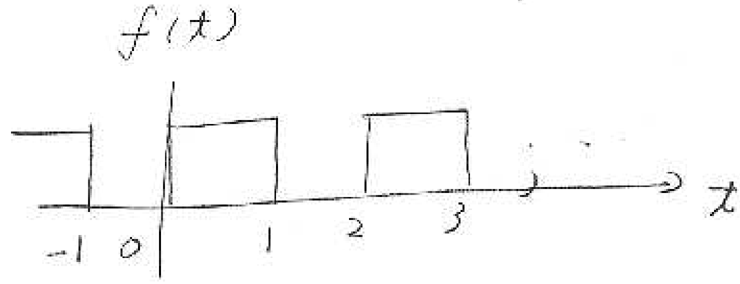
Equation $(*)$ is called the complex exponential Fourier Series.

Proof: $$\int_{t_0}^{t_0+P} f(t)\, e^{-jn\omega_0 t}\, dt = \sum_{m=-\infty}^{\infty}\int_{t_0}^{t_0+P} C_m e^{jm\omega_0 t} e^{-jn\omega_0 t}\, dt$$

$$= \sum_{m=-\infty}^{\infty} C_m \int_{t_0}^{t_0+P} e^{j(m-n)\omega_0 t}\, dt = PC_n$$

$$\therefore\ C_n = \frac{1}{p}\int_{t_0}^{t_0+p} f(t)\, e^{-jn\omega_0 t}\, dt\ , \quad n = 0, \pm 1, \pm 2,$$

Example :



$f(t)$ is depicted as above. Find the Fourier Series.

Answer :

$$p = 2 \Rightarrow \omega_0 = \frac{2\pi}{2} = \pi$$

$$f(t) = \sum_{m=-\infty}^{\infty} C_m e^{jm\pi t}$$

$$\text{with } C_m = \frac{1}{2}\int_0^2 f(t)\, e^{-jm\pi t}\, dt$$

$$= \frac{1}{2}\int_0^1 e^{-jm\pi t}\, dt$$

$$= \frac{1}{2}\left.\frac{e^{-jm\pi t}}{-jm\pi}\right|_0^1 = \frac{e^{-jm\pi}-1}{-2jm\pi}$$

$$f(t) = \underbrace{\sum_{m=-\infty}^{\infty}\left[\frac{1-e^{-jm\pi}}{2jm\pi}\, e^{jm\pi t}\right]}_{\text{Fourier Series}}$$

6.2.1 Discrete Frequency Spectrum

The discrete frequency spectrum is defined as the set of $\{C_m\}$ and generally a complex-value function of m or $m\omega_0$

Let $C_m = \alpha_m + j\beta_m$. Then the magnitude and phase of C_m are

$$|C_m| := \sqrt{\alpha_m^2 + \beta_m^2}$$

and $$\measuredangle C_m := \tan^{-1}\frac{\beta_m}{\alpha_m}$$

If $f(t)$ is a real-valued function, that is, $f^*(t) = f(t)$, then we have

$$C_m^* = \left[\frac{1}{P}\int_{t_0}^{t_0+P} f(t)\, e^{-jm\omega_0 t}\, dt\right]^*$$

$$= \frac{1}{P}\int_{t_0}^{t_0+P} f(t)\, e^{jm\omega_0 t}\, dt$$

$$= C_{-m}$$

Thus the discrete frequency spectrum of real $f(t)$ has the property

$$C_m^* = C_{-m}$$

This is called conjugate symmetry.

$$C_m^* = \alpha_m - j\beta_m = C_{-m} = \alpha_{-m} + j\beta_{-m}$$

$$\therefore \quad \alpha_m = \alpha_{-m}, \quad -\beta_m = \beta_{-m}$$

Consequently, $|C_m| = |C_{-m}|$ and

$$\measuredangle C_m = -\measuredangle C_{-m}$$

$\times$ Even and Odd Real functions

If $f(t)$ is a real and even function of t then C_m is a real and even function of m

If $f(t)$ is real and odd function of t then C_m is imaginary and odd function of m

This can be established from

$$C_m = \frac{1}{P}\int_{t_0}^{t_0+P} f(t)\, e^{-jm\omega_0 t}$$

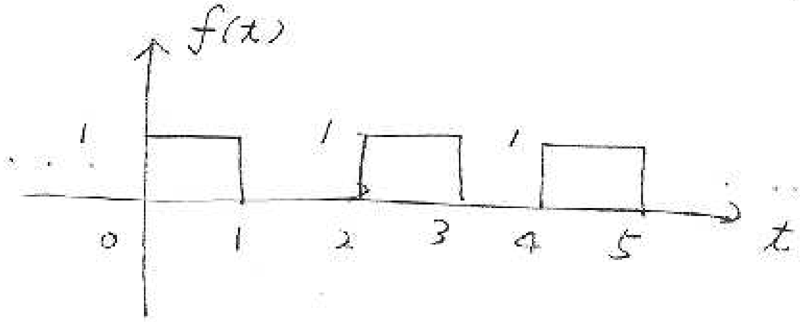
$$= \frac{1}{P}\int_{t_0}^{t_0+P} f(t)\left[\cos(m\omega_0 t) - j\sin(m\omega_0 t)\right] dt$$

If $f(t)$ is real, then

$$Re\{C_m\} = \frac{1}{P}\int_{t_0}^{t_0+P} f(t)\cos(m\omega_0 t)\, dt$$

and $\mathrm{Im}\{C_m\} = -\frac{1}{P}\int_{t_0}^{t_0+P} f(t)\sin(m\omega_0 t)\,dt$

Example: What is the discrete frequency spectrum for $f(t)$ depicted as below?



$$C_m = \frac{1}{2}\int_0^1 e^{-jm\pi t}\,dt$$

$$= \frac{e^{-jm\pi} - 1}{-2jm\pi} = \frac{e^{-j\frac{m\pi}{2}}\left(e^{j\frac{m\pi}{2}} - e^{-j\frac{m\pi}{2}}\right)}{2jm\pi}$$

$$= \frac{e^{-j\frac{m\pi}{2}}\sin\left(\frac{m\pi}{2}\right)}{m\pi}$$

$$|C_m| = \frac{\left|\sin\left(\frac{m\pi}{2}\right)\right|}{|m|\pi}$$

$$\measuredangle C_m = \frac{-m\pi}{2} + \measuredangle \sin\left(\frac{m\pi}{2}\right) - \measuredangle m$$

※ Parseval's Formula:

$$P_{av} = \frac{1}{P}\int_{t_0}^{t_0+P} f(t)\,f^*(t)\,dt = \frac{1}{P}\int_{t_0}^{t_0+P} |f(t)|^2\,dt$$

$$= \sum_{m=-\infty}^{\infty} C_m C_m^* = \sum_{m=-\infty}^{\infty} |C_m|^2$$

Proof: $$P_{av} := \frac{1}{P}\int_{t_0}^{t_0+P} f(t)\,f^*(t)\,dt$$

$$= \frac{1}{P}\int_{t_0}^{t_0+P} \left(\sum_{m=-\infty}^{\infty} C_m e^{jm\omega_0 t}\right) f^*(t)\,dt$$

$$= \sum_{m=-\infty}^{\infty} C_m \left[\frac{1}{P}\int_{t_0}^{t_0+P} f(t)\,e^{-jm\omega_0 t}\,dt\right]^*$$

$$= \sum_{m=-\infty}^{\infty} C_m C_m^*$$

6.4. Fourier Transform

For any aperiodic function $f(t)$,

the period P can be deemed as infinity.

Since the Fourier Series pair is given as

$$C_m = \frac{1}{P}\int_{-P/2}^{P/2} f(t)\,e^{-jm\omega_0 t}\,dt$$

and $$f(t) = \frac{1}{P}\sum_{m=-\infty}^{\infty} C_m P e^{jm\omega_0 t}$$

We define $F(\underbrace{m\omega_0}_{\omega}) := PC_m$

$$F(m\omega_0) := C_m P = \int_{-P/2}^{P/2} f(t)\, e^{-jm\omega_0 t}\, dt$$

$$= \int_{-P/2}^{P/2} f(t)\, e^{-j\omega t}\, dt$$

and
$$f(t) = \frac{1}{P} \sum_{m=-\infty}^{\infty} P C_m\, e^{jm\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} F(m\omega_0)\, e^{jm\omega_0 t}\, \omega_0$$

Since $f(t)$ is aperiodic, $P \to \infty$.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$f(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, \underset{\substack{\downarrow \\ d\omega}}{\omega_0}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

This is the Fourier Transform Pair.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

Sufficient conditions for $f(t)$ to have a Fourier transform are

1. $f(t)$ is absolutely integrable over $(-\infty, \infty)$, that is $\int_{-\infty}^{\infty} |f(t)|\,dt \le M < \infty$ for some finite constant M

2. $f(t)$ has a finite number of discontinuities and a finite number of minima and maxima in every finite time interval.

Example : Determine the Fourier Transform for $f(t) = e^{-a|t|}$, $a > 0$ and a is real.

Answer :

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{+at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \frac{e^{(a-j\omega)t}}{a-j\omega}\Big|_{-\infty}^{0} + \frac{e^{-(a+j\omega)t}}{-[a+j\omega]}\Big|_{0}^{\infty}$$

$$= \frac{1}{a-j\omega} - \frac{1}{-a-j\omega} = \frac{a+j\omega + a-j\omega}{a^2+\omega^2} = \frac{2a}{a^2+\omega^2}$$

$F(\omega)$ is usually called the frequency spectrum.

$$A(\omega) := |F(\omega)| = \sqrt{Re\{F(\omega)\}^2 + Im\{F(\omega)\}^2}$$

is defined as the amplitude spectrum

and

$$\theta(\omega) := \measuredangle F(\omega) = \tan^{-1} \frac{Im\{F(\omega)\}}{Re\{F(\omega)\}}$$

is defined as the phase spectrum.

If $f(t)$ is real, then $A(\omega) = A(-\omega)$ and $\theta(\omega) = -\theta(-\omega)$

If $f(t)$ is real and even function of t ($f(t) = f(-t)$), then $F(\omega)$ is a real and even function of ω. If $f(t)$ is a real and odd function of t ($f(t) = -f(-t)$), then $F(\omega)$ is an imaginary and odd function of ω.

※ The Fourier Transform is a linear operator, that is,

$$\mathcal{F}[\alpha_1 f_1(x) + \alpha_2 f_2(x)]$$

$$= \alpha_1 \mathcal{F}[f_1(x)] + \alpha_2 \mathcal{F}[f_2(x)]$$

$$= \alpha_1 F_1(\omega) + \alpha_2 F_2(\omega)$$

where α_1, α_2 are two arbitrary complex-valued constants, and

$$\mathcal{F}[f_i(x)] = F_i(\omega)$$

$$= \int_{-\infty}^{\infty} f_i(t)\, e^{-j\omega t}\, dt$$

6.4.1 From the Laplace Transform to the Fourier Transform

The Fourier Transform can be established from the Laplace Transform evaluated at the pure imaginary axis, namely $s = \pm j\omega$

Consider an absolutely integrable function $f(t)$ defined over $(-\infty, \infty)$. (Its Fourier Transform exists.)

$$f(t) = f_-(t) + f_+(t) = \overbrace{f(t)\,g(-t)}^{f_-(t)} + \overbrace{f(t)\,g(t)}^{f_+(t)}$$

$+ f(t)\,g(t)$, where $g(t)$ is redefined as

$$g(t) := \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases}$$

$$\mathcal{F}[f(t)] = \mathcal{F}[f_+(t)] + \mathcal{F}[f_-(t)]$$

$$F_+(\omega) = \mathcal{F}[f_+(t)]$$

$$= \int_{-\infty}^{\infty} f_+(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0^-}^{\infty} f_+(t)\, e^{-j\omega t}\, dt$$

Let $\bar{F}_+(s) = \mathcal{L}[f_+(t)]$

$$= \int_{0^-}^{\infty} f_+(t)\, e^{-st}\, dt$$

$$\therefore\ F_+(\omega) = \bar{F}_+(s)\Big|_{s=j\omega}$$

$$F_+(\omega) = \mathcal{F}[f(t)\,g(t)] = \mathcal{L}[f(t)\,g(t)]\Big|_{s=j\omega}$$

Similarly

$$F_-(\omega) = \int_{-\infty}^{\infty} f_-(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0^+} f_-(t)\, e^{-j\omega t}\, dt$$

let $\tau = -t$

$$F_-(\omega) = \int_{0^-}^{\infty} f_-(-\tau)\, e^{j\omega \tau}\, d\tau$$

Let $F_-(s) = \mathcal{L}[f_-(-t)]$

$$= \int_{0^-}^{\infty} f_-(-t)\, e^{-st}\, dt$$

$$\mathcal{F}[f_-(t)] = F_-(\omega) = \mathcal{L}[f_-(-t)]\Big|_{s=-j\omega}$$

$$F_-(\omega) = \mathcal{F}[f(t)\,g(-t)] = \mathcal{L}[f(-t)\,g(t)]\Big|_{s=-j\omega}$$

$$F(\omega) = F_+(\omega) + F_-(\omega)$$

$$= F_+(s)\Big|_{s=j\omega} + F_-(s)\Big|_{s=-j\omega}$$

Example: Find the Fourier Transform for $f(t) = \begin{cases} e^{-2t}, & t > 0 \\ -e^{2t}, & t < 0 \end{cases}$

Answer:

$$f_+(t) = e^{-2t}, \quad t > 0$$

$$f_-(t) = -e^{2t}, \quad t < 0$$

$$F_+(s) = \mathcal{L}[f_+(t)] = \frac{1}{s+2}$$

$$F_-(s) = \mathcal{L}[f_-(-t)] = -\frac{1}{s+2}$$

$$F_+(\omega) = \frac{1}{s+2}\Big|_{s=j\omega} = \frac{1}{j\omega+2}$$

$$F_-(\omega) = -\frac{1}{s+2}\Big|_{s=-j\omega} = -\frac{1}{-j\omega+2} = \frac{1}{j\omega-2}$$

$$F(\omega) = F_+(\omega) + F_-(\omega)$$

$$= \frac{1}{j\omega+2} + \frac{1}{j\omega-2} = \frac{-2j\omega}{4+\omega^2}$$

Since $f(t)$ is odd function,

$F(\omega)$ is a pure imaginary and odd function.

6.4.2 Fourier Transform of Periodic Functions

The impulse $F(\omega) = \delta(\omega - \omega_0)$ will cause a complex exponential function $f(t)$ $\left[f(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} F(\omega) \right]$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} e^{j\omega_0 t}$$

$\therefore$ We can conclude

$$e^{j\omega_0 t} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} 2\pi \delta(\omega - \omega_0)$$

Example: Find the Fourier Transforms for $\sin(\omega_0 t)$ and $\cos(\omega_0 t)$

Answer: $\mathcal{F}[\sin(\omega_0 t)] = \mathcal{F}\left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right]$

$$= \frac{1}{2j} \mathcal{F}[e^{j\omega_0 t}] - \frac{1}{2j} \mathcal{F}[e^{-j\omega_0 t}]$$

$$= \frac{\pi}{j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

$$\mathcal{F}[\cos(\omega_0 t)] = \mathcal{F}\left[\frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})\right]$$

$$= \frac{1}{2}\mathcal{F}[e^{j\omega_0 t}] + \frac{1}{2}[e^{-j\omega_0 t}]$$

$$= \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

If $f(t)$ is periodic function, then

$$f(t) = \sum_{m=-\infty}^{\infty} C_m e^{jm\omega_0 t} \quad \text{(Fourier Series)}$$

$$F(\omega) = \mathcal{F}[f(t)] = \sum_{m=-\infty}^{\infty} 2\pi C_m \delta(\omega - m\omega_0)$$

Example: Find the Fourier Transform for a function $f(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$

Answer:

$$f(t) = \sum_{m=-\infty}^{\infty} C_m e^{jm\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T}$

$$C_m = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jm\omega_0 t} dt$$

$$= \frac{1}{T}$$

$$\therefore \quad f(t) = \sum_{m=-\infty}^{\infty} \frac{1}{T} e^{jm\omega_0 t} \quad \text{(Fourier Series)}$$

Then $F(\omega) = \sum_{m=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - m\omega_0)$

Table 6.1 lists some Fourier Transform pairs !!

6.4.4 Spectrum density and Energy

Consider an absolutely integrable function $f(t)$, Let $F(\omega)$ be its Fourier transform or frequency spectrum. The total energy of $f(t)$ is

$$\text{total energy} := E = \int_{-\infty}^{\infty} |f(t)|^2 dt$$
$$= \int_{-\infty}^{\infty} f(t) f^*(t) dt$$

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} f(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \right]^* dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) F(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

6.5 Convolution

The zero-state response of every LTI continuous time system can be described as

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, u(\tau)\, d\tau$$

$$\mathcal{F}[y(t)] = Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t-\tau)\, u(\tau)\, d\tau\; e^{-j\omega(t-\tau)}\, e^{-j\omega\tau}\, dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega(t-\tau)}\, dt\; u(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$t' = t-\tau$$

$$= \int_{-\infty}^{\infty}\underbrace{\int_{-\infty}^{\infty} h(t')\, e^{-j\omega t'}\, dt'}_{H(\omega)}\; u(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= H(\omega) \underbrace{\int_{-\infty}^{\infty} u(\tau)\, e^{-j\omega\tau}\, d\tau}_{U(\omega)}$$

$$= H(\omega)\, U(\omega)$$

Example: Find the Fourier Transform for $y(t) = h(t) \circledast u(t)$, where $u(t) = \delta(t)$ and $h(t) = e^{-t}\delta(t)$

Answer:

$$q(t) = \frac{1}{2} + \frac{1}{2}\,sgn(t)$$

where $sgn(t) = \begin{cases} 1 & , t > 0 \\ \frac{1}{2} & , t = 0 \\ -1 & , t < 0 \end{cases}$

$$\mathcal{F}[sgn(t)] = \lim_{a \to 0} \left[-\int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt \right]$$

$$= \lim_{a \to 0} \left[\frac{-1}{a - j\omega} + \frac{1}{a + j\omega} \right]$$

$$= \frac{2}{j\omega}$$

$$\mathcal{F}\left[\frac{1}{2}\right] = \mathcal{F}\left[\frac{1}{2} e^{j0t}\right] = \pi\,\delta(\omega - 0)$$
$$= \pi\,\delta(\omega)$$

$$\therefore \quad \mathcal{F}[q(t)] = Q(\omega) = \frac{1}{j\omega} + \pi\,\delta(\omega)$$

$$H(\omega) = \mathcal{L}\left[e^{-t} q(t)\right]\Big|_{s=j\omega} + 0$$

$$= \frac{1}{s+1}\Big|_{s=j\omega} = \frac{1}{j\omega + 1}$$

$$Y(\omega) = \left[\frac{1}{j\omega} + \pi\,\delta(\omega)\right] \left(\frac{1}{j\omega + 1}\right)$$

6.6 Properties of Fourier Transform

(i) Modulation in time is equivalent to shifting in Frequency:

$$\mathcal{F}[f(t)e^{j\omega_0 t}] = F(\omega-\omega_0)$$

Proof:

$$\mathcal{F}[f(t)e^{j\omega_0 t}] = \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}dt = F(\omega-\omega_0)$$

#

(ii) Time-shifting Property:

$$\mathcal{F}[f(t-t_0)] = e^{-j\omega t_0}F(\omega)$$

Proof:

$$\mathcal{F}[f(t-t_0)] = \int_{-\infty}^{\infty} f(t-t_0)e^{-j\omega t}dt$$

$t' = t - t_0$

$$= \int_{-\infty}^{\infty} f(t')e^{-j\omega(t'+t_0)}dt'$$

$$= e^{-j\omega t_0}\underbrace{\int_{-\infty}^{\infty} f(t')e^{-j\omega t'}dt'}_{F(\omega)}$$

$$= e^{-j\omega t_0}F(\omega)$$

#

(iii) Duality Property:

If $F(\omega) = \mathcal{F}[f(t)]$

then $f(-\omega) = \frac{1}{2\pi}\mathcal{F}[F(t)]$

Proof:
$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

change t to x, change ω to t'

$$F(t') = \int_{-\infty}^{\infty} f(x)\, e^{-jxt'}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi f(x)\, e^{-jxt'}\, dx$$

let $\omega' = -x$

$$F(t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\, f(-\omega')\, e^{j\omega' t'}\, d\omega'$$

$$\Rightarrow \quad F(t') = \mathcal{F}^{-1}[\,2\pi f(-\omega')\,]$$

$$\text{or} \quad \mathcal{F}[F(t')] = 2\pi f(-\omega') \quad \#$$

(iv) Time-Scaling Property:

If $F(\omega) = \mathcal{F}[f(t)]$

then $\mathcal{F}[f(\alpha t)] = \frac{1}{|\alpha|} F\left(\frac{\omega}{\alpha}\right)$

Proof: $\mathcal{F}[f(\alpha t)] = \int_{-\infty}^{\infty} f(\alpha t)\, e^{-j\omega t}\, dt$

$$\overset{t' = \alpha t}{=} \begin{cases} \frac{1}{\alpha}\int_{-\infty}^{\infty} f(t')\, e^{-j\frac{\omega}{\alpha}t'}\, dt', & \alpha > 0 \\ -\frac{1}{\alpha}\int_{-\infty}^{\infty} f(t')\, e^{-j\frac{\omega}{\alpha}t'}\, dt', & \alpha < 0 \end{cases}$$

$$= \frac{1}{|\alpha|} F\left(\frac{\omega}{\alpha}\right) \quad \#$$

(V) Complex convolution:

$$\mathcal{F}[f_1(t) f_2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\beta) F_2(\omega-\beta)\, d\beta$$

where $F_1(\omega) = \mathcal{F}[f_1(t)]$ and $F_2(\omega) = \mathcal{F}[f_2(t)]$

Proof: $\mathcal{F}[f_1(t) f_2(t)]$

$$= \int_{-\infty}^{\infty} f_1(t) f_2(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} f_1(t) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(\beta) e^{j\beta t} d\beta\right] e^{-j\omega t} dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} f_1(t) e^{-j(\omega-\beta)t} dt}_{F_1(\omega-\beta)}\, F_2(\beta)\, d\beta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega-\beta) F_2(\beta)\, d\beta$$

Since $\mathcal{F}[f_1(t) f_2(t)] = \mathcal{F}[f_2(t) f_1(t)]$

$\mathcal{F}[f_1(t) f_2(t)] = \mathcal{F}[f_2(t) f_1(t)]$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega-\beta) F_2(\beta)\, d\beta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\beta) F_2(\omega-\beta)\, d\beta \quad \#$$

Example: What is the energy

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt \quad \text{for } f(t)$$

$$= \frac{-2jt}{4+t^2} \ ?$$

Answer: Since $\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$

$$\mathcal{F}[f(t)] = F(\omega)$$

From perivious example,

$$f(t) = \begin{cases} e^{-2t}, & t > 0 \\ -e^{2t}, & t < 0 \end{cases} \quad \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}}$$

$$F(\omega) = \frac{-2j\omega}{4+\omega^2}$$

According to Property (iii),

$$\mathcal{F}[F(t)] = \mathcal{F}\left[\frac{-2jt}{4+t^2}\right] = 2\pi f(-\omega)$$

$$= \begin{cases} 2\pi e^{2\omega}, & \omega > 0 \\ -2\pi e^{-2\omega}, & \omega < 0 \end{cases}$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

$$= 2\pi \int_{0}^{\infty} e^{4\omega} d\omega + 2\pi \int_{-\infty}^{0} e^{-4\omega} d\omega$$

$$\rightarrow \infty$$