Chapter 4 Laplace Transform

4.1 Consider an LTIL system described by the nth-order differential equation

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y^{(1)}(t) + a_0 y(t)$$
$$= b_m u^{(m)}(t) + \cdots + b_1 u^{(1)}(t) + b_0 u(t)$$

where $y^{(n)}(t) = \frac{d^n}{dt^n} y(t)$ and $u^{(m)}(t) = \frac{d^m}{dt^m} u(t)$.

How to solve $y(t)$?

Using Laplace Transform !!

4.2. Laplace Transform

Consider a function $f(t)$ defined for all t.

The Laplace Transform is defined as

$$F(s) := \mathcal{L}[f(t)] := \int_{0^-}^{\infty} f(t) e^{-st} dt$$

and

The Inverse Laplace Transform is defined

$$f(t) := \mathcal{L}^{-1}[F(s)] := \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} ds$$

for $t \geq 0$

where $j = \sqrt{-1}$ and c is a real number which belongs to the region where $F(s)$ exists

Example: Consider $f(t) = e^{2t}$. The Laplace Transform is

$$F(s) := \int_{0^-}^{\infty} e^{2t} e^{-st} \, dt$$

$$= \int_{0^-}^{\infty} e^{-(s-2)t} \, dt$$

$$= \frac{1}{-(s-2)} e^{-(s-2)t} \Big|_{t=0^-}^{\infty}$$

$$= \frac{-1}{s-2} \Big[e^{-(s-2)t} \Big]_{t=\infty} \underbrace{- e^{-(s-2)t} \Big|_{t=0^-}}_{1}$$

For what kind of s, does $F(s)$ exist? Let $s = \sigma + j\omega$

$$e^{-(s-2)t} = e^{-(\sigma-2)t} e^{-j\omega t}$$

$$= e^{-(\sigma-2)t} [\cos(\omega t) - j \sin(\omega t)]$$

$$= \begin{cases} \pm\infty, & \text{for } \sigma = \mathrm{Re}\{s\} < 2 \\ \text{undefined}, & \text{for } \sigma = \mathrm{Re}\{s\} = 2 \\ 0, & \text{for } \sigma = \mathrm{Re}\{s\} > 2 \end{cases}$$

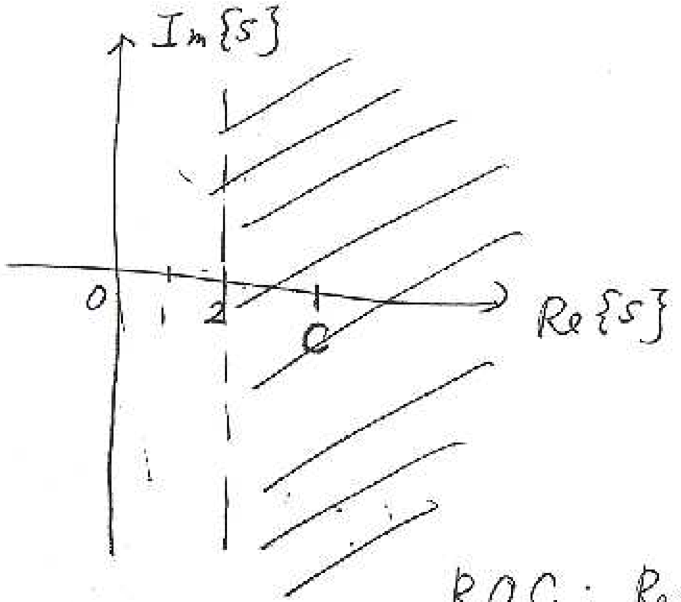
Hence $F(s)$ exists if $\sigma = \mathrm{Re}\{s\} > 2$

$$F(s) = \mathcal{L}\{e^{2t}\} = \frac{1}{s-2}, \quad \mathrm{Re}\{s\} > 2$$

The region for existing $F(s)$ is called the Region of Convergence (ROC). The ROC for the previous example is $Re\{s\} > 2$. Any Inverse Laplace Transform should be integrated in the ROC of $F(s)$!!! For instance, the inverse Laplace Transform for the previous example is

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{1}{s-2} e^{st}\, ds ,$$

where $c \in \{s \mid Re\{s\} > 2,\ s \text{ is complex}\}$



ROC: $Re\{s\} > 2$

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{1}{s-2} e^{st}\, ds = \begin{cases} e^{2t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$= e^{2t} g(t)$$

We can denote the Laplace Transform pair for previous example as

$$\begin{array}{ccc} e^{2t} g(t) & \longleftrightarrow & \dfrac{1}{s-2} \\ \| & & \| \\ f(t)\, g(t) & \longleftrightarrow & F(s) \end{array} \quad \text{or} \quad f(t) \longleftrightarrow F(s)$$

Example: What is Laplace Transform for $\delta(t)$?

Answer:

$$F(s) = \mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t)\, e^{-st}\, dt$$

$$= e^{-st}\Big|_{t=0^-} = 1$$

$$\text{or} \quad \begin{array}{ccc} \delta(t) & \longleftrightarrow & 1 \\ \| & & \| \\ f(t) & & F(s) \end{array}$$

Example: What is Laplace Transform for $f(t) = e^{-at}$ and its ROC?

Answer:

$$F(s) = \mathcal{L}\{e^{-at}\}$$

$$= \int_{0^-}^{\infty} e^{-at} e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} e^{-(s+a)t}\, dt$$

$$= -\frac{1}{s+a}\left[e^{-(s+a)t}\Big|_{t=\infty} - 1 \right]$$

$$e^{-(s+a)t}\Big|_{t=\infty} = 0 \quad \text{if} \quad Re\{s\} + Re\{a\} > 0$$

$\Rightarrow$ $Re\{s\} > -Re\{a\}$ is its ROC.

$$\Rightarrow \quad F(s) = \frac{-1}{s+a}\left[\,0 \quad -1\,\right]$$

$$= \frac{1}{s+a}, \quad \text{if} \quad Re\{s\} > -Re\{a\}$$

Example: $\mathcal{L}\{g(t)\}$

$$= \int_{0^-}^{\infty} e^{-at}\Big|_{a=0} e^{-st}\,dt$$

$$= \frac{1}{s}, \quad \text{if} \quad Re\{s\} > 0$$

Example: $\mathcal{L}\{e^{-j\omega t}\}$

$$= \int_{0^-}^{\infty} e^{-at}\Big|_{a=j\omega} e^{-st}\,dt$$

$$= \frac{1}{s+j\omega}, \quad \text{if} \quad Re\{s\} > 0$$

Hence; we obtain two typical Laplace Transform pairs:

$$g(t) \leftrightarrow \frac{1}{s}$$

$$e^{-j\omega t} \leftrightarrow \frac{1}{s+j\omega}$$

4.2.1 Region of Convergence (ROC)

If $F(s) = \mathcal{L}[e^{-at}]$, the ROC is $Re\{s\} > -Re\{a\}$ as aformentioned.

4.3 Properties of Laplace Transform

The Laplace Transform domain is often refered to as the s-domain since the Laplace Transform is a function $F(s)$.

(i) Linearity: Laplace Transform is a linear operator, that is, if

$$F_1(s) = \mathcal{L}[f_1(t)], \quad F_2(s) = \mathcal{L}[f_2(t)]$$

then, for any complex constant α_1, α_2

$$\mathcal{L}[\alpha_1 f_1(t) + \alpha_2 f_2(t)] = \alpha_1 F_1(s) + \alpha_2 F_2(s)$$

※ The overall ROC = ROC for $F_1(s)$ $\cap$ ROC for $F_2(s)$.

Example: What is the Laplace Transform for $f(t) = \sin(\omega_0 t)$?

Answer:
$$\sin(\omega_0 t) = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$

$$e^{j\omega_0 t} \longleftrightarrow \frac{1}{s-j\omega_0}, \quad ROC: Re\{s\} > 0$$

$$e^{-j\omega_0 t} \longleftrightarrow \frac{1}{s+j\omega_0}, \quad ROC: Re\{s\} > 0$$

※ The overall ROC is $Re\{s\} > 0 \cap Re\{s\} > 0$

$\Rightarrow Re\{s\} > 0$

$$\mathcal{L}\{\sin(\omega_0 t)\} = \frac{1}{2j}\left\{\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right\}$$

$$= \frac{1}{2j}\,\frac{s+j\omega_0 - s+j\omega_0}{s^2+\omega_0^2}$$

$$= \frac{\omega_0}{s^2+\omega_0^2}, \quad Re\{s\} > 0$$

(ii) Shifting in the s-domain (Multiplication by e^{-at} in the time-domain):

If $\mathcal{L}[f(t)] = F(s)$, then

$\mathcal{L}[e^{-at}f(t)] = F(s+a)$, ROC needs to be shifted by $-a$!!

proof: $$\mathcal{L}[e^{-at}f(t)] = \int_{0^-}^{\infty} e^{-at} f(t)\, e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} f(t)\, e^{-(s+a)t}\, dt = F(s+a)$$

Example: Find the Laplace Transform for

$f(t) = e^{3t}\sin\left(t - \frac{\pi}{6}\right)$

Answer :

$$f(t) = e^{3t}\sin(t)\cos(\tfrac{\pi}{6}) - e^{3t}\sin(\tfrac{\pi}{6})\cos(t)$$

$$= \frac{\sqrt{3}}{2} e^{3t}\sin(t) - \frac{1}{2} e^{3t}\cos(t)$$

$$\mathcal{L}[\cos(t)] = \mathcal{L}\left[\frac{1}{2}(e^{jt} + e^{-jt})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-j} + \frac{1}{s+j}\right]$$

$$= \frac{1}{2}\,\frac{2s}{s^2+1} = \frac{s}{s^2+1}, \quad Re[s] > 0$$

$$\mathcal{L}[e^{3t}\cos(t)] = \frac{(s-3)}{(s-3)^2+1}, \quad Re\{s\} > 3$$

$\mathcal{L}[\sin(t)] = \frac{1}{s^2+1}, \quad Re[s] > 0$ from the previous example

$$\mathcal{L}[e^{3t}\sin(t)] = \frac{1}{(s-3)^2+1}, \quad Re\{s\} > 3$$

Hence the overall ROC is $Re\{s\} > 3 \cap Re\{s\} > 3$

$\Rightarrow Re\{s\} > 3$

$$\therefore \quad F(s) = \mathcal{L}[f(t)]$$

$$= \frac{\sqrt{3}}{2}\,\frac{1}{(s-3)^2+1} - \frac{1}{2}\,\frac{(s-3)}{(s-3)^2+1}, \quad Re\{s\} > 3$$

(iii) Differentiation in the s-domain (Multiplication by t in the time-domain):

If $\mathcal{L}[f(t)] = F(s)$, then

$\mathcal{L}[t f(t)] = -\frac{d}{ds} F(s)$, ROC usually stays the same as ROC for $F(s)$!!

Proof: $\frac{d}{ds} F(s) = \frac{d}{ds} \mathcal{L}[f(t)]$

$$= \frac{d}{ds} \int_{0^-}^{\infty} f(t) e^{-st} dt = \int_{0^-}^{\infty} f(t) \left[\frac{d}{ds} e^{-st}\right] dt$$

$$= -\int_{0^-}^{\infty} t \cdot f(t) e^{-st} dt$$

Example:

$$\mathcal{L}[t] = \mathcal{L}[t\, q(t)]$$

$$= -\frac{d}{ds} \mathcal{L}[q(t)] = -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

$$\mathcal{L}[t^2] = \mathcal{L}[t \cdot t\, q(t)]$$

$$= -\frac{d}{ds} \mathcal{L}[t\, q(t)] = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3}$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} = \frac{n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1}{s^{n+1}}$$

(iv) Differentiation in the Time-domain (Multiplication by s in the s-domain):

If $\mathcal{L}[f(t)] = F(s)$, then

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] =: \mathcal{L}[\dot{f}(t)] = s\mathcal{L}[f(t)] - f(0^-)$$
$$= sF(s) - f(0^-)$$

$$\mathcal{L}\left[\frac{d^2}{dt^2} f(t)\right] =: \mathcal{L}[\ddot{f}(t)] = s^2F(s) - sf(0^-) - \dot{f}(0^-)$$

proof: $\mathcal{L}\left[\frac{d}{dt} f(t)\right] = \int_{0^-}^{\infty} \left[\frac{d}{dt} f(t)\right] e^{-st} dt$

$$= f(t)e^{-st}\Big|_{t=0^-}^{t=\infty} - \int_{0^-}^{\infty} f(t) \frac{d}{dt}[e^{-st}]dt$$

$$= \underbrace{f(t)e^{-st}\Big|_{t=\infty}}_{=0 \text{ for ROC}'} - f(0^-)$$

$$+ s \underbrace{\int_{0^-}^{\infty} f(t) e^{-st} dt}_{F(s)}$$

$$= \underbrace{sF(s)}_{\text{ROC for } F(s)} - f(0^-)$$

Hence the overall ROC is $\text{ROC}' \cap \text{ROC}$ for $F(s)$!!

However $F(s)$ exists if and only if

$$\lim_{t \to \infty} \left[f(t) e^{-st} \right] = 0 \text{, i.e., } ROC' = ROC \text{ for } F(s) \text{)}$$

Hence the overall ROC is the ROC for $F(s)$.

$$\mathcal{L}\left[\frac{d^2}{dt^2} f(t)\right] = \mathcal{L}\left[\frac{d}{dt}\frac{d}{dt} f(t)\right] = \mathcal{L}\left[\frac{d}{dt} \dot{f}(t)\right]$$

$$= s\,\mathcal{L}[\dot{f}(t)] - \dot{f}(0^-)$$

$$= s\,[sF(s) - f(0^-)] - \dot{f}(0^-)$$

$$= s^2 F(s) - s f(0^-) - \dot{f}(0^-),$$

where ROC is the ROC for $F(s)$.

In general, $\mathcal{L}\left[\frac{d^n}{dt^n} f(t)\right] =: \mathcal{L}[f^{(n)}(t)]$

$$= s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f^{(1)}(0^-)$$

$$- \cdots - f^{(n-1)}(0^-), \text{ where}$$

$$f^{(k)}(0^-) := \left. \frac{d^k}{dt^k} f(t) \right|_{t=0^-}$$

and ROC is the ROC for $F(s)$.

Example: Since $\delta(t) = \frac{d}{dt} q(t)$,

$$\mathcal{L}[q(t)] = \frac{1}{s},$$

$$\mathcal{L}[\delta(t)] = \mathcal{L}\left[\frac{d}{dt} q(t)\right]$$
$$= s\,\mathcal{L}[q(t)] - q(0^-)$$
$$= s \cdot \frac{1}{s} - 0 = 1$$

Usually, we just transform the "positive version" of a function into a Laplace Transform, and then we have $f^{(i)}(0^-) = 0$, for $i = 1, 2, \cdots$. It implies $\mathcal{L}\left[\frac{d^n}{dt^n}(f(t)q(t))\right]$ $= s^n F(s)$

(V) Integration in the time domain (Division by s in the s-domain):

If $\mathcal{L}[f(t)] = F(s)$, then

$\mathcal{L}\left[\int_{0^-}^{t} f(\tau)\, d\tau\right] = \frac{1}{s} F(s)$, ROC usually stays the same as that for $F(s)$.

Proof: Assume $g(t) := \int_{0^-}^{t} f(\tau)\, d\tau$ and

$$G(s) := \mathcal{L}[g(t)]$$

Then, $F(s) = \mathcal{L}[f(t)]$

$$= \mathcal{L}\left[\frac{d}{dt} g(t)\right]$$

$$= sG(s) - \underbrace{g(0^-)}_{=0} = sG(s)$$

$\therefore$ $G(s) = \frac{1}{s} F(s)$, ROC for $G(s)$ is usullay ROC for $F(s)$

In short, $\int_{0^-}^{t} \longleftrightarrow \frac{1}{s}$

Example: $\mathcal{L}\left[\int_0^t q(\tau)\, d\tau\right] = \mathcal{L}[t]$

$$= \frac{1}{s} \mathcal{L}[q(t)] = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

(vi) Convolution: The zero-state response for a LTI causal system is described as (causal input)

$$y(t) = \int_0^t h(t-\tau)\, u(\tau)\, d\tau$$

$$Y(s) = \mathcal{L}[y(t)] = \int_{0^-}^{\infty} y(t) e^{-st} dt$$

$$= \int_{0^-}^{\infty} \int_0^t h(t-\tau)\, u(\tau)\, d\tau\, e^{-st} dt$$

$$= \int_{0^-}^{\infty}\int_{0^-}^{\infty} h(t-\tau)\, u(\tau)\, d\tau\, e^{-st}\, dt$$

$$= \int_{0^-}^{\infty}\int_{0^-}^{\infty} h(t-\tau)\, e^{-s(t-\tau)}\, dt\;\; u(\tau)\, e^{-s\tau}\, d\tau$$

Let $x = t-\tau$, $dx = dt$

$$= \int_{0^-}^{\infty}\int_{-\tau}^{\infty} h(x)\, e^{-sx}\, dx\;\; u(\tau) e^{-s\tau}\, d\tau$$

$\tau \in [0^-, \infty)$

$$= \int_{0^-}^{\infty} \underbrace{\int_{0^-}^{\infty} h(x)\, e^{-sx}}_{H(s)}\, dx\;\; u(\tau)\, e^{-s\tau}\, d\tau$$

$$= H(s) \underbrace{\int_{0^-}^{\infty} u(\tau)\, e^{-s\tau}\, d\tau}_{U(s)}$$

$Y(s) = H(s)\, U(s)$, where

$H(s) = \mathcal{L}[h(t)]$, $U(s) = \mathcal{L}[u(t)]$

4.4. Inverse Laplace Transform - Partial Fraction Expansion

The partial fraction expansion can only be applicable if $F(s)$ is a rational function of s.

$$\boxed{\text{※ Check Table 4.1 \& 4.2 for summary of Section 4.3}}$$

Consider a rational function $F(s) = \dfrac{N(s)}{D(s)}$,
where $N(s)$ and $D(s)$ are two polynomials.
deg is to denote the degree of a polynomial.

We define:

$F(s)$ is improper $\longleftrightarrow$ $\deg N(s) > \deg D(s)$
$\longleftrightarrow$ $F(\infty) = \pm\infty$

$F(s)$ is proper $\longleftrightarrow$ $\deg N(s) \leq \deg D(s)$
$\longleftrightarrow$ $|F(\infty)| = k < \infty$ (finite)

$F(s)$ is strictly proper $\longleftrightarrow$ $\deg N(s) < \deg D(s)$
$\longleftrightarrow$ $F(\infty) = 0$

$F(s)$ is biproper $\longleftrightarrow$ $\deg N(s) = \deg D(s)$
$\longleftrightarrow$ $F(\infty) = k \neq 0 < \infty$

※ " $\longleftrightarrow$ " means if and only if

Example: Classify the following rational functions: s^2+1, 2, $\dfrac{1}{s+1}$, $\dfrac{s^2-1}{s+1}$,
$\dfrac{s-1}{3s+1}$

Answer: $s^2+1 \Rightarrow N(s) = s^2+1$, $D(s) = 1$
$\deg N(s) = 2$, $\deg D(s) = 0$
$\Rightarrow$ improper

$2 \Rightarrow N(s) = 2, \quad D(s) = 1.$

$\deg N(s) = 0, \quad \deg D(s) = 0$

$\Rightarrow$ biproper

$\frac{1}{s+1} \Rightarrow N(s) = 1, \quad D(s) = s+1$

$\deg N(s) = 0, \quad \deg D(s) = 1$

$\Rightarrow$ strictly proper

$\frac{s^2-1}{s+1} \Rightarrow N(s) = s^2-1, \quad D(s) = s+1$

$\deg N(s) = 2, \quad \deg D(s) = 1$

$\Rightarrow$ improper

$\frac{s-1}{3s+1} \Rightarrow N(s) = s-1, \quad D(s) = 3s+1$

$\deg N(s) = 1, \quad \deg D(s) = 1$

$\Rightarrow$ biproper

Poles and Zeros for $F(s)$

Consider the proper rational function

$F(s) = \frac{N(s)}{D(s)}$, where $N(s)$ and $D(s)$

have no common factors or are coprime.

The roots of $D(s)$ are called poles and the roots of $N(s)$ are called zeros of $F(s)$.

The degree of $F(s)$ is defined as the degree of $D(s)$ or the number of the poles where $N(s)$ and $D(s)$ are coprime.

Example: What is the degree of

$$F(s) = \frac{s^3-1}{s^3(s-1)}$$ and what are the poles and zeros?

Answer: Since s^3-1 and $s^3(s-1)$ have a common factor $(s-1)$, we need to cancel it from the numerator and denominator first.

$$F(s) = \frac{s^2+s+1}{s^3} = \frac{N(s)}{D(s)},$$

where s^2+s+1 and s^3 are coprime now.

The degree of $F(s)$ = $\deg s^3 = 3$

Zeros: $s^2+s+1=0$, $s = \frac{-1 \pm \sqrt{3}j}{2}$

where $j = \sqrt{-1}$

poles: $s^3 = 0$, $s = 0, 0, 0$

(three repeated poles)

Inverse Laplace Transform Using Partial Fraction Expansion

For any arbitrary rational $F(s)$; we need to break $F(s)$ into three parts, namely, ① improper term $F_{im}(s)$, ② biproper term $F_{bi}(s)$ and ③ strictly proper term $F_{st}(s)$ such that $F(s) = F_{im}(s) + F_{bi}(s) + F_{st}(s)$

Example: Break the following Laplace Transforms into three different terms; (a) $F(s) = \dfrac{s^2}{s+1}$, (b) $F(s) = \dfrac{s+2}{s+1}$, (c) $F(s) = \dfrac{3}{s+1}$

Answer: (a)

$$
\begin{array}{r|l}
 & s - 1 \\
\cline{2-2}
s+1 & s^2 \\
 & \underline{s^2 + s} \\
 & -s \\
 & \underline{-s - 1} \\
 & 1
\end{array}
$$

$$\therefore F(s) = \underbrace{s}_{F_{im}(s)} \underbrace{-1}_{F_{bi}(s)} + \underbrace{\frac{1}{s+1}}_{F_{st}(s)}$$

(b)

$$
\begin{array}{r|l}
 & 1 \\
s+1 & \overline{)\,s+2} \\
 & \underline{s+1} \\
 & 1
\end{array}
$$

$$\therefore\ F(s) = \underset{\substack{\uparrow \\ F_{bi}(s)}}{1} + \underset{\substack{\uparrow \\ F_{st}(s)}}{\frac{1}{s+1}}$$

$$\underset{\substack{\| \\ 0}}{F_{im}(s)}$$

(c) $F(s) = \underset{\substack{\uparrow \\ F_{st}(s)}}{\frac{3}{s+1}}$; $F_{im}(s) = F_{bi}(s) = 0$

※ Inverse Laplace Transform of $F_{im}(s)$ and $F_{bi}(s)$

$F_{im}(s)$ is the quotient polynomial of the long division as previous example. $F_{bi}(s)$ is a constant. Hence $F_{im}(s)$ can be described as $F_{im}(s) = \sum_{n=1}^{\deg F_{im}(s)} a_n s^n$ and $F_{bi}(s)$ can be described as $F_{bi}(s) = k_0$.

We denote $\mathcal{L}^{-1}$ as the inverse Laplace Transform operator.

$$\mathcal{L}^{-1}[F_{im}(s)] = \sum_{n=1}^{\deg F_{im}(s)} a_n \mathcal{L}^{-1}[s^n]$$

$$\boxed{\mathcal{L}^{-1}[F_{bi}(s)] = \mathcal{L}^{-1}[k_0] = k_0 \mathcal{L}^{-1}[1] = k_0 \delta(t)}$$

Since $\mathcal{L}\left[\frac{d^n}{dt^n}\delta(t)\right] = s^n \mathcal{L}[\delta(t)] = s^n$,

for $n = 0, 1, 2, \cdots$, then $\mathcal{L}^{-1}[s^n] = \frac{d^n}{dt^n}\delta(t)$

$$\boxed{\therefore \mathcal{L}^{-1}[F_{im}(s)] = \sum_{n=1}^{\deg F_{im}(s)} a_n \frac{d^n}{dt^n}\delta(t)}$$

Example: Find the inverse Laplace transform of $F(s) = 2s^2 + 3s + 7$

Answer: $\mathcal{L}^{-1}[2s^2 + 3s] = 2\frac{d^2}{dt^2}\delta(t) + 3\frac{d}{dt}\delta(t)$

$\mathcal{L}^{-1}[7] = 7\delta(t)$

$\therefore \mathcal{L}^{-1}[F(s)] = 2\ddot{\delta}(t) + 3\dot{\delta}(t) + 7\delta(t)$

X. <u>Inverse Laplace Transform of $F_{st}(s)$</u>

First, we need to write $F_{st}(s) = \frac{N(s)}{D(s)}$, where $N(s)$ and $D(s)$ are coprime. $D(s)$ can be described as $\prod_{i=1}^{M}(s-p_i)^{m_i} = (s-p_1)^{m_1} \cdot (s-p_2)^{m_2} \cdot (s-p_3)^{m_3} \cdots (s-p_M)^{m_M}$. There are

M poles of $D(s)$ and each pole $s = P_i$ has its multiplicity of m_i.

Example: $F(s) = \dfrac{s+1}{(s+1)(s+3)^2(s+9)^3(s+5)^2}$

What are poles P_i and multiplicities m_i?

Answer: $F(s) = \dfrac{1}{(s+3)^2(s+9)^3(s+5)^2}$

$D(s) = (s+3)^2(s+9)^3(s+5)^2$

Since there are three poles:

$$\begin{array}{cccc} s=-3, & s=-9, & s=-5, & M=3 \\ \downarrow & \downarrow & \downarrow & \\ P_1 & P_2 & P_3 & \\ m_1=2 & m_2=3 & m_3=2 & \end{array}$$

If we write $F_{st}(s) = \dfrac{N(s)}{D(s)}$, where $N(s)$, $D(s)$ are coprime and $D(s) = \prod_{i=1}^{M} (s-P_i)^{m_i}$, then

$$F_{st}(s) = \sum_{i=1}^{M} \sum_{l=1}^{m_i} \frac{C_{il}}{(s-P_i)^l},$$

where $C_{il} = \dfrac{1}{(m_i-l)!} \dfrac{d^{m_i-l}}{ds^{m_i-l}} \left[F_{st}(s)(s-P_i)^{m_i} \right]\Big|_{s=P_i}$

Then $\mathcal{L}^{-1}\left[\frac{C_{il}}{(s-p_i)^l}\right]$ can be obtained from Table 4.1

Example: Inverse Laplace Transform

$$F(s) = \frac{s^2}{(s+1)^2(s+2)^3}$$

Answer: Since s^2 and $(s+1)^2(s+2)^3$ are coprime, $D(s) = (s+1)^2(s+2)^3$.

$$F_{st}(s) = \frac{C_{11}}{(s+1)} + \frac{C_{12}}{(s+1)^2} + \frac{C_{21}}{(s+2)}$$

$$+ \frac{C_{22}}{(s+2)^2} + \frac{C_{23}}{(s+2)^3}$$

$$C_{11} = \frac{1}{(2-1)!}\frac{d^{(2-1)}}{ds^{(2-1)}}\left[\frac{s^2}{(s+1)^2(s+2)^3}(s+1)^2\right]\Bigg|_{s=-1}$$

$$= \frac{d}{ds}\left(\frac{s^2}{(s+2)^3}\right)\Bigg|_{s=-1}$$

$$= \frac{2s(s+2)^3 - 3(s+2)^2 s^2}{(s+2)^6}\Bigg|_{s=-1}$$

$$= \frac{-2-3}{1} = -5$$

$$C_{12} = \frac{1}{(2-2)!}\frac{d^{(2-2)}}{ds^{(2-2)}}\left[\frac{s^2}{(s+1)^2(s+2)^3}(s+1)^2\right]\Bigg|_{s=}$$

$$= \frac{(-1)^2}{(-1+2)^3} = 1$$

$$C_{21} = \frac{1}{(3-1)!} \frac{d^{(3-1)}}{ds^{(3-1)}} \left[\frac{s^2}{(s+1)^2 (s+2)^3} (s+2)^3 \right] \Bigg|_{s=-2}$$

$$= \frac{1}{2} \frac{d^2}{ds^2} \left[\frac{s^2}{(s+1)^2} \right] \Bigg|_{s=-2}$$

$$= \frac{1}{2} \times 10 = 5$$

$$C_{22} = \frac{1}{(3-2)!} \frac{d^{(3-2)}}{ds^{(3-2)}} \left[\frac{s^2}{(s+1)^2} \right] \Bigg|_{s=-2}$$

$$= \left[\frac{2s\,(s+1)^2}{(s+1)^4} - \frac{2(s+1)\,s^2}{(s+1)^4} \right] \Bigg|_{s=-2}$$

$$= -4 + 8 = 4$$

$$C_{23} = \frac{1}{(3-3)!} \frac{d^{(3-3)}}{ds^{(3-3)}} \left[\frac{s^2}{(s+1)^2} \right] \Bigg|_{s=-2}$$

$$= \left[\frac{s^2}{(s+1)^2} \right] \Bigg|_{s=-2}$$

$$= 4$$

$$\therefore \quad F(s) = F_{st}(s) = \frac{-5}{(s+1)} + \frac{1}{(s+1)^2} + \frac{5}{(s+2)} + \frac{4}{(s+2)^2} + \frac{4}{(s+2)^3}$$

From Table 4.1,

$$\mathcal{L}^{-1}[F(s)] = -5e^{-t} + te^{-t} + 5e^{-2t} + 4te^{-2t} + 2t^2 e^{-2t}$$

4.5 LTIL Differential Equations

Laplace Transform pairs can be applied to compute the system response characterized by a differential equation.

Example: An LTIL system is described as $2\ddot{y}(t) + 3\dot{y}(t) + y(t) = u(t)$ with initial condition $y(0^-) = -1$ and $\dot{y}(0^-) = 1$. What is $y(t)$ if $u(t) = \delta(t)$?

Answer: $2\,\mathcal{L}[\ddot{y}(t)] + 3\,\mathcal{L}[\dot{y}(t)] + \mathcal{L}[y(t)]$

$= \mathcal{L}[u(t)]$

Assume $\mathcal{L}[y(t)] = Y(s)$

$\mathcal{L}[u(t)] = U(s)$

Then, $2[s^2 Y(s) - s\,y(0^-) - \dot{y}(0^-)]$

$+ 3[s Y(s) - y(0^-)] + Y(s) = U(s)$

$(2s^3 + 3s + 1)\,Y(s) = 2s\,y(0^-) + 2\dot{y}(0^-)$

$+ 3\,y(0^-) + U(s)$

$$Y(s) = \underbrace{\frac{(2s+3)y(0^-)+2\dot{y}(0^-)}{2s^2+3s+1}}_{\text{zero-input response}} + \underbrace{\frac{1}{2s^2+3s+1}U(s)}_{\text{zero-state response}}$$

$$= \frac{-2s-3+2}{(2s^2+3s+1)} + \frac{1}{(2s^2+3s+1)s}$$

$$= \frac{-2s^2-s+1}{s(s+1)(2s+1)} = \frac{(-2s+1)\cancel{(s+1)}}{s\cancel{(s+1)}(2s+1)}$$

$$= \frac{-2s+1}{s(2s+1)}$$

$$= \frac{C_{11}}{s} + \frac{C_{21}}{2s+1}$$

$$C_{11} = \left[\frac{-2s+1}{2s+1}\right]\Big|_{s=0} = 1$$

$$C_{21} = \left[\frac{-2s+1}{s}\right]\Big|_{s=-\frac{1}{2}} = -4$$

$$\therefore Y(s) = \frac{1}{s} - \frac{4}{2s+1} = \frac{1}{s} - \frac{2}{s+\frac{1}{2}}$$

$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[\tfrac{1}{s}] - 2\mathcal{L}^{-1}[\tfrac{1}{s+\frac{1}{2}}]$$

From Table 4.1, $y(t) = \mathcal{L}^{-1}[Y(s)]$

$$= 1 - 2e^{-\frac{1}{2}t}$$

4.6 Zero-input Response - Characteristic Polynomial and Mode

The zero-input response of an LTIL system is

$$Y_{zi}(s) = Y(s)\Big|_{U(s)=0}$$

If $Y_{zi}(s) = \dfrac{N(s)}{D(s)}$, where $N(s)$ and $D(s)$ are co-prime, we call $D(s)$ the characteristic polynomial and its roots the modes of the system.

The form of the zero-input response excited by any initial conditions is completely determined by the modes of the system.

4.7 Zero-state Response - Transfer Function

The zero-state response of an LTIL system is

$$Y_{zs}(s) = Y(s)\Big|_{\underbrace{y(0^-) = \dot{y}(0^-) = \ddot{y}(0^-) = \cdots = 0}_{\text{no initial conditions}}}$$

If $Y(s)$ is a rational function, then

$Y_{zs}(s) = H(s)\,U(s)$, where $H(s)$ is also rational and it is called the transfer function, such that

$$H(s) = \frac{Y(s)}{U(s)}\Bigg|_{\underbrace{y(0^-) = \dot{y}(0^-) = \ddot{y}(0^-) = \cdots = 0}_{\text{: initial conditions are all zero}}}$$

Alternatively, the transfer function can be obtained through the Laplace transform of the impulse response of this system, such that

$$H(s) = \mathcal{L}[h(t)]$$

Example: An LTIL system given by

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = u(t).$$

Find the zero-input response in the s-domain, $Y_{zi}(s)$ and zero-input response in the time-domain, $y_{zi}(t)$, the zero-state response in the s-domain, $Y_{zs}(s)$ and

the zero-state response in the time-domain, $y_{zs}(t)$. (Initial Conditions: $y(0^-)=-1$, $\dot{y}(0^-)=1$)

Answer: From the previous example, we obtain

$$Y(s) = \frac{-2s-3+2}{2s^2+3s+1} + \frac{1}{s(2s^2+3s+1)}$$

where

$$Y_{zi}(s) = \frac{-2s-1}{2s^2+3s+1} = \frac{-(2s+1)}{(2s+1)(s+1)}$$

$$= \frac{-1}{s+1}$$

Hence this system has only one mode $s=-1$ and the characteristic polynomial is $s+1$.

$$y_{zi}(t) = \mathcal{L}^{-1}[Y_{zi}(s)] = -e^{-t}$$

$$Y_{zs}(s) = \frac{1}{s(2s^2+3s+1)} = \underbrace{\frac{1}{s}}_{U(s)} \underbrace{\frac{1}{(2s^2+3s+1)}}_{H(s)}$$

The transfer function $= H(s) = \frac{Y_{zs}(s)}{U(s)}$

$$= \frac{1}{2s^2+3s+1}$$

$Y_{zs}(s)$ is strictly proper, hence

$$Y_{zs}(s) = \frac{1}{s(s+1)(2s+1)} = \frac{C_{11}}{s} + \frac{C_{21}}{s+1} + \frac{C_{31}}{2s+1}$$

$$C_{11} = \left[\frac{1}{(s+1)(2s+1)}\right]\Bigg|_{s=0} = 1$$

$$C_{21} = \left[\frac{1}{s(2s+1)}\right]\Bigg|_{s=-1} = 1$$

$$C_{31} = \left[\frac{1}{s(s+1)}\right]\Bigg|_{s=-\frac{1}{2}} = -4$$

$$\therefore\ Y_{zs}(s) = \frac{1}{s} + \frac{1}{s+1} + \frac{-4}{2s+1}$$

$$y_{zs}(t) = \mathcal{L}^{-1}[Y_{zs}(s)]$$

$$= 1 + e^{-t} - 2e^{-\frac{1}{2}t}$$

The total response $y(t) = y_{zi}(t) + y_{zs}(t)$

$$= -e^{-t} + 1 + e^{-t} - 2e^{-\frac{1}{2}t}$$

$$= 1 - 2e^{-\frac{1}{2}t}$$

as the result before.

Example: An LTIL system is described as $2\ddot{y}(t) + 3\dot{y}(t) + y(t) = u(t)$, where $y(0^-) = -1$ and $\dot{y}(0^-) = 1$. Find the transfer function $H(s)$ and the impulse response $h(t)$.

Answer: From the previous example,

$$Y_{zs}(s) = H(s)\,U(s)$$

$$H(s) = \frac{Y_{zs}(s)}{U(s)} = \frac{1}{2s^2+3s+1}$$

$$= \frac{1}{(2s+1)(s+1)}$$

Assume $H(s) = \dfrac{C_{11}}{s+1} + \dfrac{C_{21}}{2s+1}$

$$C_{11} = \left[\frac{1}{2s+1}\right]\Bigg|_{s=-1} = -1$$

$$C_{21} = \left[\frac{1}{s+1}\right]\Bigg|_{s=-\frac{1}{2}} = 2$$

$$\therefore\ H(s) = \frac{-1}{s+1} + \frac{2}{2s+1}$$

$$h(t) = \mathcal{L}^{-1}[H(s)]$$

$$= -e^{-t} + e^{-\frac{1}{2}t}$$

4.7.1 Poles and Zeros of Proper Rational Transfer Functions

If we are given the values of the zeros, poles of a LTIL system as well as a value evaluated $s = s_0$, where s_0 is neither pole nor zero, we can uniquely determine the exact $H(s)$.

Example: A LTIL system transfer function has two poles at $s=3$ and one zero at $s=1$. $H(0)=8$ is given. What is $H(s)$?

Answer: $H(s) = \dfrac{a(s-1)}{(s-3)^2}$

$$H(0) = \frac{-a}{9} = 8$$

$$\Rightarrow \quad a = -72$$

$$\therefore H(s) = \frac{-72(s-1)}{(s-3)^2}$$

4.7.2 The Responses of Modes and Poles

Any pole P_i of a strictly proper $F_{st}(s)$

$\doteq \sum_{i=1}^{M} \sum_{l=1}^{m_i} \dfrac{C_{il}}{(s-P_i)^l}$ can be written as

$$P_i = \sigma_i + j\omega_i$$

$$\mathcal{L}^{-1}\left[\frac{C_{il}}{(s-P_i)^l}\right] = \frac{C_{il}}{(l-1)!} t^{(l-1)} e^{P_i t}, \quad l = 1, 2, \ldots, m_i$$

Discuss the asymptotical behavior of

$$\frac{C_{il}}{(l-1)!} t^{(l-1)} e^{P_i t}, \quad l = 1, 2, \ldots, m_i$$

① $l=1$, $\sigma_i = \omega_i = 0 \Rightarrow p_i = 0$

$$\mathcal{L}^{-1}\left[\frac{c_{i1}}{s}\right] = c_{i1}$$

The time response for a single mode (pole) located at $s=0$ approaches a nonzero constant as $t \to \infty$

② $l \geq 2$, $\sigma_i = \omega_i = 0 \Rightarrow p_i = 0$

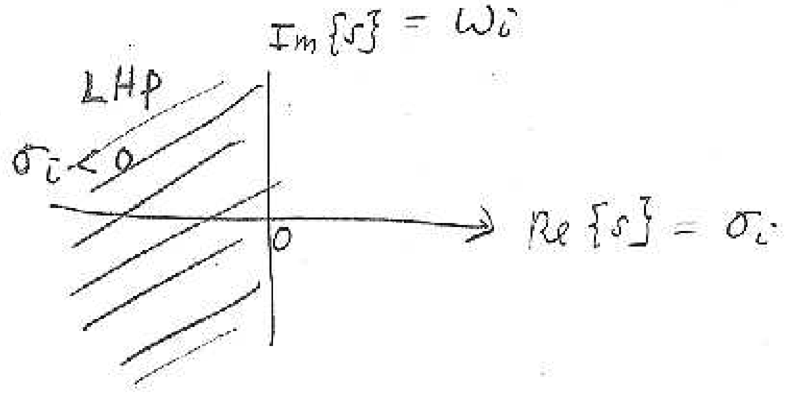
$$\mathcal{L}^{-1}\left[\frac{c_{il}}{s^l}\right] = \frac{c_{il}}{(l-1)!} t^{(l-1)}$$

The time response for a repeated mode (pole) located at $s=0$ approaches $\pm\infty$ as $t \to \infty$

③ $\sigma_i < 0$, ω_i is arbitrary

$$\mathcal{L}^{-1}\left[\frac{c_{il}}{(s-p_i)^l}\right] = \frac{c_{il}}{(l-1)!} t^{(l-1)} e^{\sigma_i t} e^{j\omega_i t}, \quad l=1,2,\cdots m_i$$

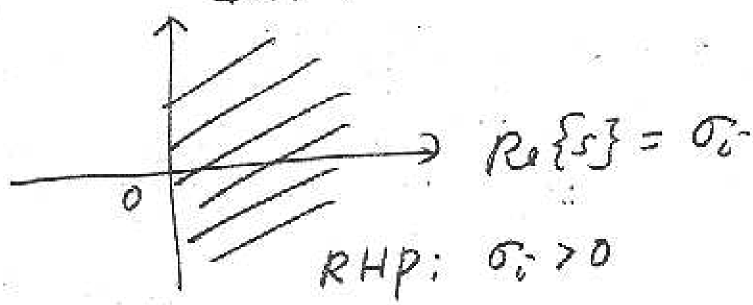
The time response for a single or repeated mode (pole) located at open LHP approaches zero as $t \to \infty$



④ $\sigma_i > 0$, ω_i is arbitrary

$$\mathcal{L}^{-1}\left[\frac{C_{il}}{(s-P_i)^l}\right] = \frac{C_{il}}{(l-1)!} t^{(l-1)} e^{\sigma_i t} e^{j\omega_i t}$$

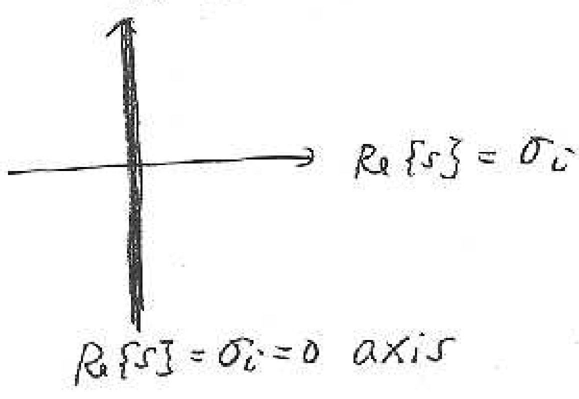
The time response for a single or repeated mode (pole) located at open RHP approaches $\pm\infty$ as $t \to \infty$



⑤ $l \geq 2$, $\sigma_i = 0$, $\omega_i \neq 0 \Rightarrow P_i = j\omega_i$

$$\mathcal{L}^{-1}\left[\frac{C_{il}}{(s-P_i)^l}\right] = \frac{C_{il}}{(l-1)!} t^{(l-1)} e^{j\omega_i t}$$

The time response for a repeated mode (pole) located at the Imaginary-S axis or $Re\{s\} = \sigma_i = 0$ axis approaches $\pm\infty$ as $t \to \infty$



Example: What values will the impulse responses and unit-step responses approach as $t \to \infty$? (a) $H(s) = \frac{1}{(s+2)(s+3)}$;

(b) $H(s) = \frac{s}{s^2-1}$

Answer: (a) Impulse Response

$$Y(s) = H(s) \times 1 = H(s)$$

$$Y(s) = \frac{C_{11}}{s+2} + \frac{C_{21}}{s+3}$$

Since both $s=-2$, and $s=-3$ modes are at LHP, both time responses for these two poles will approach zero as $t \to \infty$.

$$\therefore \mathcal{L}^{-1}[Y(s)] = 0 + 0 = 0 \quad \text{as } t \to \infty$$

Unit-step Response

$$Y(s) = H(s) \times \frac{1}{s} = \frac{1}{s(s+2)(s+3)}$$

$$Y(s) = \frac{C_{11}}{s} + \frac{C_{21}}{s+2} + \frac{C_{21}}{s+3}$$

$$\mathcal{L}^{-1}[Y(s)] = C_{11} + 0 + 0 \quad \text{as } t \to \infty$$

Since first pole is $s=0$

$$\therefore \mathcal{L}^{-1}[Y(s)] = C_{11} = \left[\frac{1}{(s+2)(s+3)}\right]\Bigg|_{s=0}$$

$$= \frac{1}{6} \quad \text{as } t \to \infty$$

(b) Impulse Response

$$Y(s) = H(s) \times 1 = H(s) = \frac{s}{s^2-1}$$

$$Y(s) = \frac{C_{11}}{s-1} + \frac{C_{21}}{s+1}$$

Since the first mode $s=1$ is at RHP and the second mode $s=-1$ is at LHP, hence

$$\mathcal{L}^{-1}[Y(s)] = C_{11} \cdot \infty + 0 \quad \text{as } t \to \infty$$

$$C_{11} = \left[\frac{s}{s+1}\right]\Big|_{s=1} = \frac{1}{2}$$

$$\therefore \mathcal{L}^{-1}[Y(s)] = \infty + 0 = \infty \quad \text{as } t \to \infty$$

Unit-step Response

$$Y(s) = H(s) \times \frac{1}{s} = \frac{1}{s^2-1}$$

$$Y(s) = \frac{C_{11}}{s-1} + \frac{C_{21}}{s+1}$$

Similarly to impulse response,

$$\mathcal{L}^{-1}[Y(s)] = C_{11} \cdot \infty + 0 \quad \text{as } t \to \infty$$

$$C_{11} = \left[\frac{1}{s+1}\right]\Big|_{s=1} = \frac{1}{2}$$

$$\therefore \mathcal{L}^{-1}[Y(s)] = \infty + 0 = \infty \quad \text{as } t \to \infty$$

4.10 Transfer Function Representation

For a causal LTIL system excited by a causal input, we can depict the system response in the time-domain (convolution form) or in the s-domain (Laplace Transform product).

Time-domain:

$$\xrightarrow{u(t)} \boxed{h(t)} \longrightarrow y(t) = \int_0^t h(t-\tau)\, u(\tau)\, d\tau = h(t) \circledast u(t)$$

$$\mathcal{L} \downarrow \qquad \uparrow \mathcal{L}^{-1}$$

s-domain:

$$\xrightarrow{U(s)} \boxed{H(s)} \longrightarrow Y(s) = H(s)\, U(s)$$

<u>Equivalence between time-domain and s-domain</u>

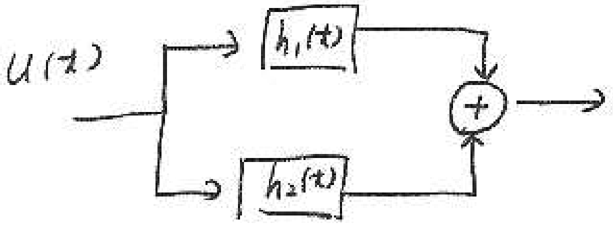
(i) Convolution of two impulse responses

$$u(t) \rightarrow \boxed{h_1(t)} \rightarrow \boxed{h_2(t)} \rightarrow y(t) = h_1(t) \circledast h_2(t) \circledast u(t)$$

$$\mathcal{L} \uparrow \qquad \uparrow \mathcal{L}^{-1}$$

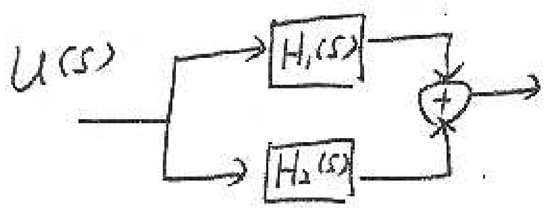
$$U(s) \rightarrow \boxed{H_1(s)} \rightarrow \boxed{H_2(s)} \rightarrow Y(s) = H_1(s)\, H_2(s)\, U(s)$$

(ii) Parallel addition



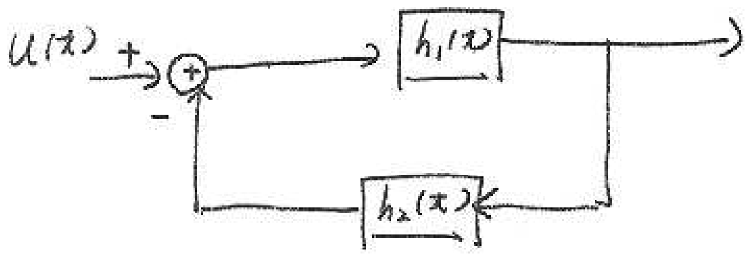
$$y(t) = u(t) \circledast h_1(t) + u(t) \circledast h_2(t)$$

$$\mathcal{L} \downarrow \quad \uparrow \mathcal{L}^{-1}$$



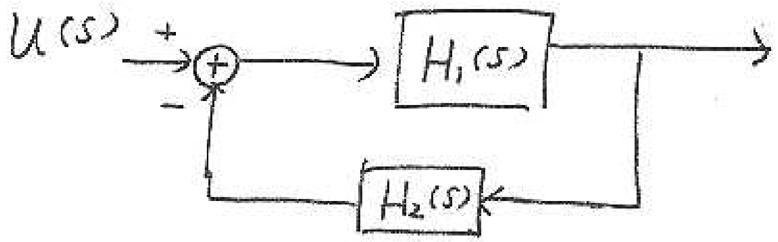
$$Y(s) = H_1(s)U(s) + H_2(s)U(s)$$

(iii) Negative Feedback



$$y(t) = u(t) \circledast h_1(t) - h_1(t) \circledast h_2(t) \circledast y(t)$$

$$\mathcal{L} \downarrow \quad \uparrow \mathcal{L}^{-1}$$

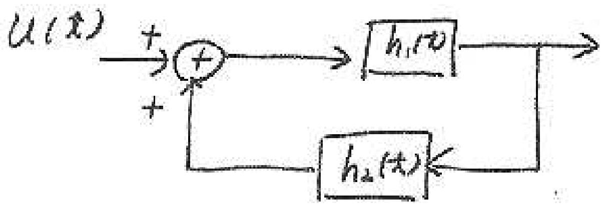


$$Y(s) = U(s)H_1(s) - H_1(s)H_2(s)Y(s)$$

$$(1 + H_1(s)H_2(s))\,Y(s) = H_1(s)U(s)$$

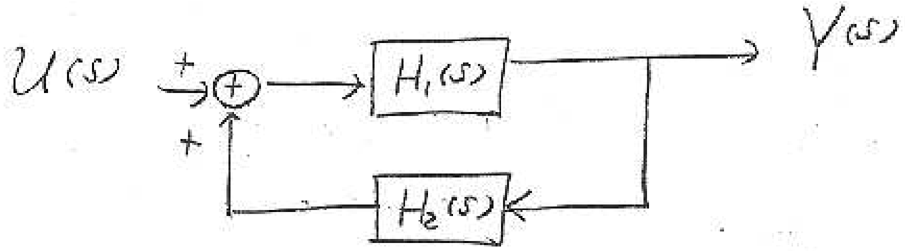
$$\therefore\ Y(s) = \frac{H_1(s)U(s)}{1 + H_1(s)H_2(s)}$$

(iv) Positive Feedback



$$y(t) = u(t) \circledast h_1(t) + h_1(t) \circledast h_2(t) \circledast y(t)$$

$$\mathcal{L} \downarrow \qquad \uparrow \mathcal{L}^{-1}$$



$$Y(s) = U(s)\,H_1(s) + H_1(s)\,H_2(s)\,Y(s)$$

$$(1 - H_1(s)\,H_2(s))\,Y(s) = H_1(s)\,U(s)$$

$$\therefore \quad Y(s) = \frac{H_1(s)\,U(s)}{1 - H_1(s)\,H_2(s)}$$

※ From (iii) & (iv), we find that the Laplace Transform representation for LTIL feedback systems can easily show the input/output relationship in rational transfer function form. Hence the Laplace Transform is widely used to represent any LTIL system.

4.11 Final-value and initial-value Theorems

<u>The final-value Theorem</u>: Let $F(s)$ be the Laplace transform of $f(t)$ and $F(s)$ is a proper rational function.

Then $f(t)$ approaches a zero or nonzero constant as $t \to \infty$ if and only if all poles of $sF(s)$ have negative real parts.

Under this condition, we have

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Example: What is $\lim_{t \to \infty} f(t)$ if $f(t) = \mathcal{L}^{-1}[F(s)] = \frac{s-2}{s(s+1)}, \frac{s-1}{s(s^2-1)}$

Answer: (a) $F(s) = \frac{s-2}{s(s+1)}$

$$sF(s) = \frac{s-2}{s+1}$$

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

$$= \lim_{s \to 0} \frac{s-2}{s+1} = -2$$

(b) $F(s) = \dfrac{s-1}{s(s^2-1)} = \dfrac{1}{s(s+1)}$

$sF(s) = \dfrac{1}{s+1}$

$\lim_{t\to\infty} f(t) = \lim_{s\to 0} \dfrac{1}{s+1} = 1$

<u>The initial-value Theorem</u>: Let $F(s)$ be the Laplace transform of $f(t)$ and $F(s)$ is a rational function and strictly proper. Then

$$f(0^+) = \lim_{s\to\infty} sF(s)$$

Example: Find $f(0^+)$ if

$F(s) = \dfrac{1}{s^3+2s+1}$ and $F(s) = \dfrac{2s+1}{s^2-1}$

Answer: (a) $f(0^+) = \lim_{s\to\infty} sF(s)$

$= \lim_{s\to\infty} \dfrac{s}{s^3+2s+1} = 0$

(b) $f(0^+) = \lim_{s\to\infty} sF(s)$

$= \lim_{s\to\infty} \dfrac{2s^2+s}{s^2-1} = 2$