Chapter 5 The z-Transform and Discrete-Time System Analysis

5.1 An LTIL system can be described as a difference equation. For example,

$$2y[k] + 3y[k-1] + y[k-2] = u[k] + u[k-1] - u[k-2]$$

Such a difference equation can be z-transformed into a rational function of z.

5.2 Z-transform

Consider a discrete-time signal $f[k] := f(kT)$, where k is an integer ranging from $(-\infty, \infty)$.

Definition: Consider a discrete-time sequence $f[k] := f[kT]$ defined for all integer k and $T > 0$, The z-transform is defined as

$$F(z) := \mathcal{Z}[f[k]] := \sum_{k=0}^{\infty} f[k] z^{-k}$$

and the inverse z-transform is

$$f[k] := \mathcal{Z}^{-1}[F(z)] := \frac{1}{2\pi j} \oint F(z)\, z^{k-1}\, dz$$

for $k \geq 0$,

where z is a complex variable, called the z-transform variable. the small circle on the integral symbol is a contour discussed later.

The z-transform is therefore defined as an infinite power series of z^{-1}! It can converge to a rational function of z usually. If such a rational function exists in a specific region on the z-plane, that region is called the region of convergence (ROC)

Example: Determine the z-transform for $f[k] = b^k$.

Answer: $F(z) = \sum_{k=0}^{\infty} f[k]\, z^{-k} = \sum_{k=0}^{\infty} b^k z^{-k}$

$$= \sum_{k=0}^{\infty} (bz^{-1})^k = \frac{1}{1 - bz^{-1}} \quad \text{if } |bz^{-1}| < 1$$

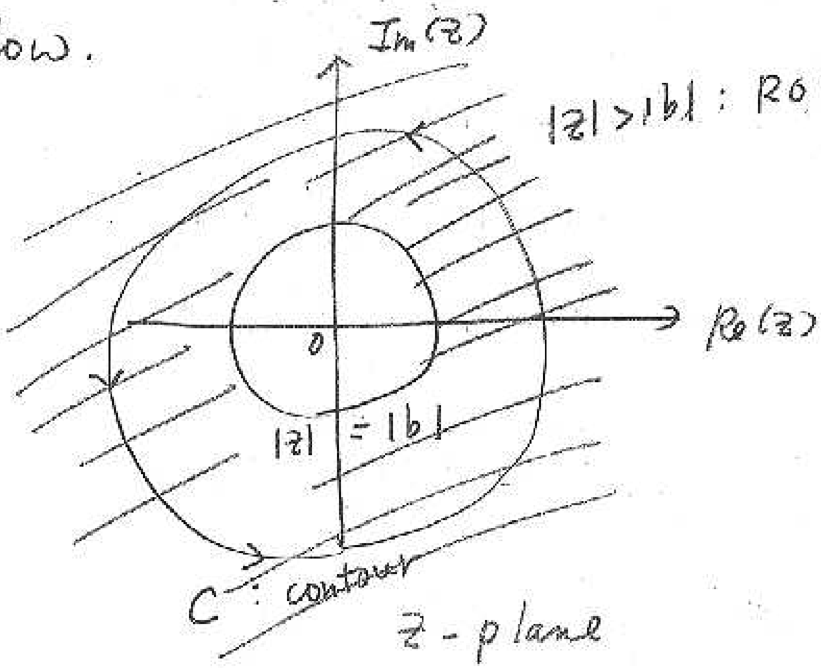
Hence $F(z)$ exists when $|bz^{-1}|<1$.

$|bz^{-1}|<1 \Rightarrow |z|>|b|$ is the ROC of

$$F(z) = \frac{1}{1-bz^{-1}} = \frac{z}{z-b}.$$

※ Whenever $F(z)$ is computed in a rational function form, its ROC has to be included!!

The ROC of previous example is depicted as below.



The contour C has to be a close path counterclockwise within ROC for the inverse Z-transform!!

$\mathcal{Z}$-transforms for typical sequences

※ Unit step sequence:

$$q[k] = \begin{cases} 1, & k = 0, 1, 2, \cdots \\ 0, & k < 0 \end{cases}$$

$$Q(z) := \mathcal{Z}[q[k]] = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

and ROC: $|z^{-1}| < 1 \Rightarrow |z| > 1$

※ Exponential sequence:

$$f[k] = e^{akT}$$

$$F(z) = \mathcal{Z}[e^{akT}] = \sum_{k=0}^{\infty} e^{akT} z^{-k} = \frac{1}{1-e^{aT}z^{-1}}$$

$$= \frac{z}{z-e^{aT}}, \quad \text{and ROC}: |e^{aT}z^{-1}| < 1$$

$$\Rightarrow |z| > |e^{aT}|$$

※ Kronecker delta sequence:

$$\delta[k] = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

$$\Delta[k] = \mathcal{Z}[\delta[k]] = \sum_{k=0}^{\infty} \delta[k] z^{-k} = 1, \text{ and}$$

ROC: the entire z-plane.

Similarly, $\mathcal{Z}[\delta[k-i]] = z^{-i}$ and ROC: the entire z-plane.

Example: Find the z-transforms of the following:

(a) $f[k] = 2^k$ (b) $f[k] = (-2)^k$ (c) $f[k] = \delta[k-2] - 2\delta[k-5]$

Answer:

(a) $$F(z) = \sum_{k=0}^{\infty} f[k] z^{-k}$$

$$= \sum_{k=0}^{\infty} 2^k z^{-k} = \frac{1}{1-2z^{-1}} = \frac{z}{z-2}$$

ROC: $|2z^{-1}| < 1 \Rightarrow |z| > 2$

(b) $$F(z) = \sum_{k=0}^{\infty} f[k] z^{-k}$$

$$= \sum_{k=0}^{\infty} (-2)^k z^{-k} = \frac{1}{1+2z^{-1}} = \frac{z}{z+2}$$

ROC: $|2z^{-1}| < 1 \Rightarrow |z| > 2$

(c) $$F(z) = \sum_{k=0}^{\infty} f[k] z^{-k}$$

$$= \sum_{k=0}^{\infty} \delta[k-2] z^{-k} - 2\sum_{k=0}^{\infty} \delta[k-5] z^{-k}$$

$$= z^{-2} - 2z^{-5} = \frac{z^3-2}{z^5}$$

ROC: entire z-plane except $z = 0$

5.2.2 From Laplace Transform to the Z-transform.

A sampling function $\sum_{k=0}^{\infty} \delta(t-kT)$ is a continuous-time function having an impulse at each sampling point $t = kT$.

The sampled version of a continuous-time function $f(t)$ is defined as

$$f_s(t) := f(t) \sum_{k=0}^{\infty} \delta(t-kT)$$
$$= \sum_{k=0}^{\infty} f(kT)\, \delta(t-kT)$$

The Laplace Transform of $f_s(t)$ is

$$F_s(s) = \mathcal{L}[f_s(t)]$$
$$= \sum_{k=0}^{\infty} f(kT)\, \mathcal{L}[\delta(t-kT)]$$
$$= \sum_{k=0}^{\infty} f(kT)\, e^{-kTs}$$

If we define $z = e^{Ts}$, then

$$F_s(s)\Big|_{z=e^{Ts}} = \sum_{k=0}^{\infty} f(kT)\, z^{-k} = \mathcal{Z}[f(kT)]$$

or $s = \frac{1}{T}\ln(z)$

$$z = re^{j\theta} = e^{Ts} = e^{T(\sigma + j\omega)},$$

where $\sigma = Re\{s\}$, and $\omega = Im\{s\}$

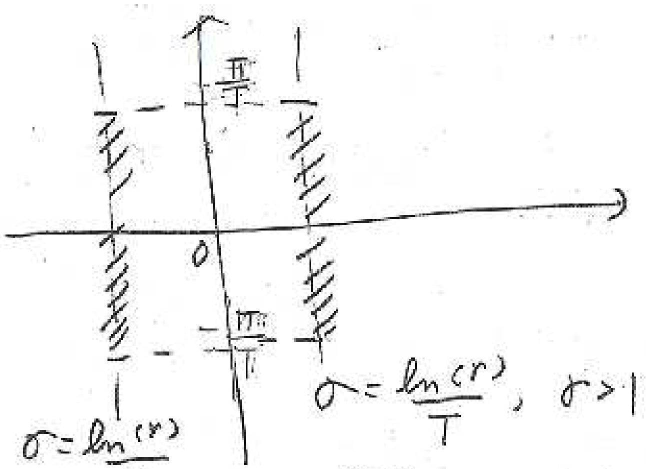
$$z = re^{j\theta} = e^{T\sigma} e^{j\omega T}$$

$\therefore \quad r = e^{T\sigma}$ and $\theta = \omega T$, $\sigma = \dfrac{\ln(r)}{T}$, $\omega = \dfrac{\theta}{T}$

$\therefore \quad |z| = r = e^{\sigma T}$

and $\measuredangle z = \theta = \omega T$

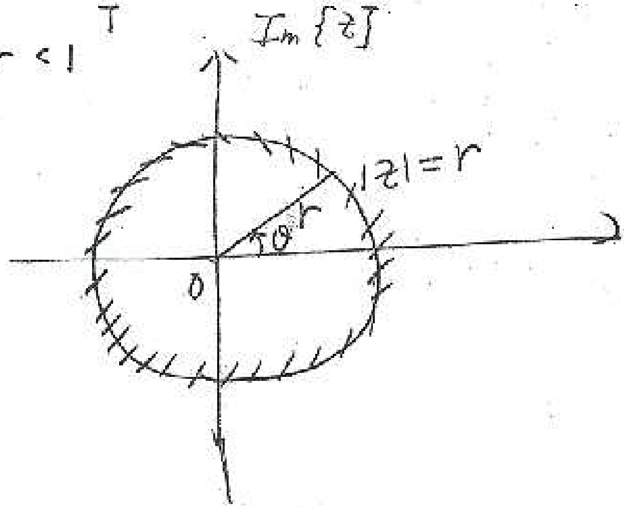
$Im\{s\} = \omega$



$Re\{s\} = \sigma$

s-plane

$Im\{z\}$



$Re\{z\}$

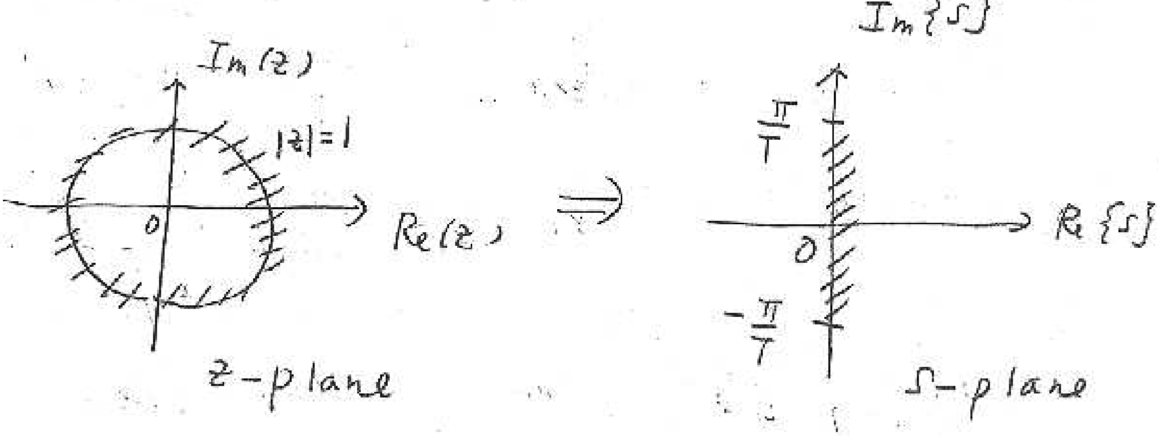
z-plane

Example: plot the corresponding region in the s-plane for $|z|<1$ in the z-plane

Answer: $|z| = 1 \Rightarrow r = 1$

$$\sigma = Re\{s\} = \frac{\ln(1)}{T} = 0$$

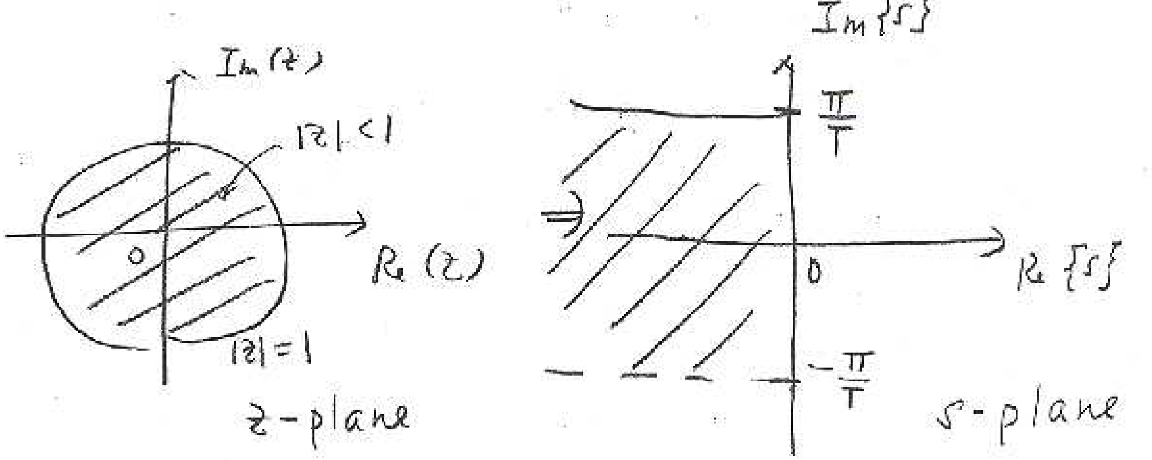
$$\omega = \frac{\theta}{T}, \quad -\pi < \theta \le \pi$$



$|z| < 1 \Rightarrow r < 1$

$$\sigma = Re\{s\} = \frac{\ln(r)}{T} < 0$$

$$\omega = \frac{\theta}{T}, \quad -\pi < \theta \le \pi$$



5.3 Properties of Z-Transform

(i) Linearity: If $F_i(z) = \mathcal{Z}[f_i[k]]$, $i = 1, 2$

then for any complex constants α_1 and α_2,

$$\mathcal{Z}[\alpha_1 f_1[k] + \alpha_2 f_2[k]]$$
$$= \alpha_1 \mathcal{Z}[f_1[k]] + \alpha_2 \mathcal{Z}[f_2[k]]$$
$$= \alpha_1 F_1(z) + \alpha_2 F_2(z), \quad \text{ROC is } \underset{(\text{for } F_1(z))}{ROC_1} \cap \underset{(\text{for } F_2(z))}{ROC_2}$$

Example: Determine $\mathcal{Z}[\sin(k\omega_0 T)]$.

Answer: $$\sin(k\omega_0 T) = \frac{e^{jk\omega_0 T} - e^{-jk\omega_0 T}}{2j}$$

Using the result in the previous example,

$$\mathcal{Z}[e^{\pm jk\omega_0 T}] = \mathcal{Z}[(e^{\pm j\omega_0 T})^k]$$

$$= \frac{z}{z - e^{\pm j\omega_0 T}} = \frac{1}{1 - e^{\pm j\omega_0 T} z^{-1}},$$

ROC: both are $|e^{\pm j\omega_0 T} z^{-1}| < 1$

$$\Rightarrow |z| > 1$$

Therefore, we have

$$\mathcal{Z}\left[\sin(k\omega_0 T)\right] = \frac{1}{2j}\left[\frac{z}{z-e^{j\omega_0 T}} - \frac{z}{z-e^{-j\omega_0 T}}\right]$$

$$= \frac{(e^{j\omega_0 T} - e^{-j\omega_0 T})z}{2j\left[z^2 - (e^{j\omega_0 T} + e^{-j\omega_0 T})z + 1\right]}$$

$$= \frac{\sin(\omega_0 T)z}{z^2 - 2\cos(\omega_0 T)z + 1}, \quad \text{ROC: } |z|>1 \cap |z|>1$$

$$\Rightarrow |z|>1$$

(ii) Multiplication by k in the time domain:

Let $F(z) = \mathcal{Z}[f[k]]$. Then

$$\mathcal{Z}[kf[k]] = -z\frac{dF(z)}{dz},$$ ROC stays the same as that for $F(z)$

Proof: $$F(z) = \sum_{k=0}^{\infty} f[k]z^{-k}$$

$$\frac{dF(z)}{dz} = \sum_{k=0}^{\infty} -k f[k] z^{-k-1}$$

$$-z\frac{dF(z)}{dz} = \sum_{k=0}^{\infty} k f[k] z^{-k}$$

$$= \mathcal{Z}[kf[k]]$$

Example: Determine $\mathcal{Z}[kb^k]$

Answer: Assume $F(z) = \mathcal{Z}[b^k] = \dfrac{z}{z-b}$

$$\mathcal{Z}[kb^k] = -z\frac{d}{dz}\left(\frac{z}{z-b}\right) = \frac{bz}{(z-b)^2}$$

(iii)

Multiplication by a^k in the time domain:

Let $F(z) = \mathcal{Z}[f[k]]$. Then

$$\mathcal{Z}[a^k f[k]] = F\left(\frac{z}{a}\right)$$

Proof:

$$\mathcal{Z}[a^k f[k]] = \sum_{k=0}^{\infty} f[k]\, a^k z^{-k}$$

$$= \sum_{k=0}^{\infty} f[k] \left(\frac{z}{a}\right)^{-k} = F\left(\frac{z}{a}\right)$$

Example: Determine $\mathcal{Z}[\cos(k\omega_0 T)\, f[k]]$.

Answer:

$$\mathcal{Z}[\cos(k\omega_0 T)\, f[k]]$$

$$= \mathcal{Z}\left[\left\{\frac{1}{2} e^{jk\omega_0 T} + \frac{1}{2} e^{-jk\omega_0 T}\right\} f[k]\right]$$

$$= \frac{1}{2}\mathcal{Z}\left[e^{jk\omega_0 T} f[k]\right] + \frac{1}{2}\mathcal{Z}\left[e^{-jk\omega_0 T} f[k]\right]$$

$$= \frac{1}{2} F\left(\frac{z}{e^{j\omega_0 T}}\right) + \frac{1}{2} F\left(\frac{z}{e^{-j\omega_0 T}}\right)$$

$$= \frac{1}{2} F\left(z e^{-j\omega_0 T}\right) + \frac{1}{2} F\left(z e^{j\omega_0 T}\right)$$

(iv) Time delay (right shift):

Let $\mathcal{Z}[f[k]] = F(z)$. Then

$$\mathcal{Z}[f[k-i]] = z^{-i}F(z) + \sum_{n=1}^{i} f[-n]\, z^{-i+n}$$

Proof: $\mathcal{Z}[f[k-i]]$

$$= \sum_{k=0}^{\infty} f[k-i]\, z^{-k}, \quad m = k-i$$

$$= \sum_{m=-i}^{\infty} f[m]\, z^{-m-i}$$

$$= \sum_{m=-i}^{-1} f[m]\, z^{-m-i} + \sum_{m=0}^{\infty} z^{-i} f[m]\, z^{-m}$$

$$\overset{n=-m}{=} \sum_{n=1}^{i} f[-n]\, z^{n-i} + z^{-i} \underbrace{\sum_{m=0}^{\infty} f[m]\, z^{-m}}_{F(z)}$$

$$= z^{-i}F(z) + \sum_{n=1}^{i} f[-n]\, z^{-i+n}$$

(v) Time advance (left shift):

Let $\mathcal{Z}[f[k]] = F(z)$. Then

$$\mathcal{Z}[f[k+i]] = z^{i}\left(F(z) - \sum_{n=0}^{i-1} f[n]\, z^{-n}\right)$$

Proof: $\mathcal{Z}[f[k+i]] = \sum_{k=0}^{\infty} f[k+i]\, z^{-k}$

$$\overset{m=k+i}{=} \sum_{m=i}^{\infty} f[m]\, z^{-m+i} = \sum_{m=0}^{\infty} f[m]\, z^{-m+i}$$

$$- \sum_{m=0}^{i-1} f[m]\, z^{-m+i} = z^{i}F(z) - \sum_{n=0}^{i-1} f[n]\, z^{-n+i}$$

$\boxed{\text{Table 5.1 lists all z-transform pairs!!}}$

(vi) Discrete Convolution: If $y[k] = \sum_{i=0}^{k} h[k-i]\,u[i]$

$= \sum_{i=0}^{k} h[i]\,u[k-i]$, then $Y(z) = H(z)\,U(z)$

where
$$U(z) = \mathcal{Z}[u[k]]$$
$$H(z) = \mathcal{Z}[h[k]]$$
$$Y(z) = \mathcal{Z}[y[k]].$$

Proof:

$$Y(z) = \mathcal{Z}[y[k]]$$
$$= \mathcal{Z}\left[\sum_{i=0}^{k} h[k-i]\,u[i]\right]$$
$$= \sum_{k=0}^{\infty} \sum_{i=0}^{k} h[k-i]\,u[i]\,z^{-k}$$

since it is causal

$$= \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} h[k-i]\,z^{-(k-i)}\,u[i]\,z^{-i}$$
$$= \sum_{i=0}^{\infty} u[i]\,z^{-i} \sum_{k=0}^{\infty} h[k-i]\,z^{-(k-i)}$$
$$= U(z) \sum_{m=-i}^{\infty} h[m]\,z^{-m}$$

$m = k-i$

$$= U(z) \sum_{m=0}^{\infty} h[m]\,z^{-m}$$
$$= U(z)\,H(z)$$

Example: If $u[k] = h[k] = \delta[k]$, and $y[k] = u[k] \circledast h[k]$, then determine $Y(z)$.

Answer: $Y(z) = H(z)\,U(z)$

$$U(z) = \sum_{k=0}^{\infty} \delta[k] z^{-k} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$$= H(z)$$

$$\therefore\ Y(z) = U(z)H(z) = \left(\frac{z}{z-1}\right)^2$$

$$= \frac{z^2}{(z-1)^2}$$

$$\boxed{\text{Table 5.2 lists all } z\text{-transform Properties}}$$

5.4 Inverse Z-transform - Direct Division

Example: Compute the inverse - Z-transform Using Direct Division for

$$F(z) = \frac{3}{z^2 - z - 2}$$

$$
\begin{array}{r l}
 & 3z^{-2} + 3z^{-3} + 9z^{-4} + \cdots \\
z^{2} - z - 2 \,\big) & 3 \\
 & 3 - 3z^{-1} - 6z^{-2} \\
\hline
 & 3z^{-1} + 6z^{-2} \\
 & 3z^{-1} - 3z^{-2} - 6z^{-3} \\
\hline
 & 9z^{-2} + 6z^{-3} \\
 & 9z^{-2} - 9z^{-3} - 18z^{-4} \\
\hline
 & 15z^{-3} + 18z^{-4}
\end{array}
$$

$$\therefore\ F(z) = 0 + 0z^{-1} + 3z^{-2} + 3z^{-3} + 9z^{-4} + \cdots$$

$$f[0] \quad f[1] \quad f[2] \quad f[3] \quad f[4] \quad \cdots$$

$$\Rightarrow f[k] = 3\delta[k-2] + 3\delta[k-3] + 9\delta[k-4] + \cdots$$

The disadvantage of Direct Division is that only a subset of the sequence $f[k]$ can be derived.

5.4.1 Partial-Fraction Expansion and Table
5.4.2 lookup

Similar to Partial-Fraction Expansion in Laplace Transforms, Any rational z-transform $F(z)$ can be broken into three parts, namely, improper $F_{im}(z) = \sum_{k=-m}^{-1} f[k] z^{-k}$, biproper $F_{bi}(z) = k_0$ and strictly proper $F_{st}(z)$.

The inverse Z-transform of $F(z)$ can be determined as

$$f[k] = Z^{-1}[F(z)] = \begin{cases} f[k], & -m \le k \le -1 \\ k_0 \delta[k], & k = 0 \\ f_{st}[k] = Z^{-1}[F_{st}(z)], & k > 0 \end{cases}$$

We can determine $f_{st}[k] = Z^{-1}[F_{st}(z)]$, $k > 0$ through the Partial-Fraction Expansion Technique. $\frac{F_{st}(z)}{z}$ can be factorized as $\frac{N(z)}{D(z)}$ where $N(z)$ and $D(z)$ are coprime. Then the poles of $\frac{F_{st}(z)}{z}$ are the roots of $D(z) = 0$.

Assume $D(z)$ can be factorized as

$$D(z) = \prod_{i=1}^{M} (z - p_i)^{m_i}$$

Example: $$F_{st}(z) = \frac{\cancel{(z-1)}}{\cancel{(z-1)}(z-2)^2(z-3)}$$

$$\frac{F_{st}(z)}{z} = \frac{1}{z(z-2)^2(z-3)}$$

$$\Rightarrow D(z) = z(z-2)^2(z-3)$$

$$\Rightarrow \begin{cases} p_1 = 0, & m_1 = 1 \\ p_2 = 2, & m_2 = 2 \\ p_3 = 3, & m_3 = 1 \end{cases}, \quad M = 3$$

Once we obtain p_i's and m_i's, we can write $\frac{F_{st}(z)}{z}$ as $\frac{F_{st}(z)}{z} = \sum_{i=1}^{M} \sum_{l=1}^{m_i} \frac{C_{il}}{(z-p_i)^l}$

where $$C_{il} = \frac{1}{(m_i - l)!} \frac{d^{m_i - l}}{dz^{m_i - l}} \left[\frac{F_{st}(z)}{z} (z - p_i)^{m_i} \right] \Bigg|_{z=p_i}$$

Then $F_{st}(z) = \sum_{i=1}^{M} \sum_{l=1}^{m_i} \frac{C_{il} z}{(z-p_i)^l}$ and each term $\frac{C_{il} z}{(z-p_i)^l}$ can be inverse z-transformed to generate $f_{st}[k]$, $k > 0$ totally by looking up into Table 5.1

Example: Determine the inverse z-transform for $F_{st}(z) = \dfrac{(z-1)}{(z-1)(z-2)^2(z-3)}$

Answer:

$$\frac{F_{st}(z)}{z} = \frac{1}{z(z-2)^2(z-3)}$$

$$= \frac{C_{11}}{z} + \frac{C_{21}}{(z-2)} + \frac{C_{22}}{(z-2)^2} + \frac{C_{31}}{z-3}$$

$$C_{11} = \left[\frac{1}{(z-2)^2(z-3)}\right]\Bigg|_{z=0} = -\frac{1}{12}$$

$$C_{21} = \frac{1}{(2-1)!}\frac{d^{(2-1)}}{dz^{(2-1)}}\left[\frac{1}{z(z-3)}\right]\Bigg|_{z=2}$$

$$= \frac{d}{dz}\left[\frac{1}{z(z-3)}\right]\Bigg|_{z=2}$$

$$= \left[\frac{-z+3-z}{z^2(z-3)^2}\right]\Bigg|_{z=2} = \frac{-1}{4}$$

$$C_{22} = \frac{1}{(2-2)!}\frac{d^{(2-2)}}{dz^{(2-2)}}\left[\frac{1}{z(z-3)}\right]\Bigg|_{z=2}$$

$$= \left[\frac{1}{z(z-3)}\right]\Bigg|_{z=2} = -\frac{1}{2}$$

$$C_{31} = \left[\frac{1}{z(z-2)^2}\right]\Bigg|_{z=3} = \frac{1}{3}$$

$$\therefore \quad F_{st}(z) = -\frac{1}{12}\,\frac{\cancel{z}}{\cancel{z}} - \frac{1}{4}\,\frac{z}{z-2} - \frac{1}{2}\,\frac{z}{(z-2)^2} + \frac{1}{3}\,\frac{z}{z-3}$$

$$f_{st}[k] = \mathcal{Z}^{-1}\left[F_{st}(z)\right]$$

$$= -\frac{1}{12}\,\delta[k] - \frac{1}{4}\,(2^k) - \frac{1}{4}\,k\,2^k + \frac{1}{3}\,(3^k),$$

$k \geq 0$

5.5 LTIL Difference Equations

Any difference equation can be easily transformed into a rational Z-transform function. We need to always write the difference equations in a delayed form before we do the Z-transform.

Example: A LTIL system is characterized by the difference equation

$$2y[k] + 3y[k-1] + y[k-2]$$

$$= u[k] + u[k-1] - u[k-2]$$

Take Z-transforms at each term, we have (causal system)

unit step input

$$2Y(z) + 3\left(y[-1] + z^{-1}Y(z)\right) + \left(y[-2] + y[-1]z^{-1} + z^{-2}Y(z)\right) = U(z) + z^{-1}U(z) - z^{-2}U(z)$$

$$(2 + 3z^{-1} + z^{-2})\,Y(z) = -3y[-1] - y[-2] - y[-1]z^{-1} + (1 + z^{-1} - z^{-2})\,U(z)$$

Hence

$$Y(z) = \underbrace{\frac{-3y[-1] - y[-2] - y[-1]z^{-1}}{3 + 3z^{-1} + z^{-2}}}_{\text{zero-input response } Y_{zi}(z)} + \underbrace{\frac{1 + z^{-1} - z^{-2}}{3 + 3z^{-1} + z^{-2}}\,U(z)}_{\text{zero-state response } Y_{zs}(z)}$$

If the initial conditions are $y[-2] = -1$ and $y[-1] = 2$, and $U(z) = \mathcal{Z}[q[k]]$, then

$$Y(z) = \frac{-5 - 2z^{-1}}{2 + 3z^{-1} + z^{-2}} + \frac{1 + z^{-1} - z^{-2}}{2 + 3z^{-1} + z^{-2}}\,\frac{1}{1 - z^{-1}}$$

$$= \frac{-5z^2 - 2z}{2z^2 + 3z + 1} + \frac{z(z^2 + z - 1)}{(z-1)(2z^2 + 3z + 1)}$$

$$= \frac{-4z^3 + 4z^2 + z}{(z-1)(2z+1)(z+1)}$$

Example : Find the response of

$$y[k] + y[k-1] - 2y[k-2] = u[k-1] + 2u[k-2]$$ excited by $y[-1] = -\frac{1}{2}$

$y[-2] = \frac{1}{4}$ and $u[k] = 1$, $k \geq 0$

Answer : Take z - transform of each sequence in the difference equation; we have

$$Y(z) + (y[-1] + z^{-1}Y(z)) - 2(y[-2] + y[-1]z^{-1} + z^{-2}Y(z)) = z^{-1}U(z) + 2z^{-2}U(z)$$

$$(1 + z^{-1} - 2z^{-2})Y(z) = \frac{1}{2} + \frac{1}{2} - z^{-1} + (z^{-1} + 2z^{-2})\frac{1}{1-z^{-1}}$$

$$Y(z) = \frac{(1 - z^{-1})}{1 + z^{-1} - 2z^{-2}} + \frac{z^{-1} + 2z^{-2}}{(1+z^{-1}-2z^{-2})(1-z^{-1})}$$

$$= \frac{z^3 - z^2 + 3z}{(z-1)^2(z+2)}$$

※ Since the numerator doesn't have a constant term (z^0), we can directly do Partial fraction expansion of $\frac{Y(z)}{z}$ without further breaking $Y(z)$ into $Y_{0i}(z)$

and $Y_{st}(z)$ when $Y(z)$ is proper

$$\frac{Y(z)}{z} = \frac{z^2 - z + 3}{(z-1)^2 (z+2)}$$

$$= \frac{C_{11}}{z-1} + \frac{C_{12}}{(z-1)^2} + \frac{C_{21}}{z+2}$$

$$C_{11} = \frac{d}{dz}\left[\frac{z^2 - z + 3}{z+2}\right]\Bigg|_{z=1}$$

$$= \left[\frac{(2z-1)(z+2) - (z^2 - z + 3)}{(z+2)^2}\right]\Bigg|_{z=1}$$

$$= 0$$

$$C_{12} = \left[\frac{z^2 - z + 3}{z+2}\right]\Bigg|_{z=1} = 1$$

$$C_{21} = \left[\frac{z^2 - z + 3}{(z-1)^2}\right]\Bigg|_{z=-2} = 1$$

$$\therefore Y(z) = \frac{z}{(z-1)^2} + \frac{z}{z+2}$$

Look up table 5.1,

$$y[k] = \mathcal{Z}^{-1}[Y(z)]$$

$$= \mathcal{Z}^{-1}\left[\frac{z}{(z-1)^2}\right] + \mathcal{Z}^{-1}\left[\frac{z}{z+2}\right]$$

$$= k + (-2)^k, \quad k \geq 0$$

$$= (k + (-2)^k)\, q[k]$$

5.7 Zero-State Response - Transfer Function

The transfer function for a discrete-time system is defined as

$$H(z) = \left.\frac{Y(z)}{U(z)}\right|_{\text{initial conditions} = 0}$$

$$= \frac{Y_{zs}(z)}{U(z)}$$

Example: Determine the transfer function for a causal LTIL system:

$2y[k] + 3y[k-1] + y[k-2] = u[k] + u[k-1] - u[k-2]$. Then determine the impulse response $h[k]$. Is this system FIR or IIR?

Answer: Set $y[-1] = y[-2] = 0$, $u[-1] = u[-2] = 0$

Take Z-transform of each term in the difference equation.

$$2Y_{zs}(z) + 3z^{-1}Y_{zs}(z) + z^{-2}Y_{zs}(z) =$$

$$U(z) + z^{-1}U(z) - z^{-2}U(z)$$

$$H(z) = \frac{Y_{zs}(z)}{U(z)} = \frac{1+z^{-1}-z^{-2}}{2+3z^{-1}+z^{-2}} = \frac{z^2+z-1}{2z^2+3z+1} \quad \#$$

$$= \underbrace{\frac{1}{2}}_{H_{bi}(z)} + \underbrace{\frac{-\frac{1}{2}z - \frac{3}{2}}{2z^2+3z+1}}_{H_{st}(z)}$$

$$\frac{H_{st}(z)}{z} = \frac{-\frac{1}{2}z - \frac{3}{2}}{z(2z+1)(z+1)}$$

$$= \frac{C_{11}}{z} + \frac{C_{21}}{2z+1} + \frac{C_{31}}{z+1}$$

$$C_{11} = \left[\frac{-\frac{1}{2}z - \frac{3}{2}}{(2z+1)(z+1)}\right]\Bigg|_{z=0} = -\frac{3}{2}$$

$$C_{21} = \left[\frac{-\frac{1}{2}z - \frac{3}{2}}{z(z+1)}\right]\Bigg|_{z=-\frac{1}{2}} = 5$$

$$C_{31} = \left[\frac{-\frac{1}{2}z - \frac{3}{2}}{z(2z+1)}\right]\Bigg|_{z=-1} = -1$$

$$H_{st}(z) = -\frac{3}{2}\frac{z}{z} + 5\frac{z}{2z+1} - \frac{z}{z+1}$$

$$h[k] = \mathcal{Z}^{-1}[H(z)] = \frac{1}{2}\delta[k] + \mathcal{Z}^{-1}[H_{st}(z)]$$

$$= \frac{1}{2}\delta[k] - \frac{3}{2}\delta[k] + \frac{5}{2}\left(-\frac{1}{2}\right)^k - (-1)^k, \quad k \geq 0$$

$$= -\delta[k] + \frac{5}{2}\left(-\frac{1}{2}\right)^k - (-1)^k, \quad k \geq 0 \quad \#$$

$$\lim_{k\to\infty} h[k] = \begin{cases} 1, & k \text{ is odd} \\ -1, & k \text{ is even} \end{cases} \neq 0$$

$\Rightarrow$ $h[k]$ is an IIR #

5.7.1 Poles and zeros of proper rational transfer function

A causal LTIL system can always be described as a proper rational transfer function.

Consider a proper transfer function $H(z) = \frac{N(z)}{D(z)}$, where $N(z)$ and $D(z)$ are co-prime

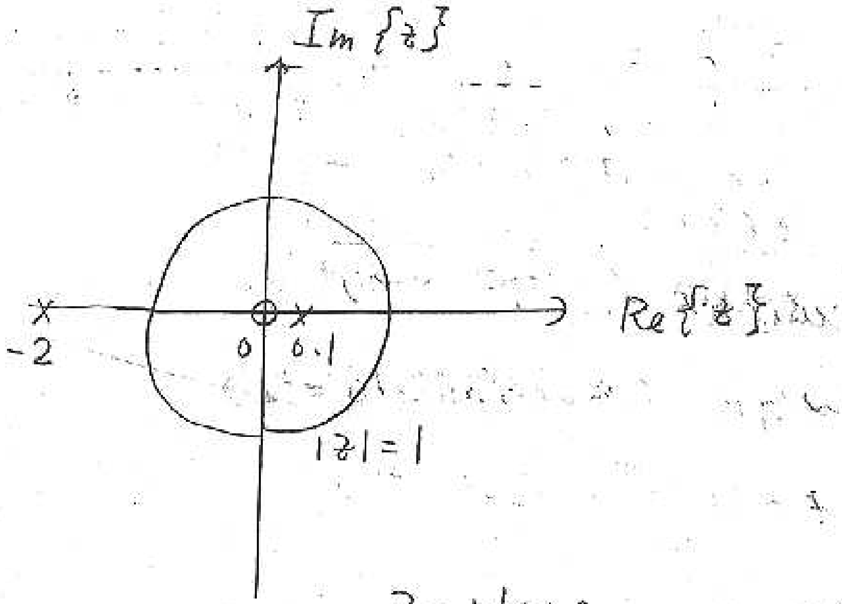
The zeros are defined as the roots of $N(z) = 0$ and the poles are defined as the roots of $D(z) = 0$.

Example: Plot the poles and zeros of $H(z) = \frac{z}{(z-0.1)(z+2)}$ on the z-plane

Answer:

poles: $z = 0.1$, $z = -2$ (denoted as crosses)

zeros: $z = 0$ (denoted as circles)



z-plane

Example: What is the transfer function for $y[k] + y[k-1] - 2y[k-2] = u[k-1] + 2u[k-2]$? What are the zeros, poles of the zero-state response $Y_{zs}(z)$ and the zero-input response $Y_{zi}(z)$? (input is $g[k]$)

Answer: From previous example,

$$Y(z) = \underbrace{\frac{1 - z^{-1}}{1 + z^{-1} - 2z^{-2}}}_{\text{zero-input response } Y_{zi}(z)} + \underbrace{\frac{z^{-1} + 2z^{-2}}{(1 + z^{-1} - 2z^{-2})(1 - z^{-1})}}_{\text{zero-state response } Y_{zs}(z)}$$

$$Y_{zi}(z) = \frac{1 - z^{-1}}{1 + z^{-1} - 2z^{-2}} = \frac{z^2 - z}{z^2 + z - 2} = \frac{z\cancel{(z-1)}}{(z+2)\cancel{(z-1)}} = \frac{z}{z+2}$$

poles: $z = -2$,

zeros: $z = 0$,

$$Y_{zs}(z) = \frac{z^{-1} + 2z^{-2}}{(1 + z^{-1} - 2z^{-2})(1 - z^{-1})} = \frac{z^2 + 2z}{(z^2 + z - 2)(z-1)}$$

$$= \frac{z(z+2)}{(z-1)^2(z+2)} = \frac{z}{(z-1)^2}$$

poles: $z = 1$ (multiplicity $= 2$)

zeros: $z = 0$

$$H(z) = \frac{Y_{zs}(z)}{U(z)} = \frac{\cancel{z}}{(z-1)^2} \cdot \frac{\cancel{z-1}}{\cancel{z}} = \frac{1}{z-1}$$

$$y[k] = \{k + (-2)^k\}\, g[k]$$

due to pole $z=1$ of $H(z)$ and pole $z=1$ of $u(z)$

due to pole of $Y_{zi}(z)$

5.7.2 Time responses of Modes and Poles

The response of LTIL difference equations can be decomposed as the zero-input and zero-state responses. The form of the zero-input response is determined by the modes (the poles of $Y_{zi}(z)$) and the form of the zero-state response is determined by the poles of $Y_{zs}(z)$.

The following conclusions can be made:

1. The time response of a pole (mode), simple or repeated, approaches to zero as $k \to \infty$ if and only if the pole (mode) lies inside the unit circle or its magnitude is smaller than 1.

2. The time response of a pole (mode), approaches a nonzero constant or an oscillating exponential with constant magnitude as $k \to \infty$ if and only if the pole (mode) is simple and located at $|z| = 1$.

3. The time response of a pole (mode) approaches $\pm\infty$ or an oscillating exponential with magnitude $\pm\infty$ as $k \to \infty$ if and only if the pole (mode) is repeated and located at $|z| = 1$.

Table 5.3 Time responses of poles and modes as $k \to \infty$

poles or modes location	simple	repeated
$p_i \in |z| < 1$	0	0
$|z| > 1$	$\pm\infty$	$\pm\infty$
$|z| = 1$	A constant or oscillating constant	$\pm\infty$ or oscillating $\pm\infty$

Example: An LTIL system can be described as $y[k] - 2y[k-1] + y[k-2] = u[k] + u[k-1]$; $y[-1] = \frac{1}{2}$ and $y[-2] = 1$, $u[k] = \delta[k]$.

What are the modes? What are the time response for $Y_{zi}(z)$ and $Y_{zs}(z)$ as $k \to \infty$?

Answer: $Y_{zs}(z) - 2z^{-1}Y_{zs}(z) + z^{-2}Y_{zs}(z) = U(z) - z^{-1}U(z)$

$$Y_{zs}(z) = \frac{1+z^{-1}}{1-2z^{-1}+z^{-2}} \cdot \frac{1}{1-z^{-1}}$$

$$\frac{z^2}{(z-1)^2}$$

Since the poles of $Y_{zs}(z)$ are $z = 1, 1$, it belongs to the region $|z| = 1$, and hence $\lim_{k\to\infty} y_{zs}[k] = \pm\infty$ according to

Table 5.3

Verify it:

$$y_{zs}[k] = \mathcal{Z}^{-1}[Y_{zs}(z)] = \mathcal{Z}^{-1}\left[\frac{z^2}{(z-1)^2}\right]$$

$$= (k+1)\, q[k+1]$$

$$\lim_{k\to\infty} y_{zs}[k] = \infty$$

$$Y_{zi}(z) - 2\{y[-1] + z^{-1}Y_{zi}(z)\} + \{y[-2] + y[-1]z^{-1} + z^{-2}Y_{zi}(z)\} = 0$$

$$(1 - 2z^{-1} + z^{-2})Y_{zi}(z) = -\frac{1}{2}z^{-1}$$

$$Y_{zi}(z) = \frac{-\frac{1}{2}z}{z^2 - 2z + 1} = \frac{-\frac{1}{2}z}{(z-1)^2}$$

Since the poles of $Y_{zi}(z)$ (the modes) $z = 1, 1$ are $z = 1, 1$, it belongs to the region $|z| = 1$ and hence $\lim_{k\to\infty} y_{zi}[k] = \pm\infty$ according to Table 5.3.

Verify it:

$$y_{zi}[k] = \mathcal{Z}^{-1}[Y_{zi}(z)] = \mathcal{Z}^{-1}\left[\frac{-\frac{1}{2}z}{(z-1)^2}\right]$$

$$= -\frac{1}{2}k\, q[k]$$

$$\lim_{k \to \infty} y_{zi}[k] = -\infty$$

5.8 Transfer function Representation – Complete Characterization

Time domain

$$u[k] \longrightarrow \boxed{h[k]} \longrightarrow y[k] = \sum_{i=0}^{k} h[k-i]\,u[i]$$

Z-domain

$$U(z) \rightarrow \boxed{H(z)} \longrightarrow Y(z) = H(z)\,U(z)$$

Every LTIL system (causal input, causal impulse response) can be described as above, either through a discrete convolution in the time domain, or a product form in the Z-domain.

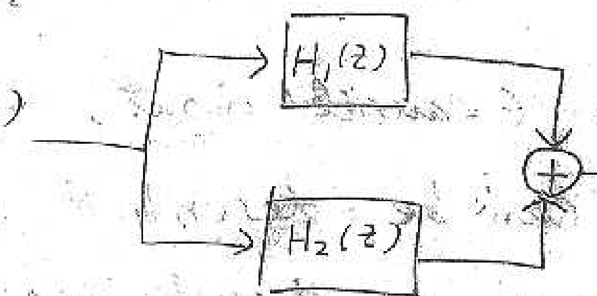
The advantage of using transfer functions can be observed through some typical block diagrams, which are difficult to obtain input/output relationships in the time domain.

Using transfer functions in the z-domain, we can easily derive the input / output relationships in the following figures.

(i) cascade connection of two systems

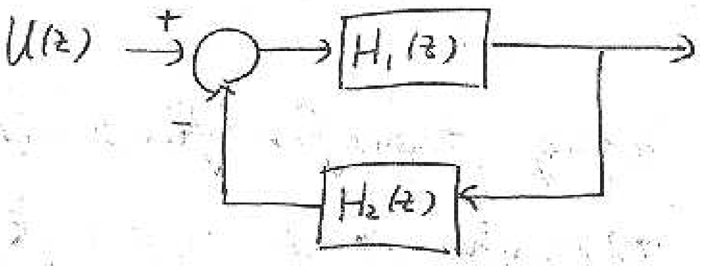
$$Y(z) = H_1(z) H_2(z) U(z)$$

(ii) parallel connection of two systems



$$Y(z) = \{H_1(z) + H_2(z)\} U(z)$$
$$= H_1(z) U(z) + H_2(z) U(z)$$

(iii) Negative feedback system

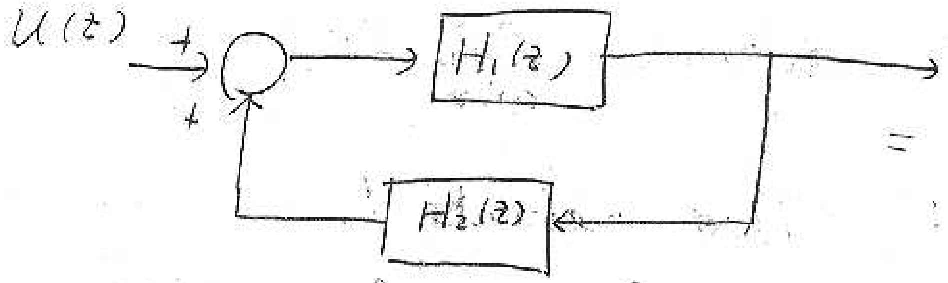


$$Y(z) = \frac{H_1(z)}{1 + H_1(z) H_2(z)} U(z)$$

$$Y(z) = H_1(z) U(z) - H_1(z) H_2(z) Y(z)$$

$$Y(z) = \frac{H_1(z) U(z)}{1 + H_1(z) H_2(z)}$$

(iv) positive feedback system



$$= \frac{H_1(z)}{1 - H_1(z) H_2(z)} U(z)$$

$$Y(z) = H_1(z)\, U(z) + H_1(z) H_2(z)\, Y(z)$$

$$Y(z) = \frac{H_1(z)\, U(z)}{1 - H_1(z) H_2(z)}$$

5.9 The Final-Value and Initial-Value Theorems

※ The Final-value Theorem: Let $F(z)$ be the z-transform of a positive-time sequence $f[k]$, $k \geq 0$ and let $F(z)$ be a proper rational function. If every pole of $(z-1)F(z)$ has a magnitude smaller than 1, or, equivalently, lies inside the unit circle of the z-plane, then $f[k]$ approaches a constant (zero or nonzero) and

$$\lim_{k \to \infty} f[k] = \lim_{z \to 1} (z-1) F(z)$$

Example: If $f[k] = \mathcal{Z}^{-1}\left[\dfrac{z}{(2z+1)(3z-1)} \right]$,

determine $\lim\limits_{k \to \infty} f[k]$.

Answer: Two methods !!

(i) Directly compute $f[k]$:

$$\frac{F(z)}{z} = \frac{1}{(2z+1)(3z-1)} \quad \text{is strictly proper}$$

$$\frac{F(z)}{z} = \frac{C_{11}}{2z+1} + \frac{C_{21}}{3z-1}$$

$$C_{11} = \left[\frac{1}{3z-1} \right] \bigg|_{z=-\frac{1}{2}} = -\frac{2}{5}$$

$$C_{21} = \left[\frac{1}{2z+1} \right] \bigg|_{z=\frac{1}{3}} = \frac{3}{5}$$

$$\therefore \quad F(z) = -\frac{2}{5}\,\frac{z}{2z+1} + \frac{3}{5}\,\frac{z}{3z-1}$$

$$f[k] = \mathcal{Z}^{-1}[F(z)]$$
$$= -\frac{1}{5}\left(-\frac{1}{2}\right)^k g[k] + \frac{1}{5}\left(\frac{1}{3}\right)^k g[k]$$

$$\therefore \quad \lim_{k \to \infty} f[k] = 0 + 0 = 0$$

(ii) Applying Final-value theorem:

The poles of $(z-1)F(z) = \dfrac{z(z-1)}{(2z+1)(3z-1)}$

are both inside the unit-circle.

Then $\lim_{k \to \infty} f[k] = \lim_{z \to 1} \dfrac{z(z-1)}{(2z+1)(3z-1)}$

$= 0$

※ The Initial-value Theorem: Let $F(z)$ be the z-transform of a positive-time sequence $f[k]$, $k \geq 0$ and let $F(z)$ be a proper rational function. Then we have

$$f[0] = \lim_{z \to \infty} F(z)$$

Example: Check the initial-value theorem for a sequence $q[k]$.

Answer:

$$Q(z) = \mathcal{Z}[q[k]]$$

$$= \frac{z}{z-1}$$

Since $Q(z)$ is proper,

$$q[0] = \lim_{z \to \infty} Q(z) = \lim_{z \to \infty} \frac{z}{z-1} = 1$$