Chapter 3 Convolution, Difference and Differential Equations

3.1 Linear Time-invariant Systems with Memory

The response of a LTI system can be always decomposed as

Total response = zero-state response + zero-input response.

3.2 LTI Discrete-time Systems and Impulse Responses.

For an LTI discrete-time system, a permissibe pair is $\{u[k]\} \rightarrow \{y[k]\}$.

The discrete-time impulse response or simply impulse response is defined as

$$h[k] = \text{impulse response} := \text{output excited by } \delta[k] = \begin{cases} 1, & \text{for } k=0 \\ 0, & \text{for } k \neq 0 \end{cases}$$

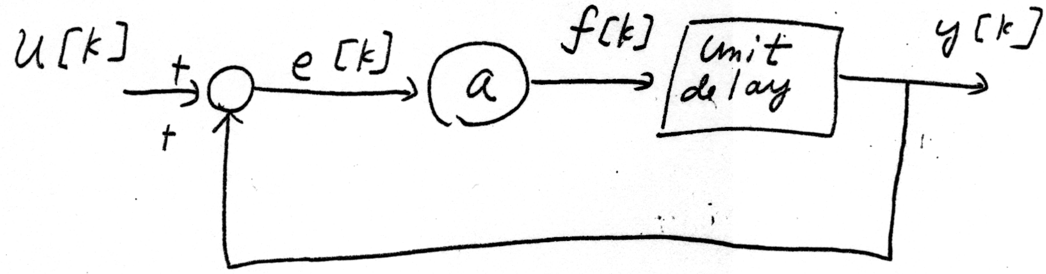
Since different LTI discrete-time systems have different impulse responses, impulse responses are the system characteristics.

A necessary and sufficient condition for a LTI system to be causal is

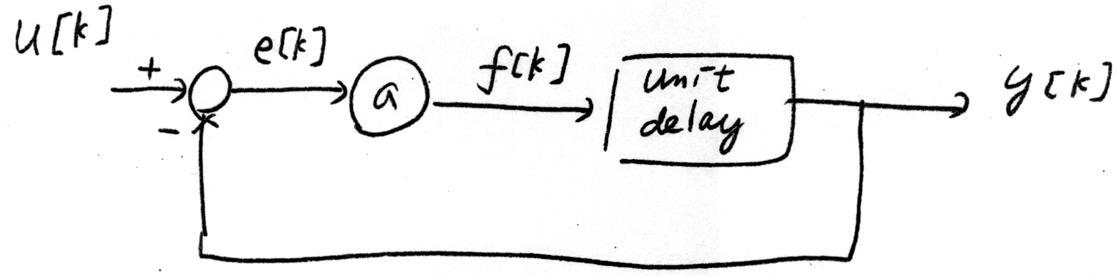
$$h[k] = 0, \quad \text{for } k < 0.$$

Example: Consider two systems depicted as below. Compute their corresponding impulse responses.

(a)



(b)



Answer:

(a) $e[k] = u[k] + y[k]$

$y[k] = f[k-1]$

$f[k] = a\,e[k]$

$y[k] = a\,e[k-1]$

$= a\,u[k-1] + a\,y[k-1]$

※ The impulse response describes only the zero-state response!

$\{\delta[k]\} \rightarrow \{h[k]\}$

$h[0] = a\delta[-1] + a\,h[-1] = 0$

$h[1] = a\delta[0] + a\,h[0] = a$

$h[2] = a\delta[1] + a\,h[1] = a^2$

$\vdots$

$h[k] = a\delta[k-1] + a\,h[k-1]$

$= a\,h[k-1] \quad , \quad k \geq 2$

$$h[k] = \begin{cases} a^k, & k \geq 1 \\ 0, & k \leq 0 \end{cases}$$

(b)

$e[k] = u[k] - y[k]$

$y[k] = f[k-1]$

$f[k] = a e[k]$

$y[k] = a e[k-1]$

$= a u[k-1] - a y[k-1]$

$\{\delta[k]\} \rightarrow \{h[k]\}$

$h[0] = a\delta[-1] - a h[-1] = 0$

$h[1] = a\delta[0] - a h[0] = a$

$h[2] = a\delta[1] - a h[1] = -a^2$

$\vdots$

$h[k] = a\delta[k-1] - a h[k-1]$

$= -a h[k-1], \quad k \geq 2$

$$h[k] = \begin{cases} -(-a)^k, & k \geq 1 \\ 0, & k \leq 0 \end{cases}$$

$$= \begin{cases} (-1)^{k-1} a^k, & k \geq 1 \\ 0, & k \leq 0 \end{cases}$$

3.2-1

Discrete - Convolution

The input of a discrete-time system can be written as ($u[k]=0$, $k<0$)

$$u[k] = u[0]\,\delta[k] + u[1]\,\delta[k-1] + u[2]\,\delta[k-2] + \cdots$$
$$= \sum_{i=0}^{\infty} u[i]\,\delta[k-i] .$$

For an LTI discrete-time system, assume $\{\delta[k]\} \rightarrow \{h[k]\}$ is a permissible pair.

Then we have

$$\{\delta[k-i]\} \rightarrow \{h[k-i]\} \quad \text{(time-invariance)}$$

and

$$\{u[i]\,\delta[k-i]\} \rightarrow \{u[i]\,h[k-i]\} \quad \text{(homogeneity)}$$

and

$$\left\{\sum_{i=0}^{\infty} u[i]\,\delta[k-i]\right\} \rightarrow \left\{\sum_{i=0}^{\infty} u[i]\,h[k-i]\right\} \quad \text{(additivity)}$$

Thus, if the system is linear and time-invariant, the zero-state output $y[k]$ excited by the input $u[k]$ equals

$$y[k] = \sum_{i=0}^{\infty} h[k-i]\, u[i]$$

If the system is causal as well, i.e.,

$h[k-i] = 0, \quad \forall k < i$, then

$$y[k] = \sum_{i=0}^{k} h[k-i]\, u[i] \quad \cdots \text{ for } \begin{cases} u[k]=0, & k<0 \\ h[k]=0, & k<0 \end{cases}$$

$$\begin{aligned} y[0] &= h[0]\, u[0] \\ y[1] &= h[1]\, u[0] + h[0]\, u[1] \\ y[2] &= h[2]\, u[0] + h[1]\, u[1] + h[0]\, u[2] \\ &\vdots \end{aligned}$$

Discrete convolution is often written more generally as (for any LTI non causal system)

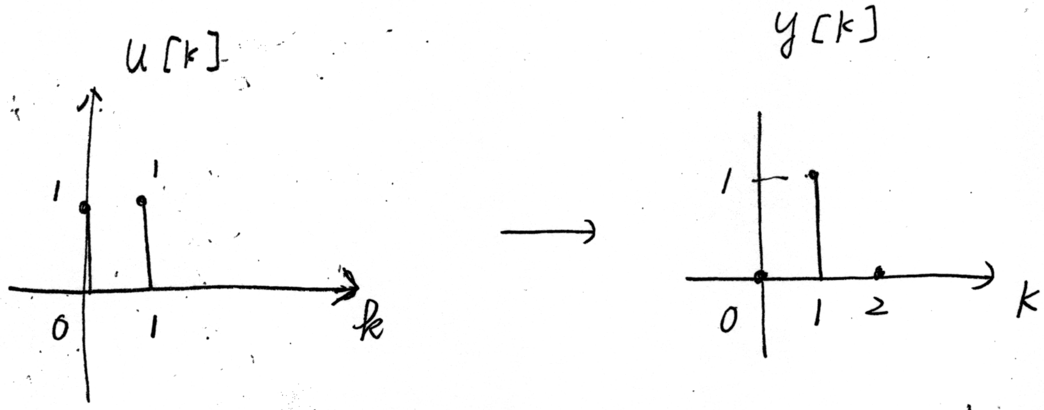
$$y[k] = \sum_{i=-\infty}^{\infty} h[k-i]\, u[i] := h[k] \circledast u[k]$$

$$= \sum_{i=-\infty}^{\infty} u[k-i]\, h[i] := u[k] \circledast h[k]$$

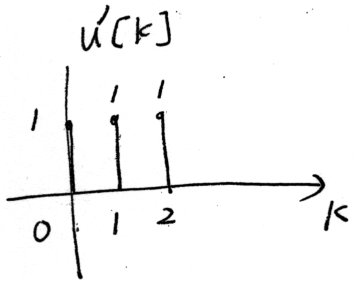
for all integer $k \in (-\infty, \infty)$

If $\{\delta[k]\} \to \{h[k]\}$ is a permissible pair for a LTI system, $h[k]$ is called the impulse response

Example : A permissible pair for a causal LTI system is depicted as below. ($h[k]=0$, $k<0$)



What is the impulse response $h[k]$ for this system? What is the output $y'[k]$ if the input $u'[k]$ is applied which is depicted as below?



Answer :

$$u[k] = \delta[k] + \delta[k-1]$$

$$y[k] = h[k] + h[k-1]$$

$$h[k] = y[k] - h[k-1], \quad k \geq 0$$

$$h[0] = y[0] - h[-1] = 0$$

$$h[1] = y[1] - h[0] = 1 - 0 = 1$$

$$h[2] = y[2] - h[1] = 0 - 1 = -1$$

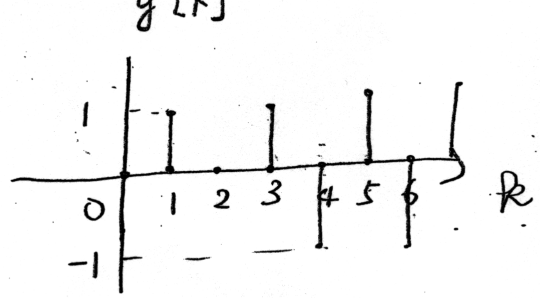
$$h[3] = y[3] = h[2] = 0 + 1 = 1$$

$$\vdots$$

$$h[k] = \begin{cases} (-1)^{k-1}, & k \geq 1 \\ 0, & k \leq 0 \end{cases} = (-1)^{k-1} g[k-1]$$

$$u'[k] = \delta[k] + \delta[k-1] + \delta[k-2]$$

$$\begin{aligned} y'[k] &= u'[k] \circledast h[k] \\ &= (\delta[k] + \delta[k-1] + \delta[k-2]) \circledast h[k] \\ &= \delta[k] \circledast h[k] + \delta[k-1] \circledast h[k] + \delta[k-2] \circledast h[k] \\ &= h[k] + h[k-1] + h[k-2] \\ &= (-1)^{k-1} g[k-1] + (-1)^{k-2} g[k-2] + (-1)^{k-3} g[k-3] \\ &= \begin{cases} (-1)^0, & k = 1 \\ (-1)^1 + (-1)^0, & k = 2 \\ (-1)^{k-1} + (-1)^{k-2} + (-1)^{k-3}, & k \geq 3 \\ 0, & k < 1 \end{cases} \\ &= \begin{cases} 1, & k = 1 \\ 0, & k = 2 \\ (-1)^{k-3}, & k \geq 3 \\ 0, & k < 1 \end{cases} \end{aligned}$$



Example: Given the impulse response $h[k]$ of an LTI system, $h[k] = (\frac{1}{2})^{|k|}$, what is the output $y[k]$ if an unit step input sequence $q[k]$ is applied?

Answer: It is not a causal system since $h[k] \neq 0$, $k<0$. We need to use the following equation to compute $y[k]$.

$$y[k] = \sum_{i=-\infty}^{\infty} u[k-i]\, h[i]$$

$$= \sum_{i=-\infty}^{\infty} q[k-i] \left(\frac{1}{2}\right)^{|i|}$$

$$= \sum_{i=-\infty}^{k} \left(\frac{1}{2}\right)^{|i|}$$

since $q[k-i] = \begin{cases} 1, & k \geq i \\ 0, & k < i \end{cases}$

$k \geq 0$, $$y[k] = \sum_{i=-\infty}^{-1} \left(\frac{1}{2}\right)^{-i} + \sum_{i=0}^{k} \left(\frac{1}{2}\right)^{i}$$

$$= \sum_{i'=1}^{\infty} \left(\frac{1}{2}\right)^{i'} + \sum_{i=0}^{k} \left(\frac{1}{2}\right)^{i}$$

$i' = -i$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} + \frac{1\left(1-\left(\frac{1}{2}\right)^{k+1}\right)}{1-\frac{1}{2}}, \quad k \geq 0$$

$$= 1 + 2 - 2\left(\frac{1}{2}\right)^{k+1}, \quad k \geq 0$$

$$= 3 - \left(\frac{1}{2}\right)^{k}, \quad k \geq 0$$

$$k<0, \quad y[k] = \sum_{i=-\infty}^{k} \left(\frac{1}{2}\right)^{-i}$$

$$= \sum_{i'=-k}^{\infty} \left(\frac{1}{2}\right)^{i'} = \frac{\left(\frac{1}{2}\right)^{-k}}{1-\frac{1}{2}}$$

$$i' = -i$$

$$= \left(\frac{1}{2}\right)^{-k-1}, \quad k<0$$

$$\therefore \quad y[k] = \begin{cases} \left(\frac{1}{2}\right)^{-k-1}, & k<0 \\ 3-\left(\frac{1}{2}\right)^{k}, & k \geq 0 \end{cases}$$

$$\text{or} = \left(\frac{1}{2}\right)^{-k-1} g[-k-1] + \left[3-\left(\frac{1}{2}\right)^{k}\right] g[k]$$

3.2.2 Graphical Computation of Discrete Convolution

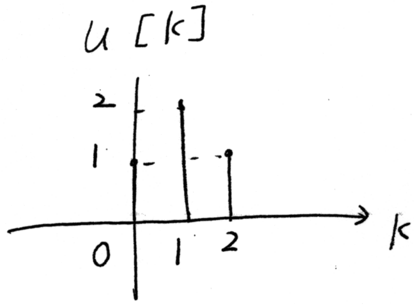
For a causal LTI system, the input $u[k]$ is applied at $k=0$ and after. The output can be computed through the impulse response such that

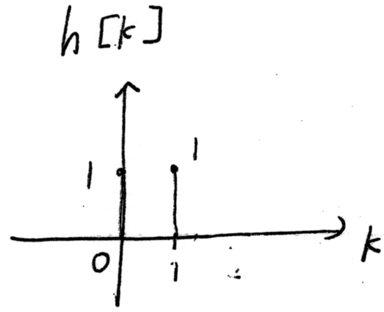
$$y[k] = \sum_{i=0}^{k} h[k-i]\, u[i]$$

To compute graphically $y[k]$ for a given k, $0 \le k \le L$
(L = the length of $h[i]$ plus the length of $u[i]$ minus 2)

1. Flipping $h[i]$ to yield $h[-i]$
2. Shifting $h[-i]$ to yield $h[k-i]$
3. Multiplication of $h[k-i]$ and $u[i]$
4. Summation of $h[k-i]\,u[i]$ for $i = 0, 1, 2, \cdots, k$
5. next k

Example: A causal LTI system with impulse response $h[k]$. What is the output $y[k]$ if an input sequence $u[k]$ is applied? $u[k]$ and $h[k]$ are depicted as below.

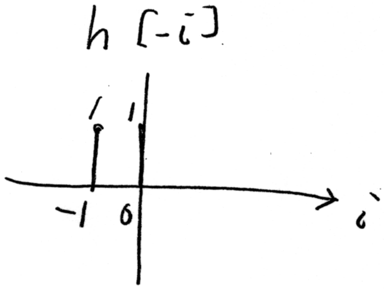




Answer: Following the four steps: $0 \le k \le 2+3-2 = 3$

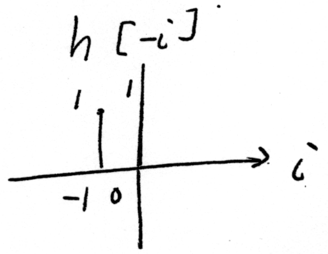
$k = 0$

1. Flipping $h[i]$ to yield $h[-i]$

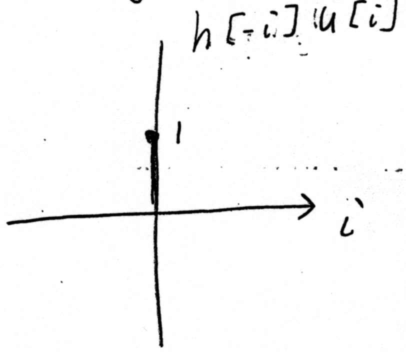


2. Shifting $h[-i]$ to yield $h[k-i]$, $0 \le k \le 3$

Since $k=0$,



3. Multiplication of $h[-i]$ and $u[i]$

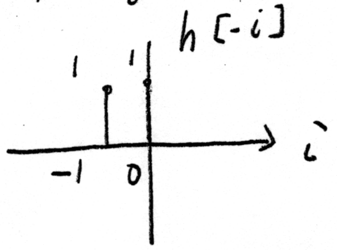


4. Summation of $h[-i]u[i]$ for $i = 0, 1, 2, \dots, k$ ($k \to 0$)

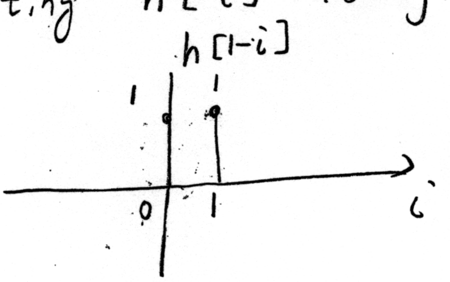
$$y[0] = \sum_{i=0}^{0} h[-i]\,u[i] = 1$$

5. next k, $k = 1$

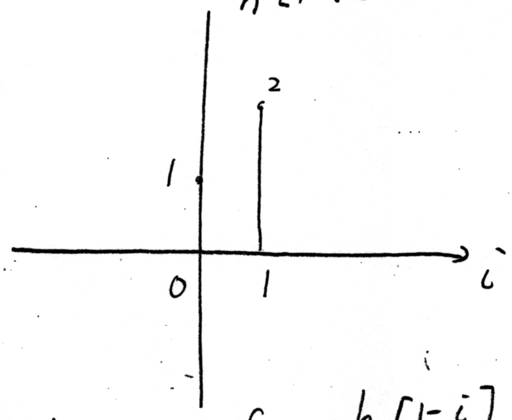
1. Flipping $h[i]$ to yield $h[-i]$



2. Shifting $h[-i]$ to yield $h[1-i]$



3. Multiplication of $h[1-i]$ and $u[i]$

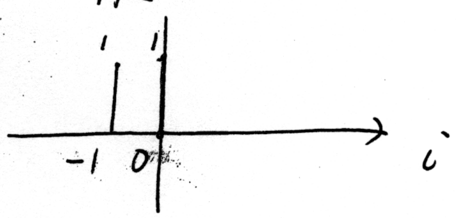


4. Summation of $h[1-i]\,u[i]$ for $i = 0, 1$

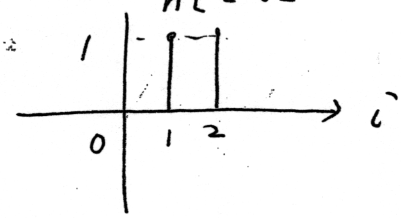
$$y[1] = \sum_{i=0}^{1} h[1-i]\,u[i] = 1+2 = 3$$

5. next k, $k = 2$

1. Flipping $h[i]$ to yield $h[-i]$

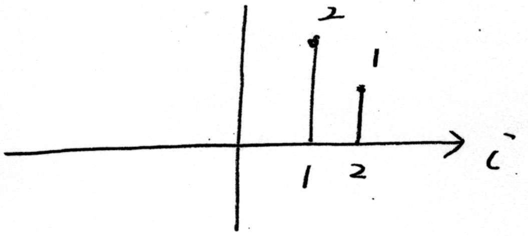


2. Shifting $h[-i]$ to yield $h[2-i]$



3. Multiplication of $h[2-i]$ and $u[i]$

$h[2-i]\,u[i]$



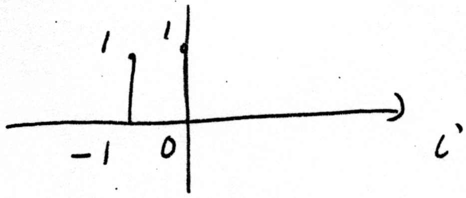
4. Summation of $h[1-i]\,u[i]$ for $i = 0, 1, 2$

$$y[2] = \sum_{i=0}^{2} h[2-i]\,u[i] = 0+2+1 = 3$$

5. Next k, $k = 3$

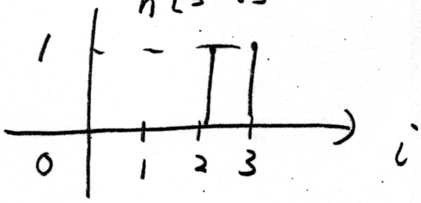
1. Flipping $h[i]$ to yield $h[-i]$

$h[-i]$



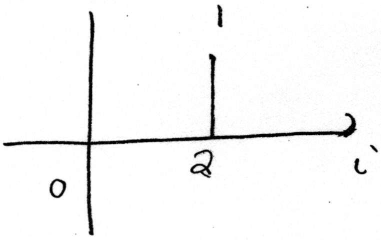
2. shifting $h[-i]$ to yield $h[3-i]$

$h[3-i]$



3. Multiplication of $h[3-i]$ and $u[i]$

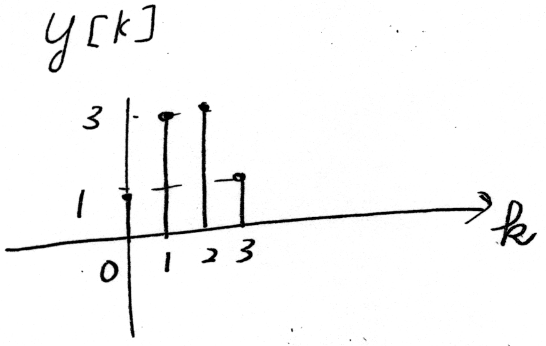
$h[3-i]\,u[i]$



4. Summation of $h[3-i]\,u[i]$ for $i = 0, 1, 2, 3$

$$y[3] = \sum_{i=0}^{3} h[3-i]\,u[i] = 0+0+1+0 = 1$$

Done !!



$(L = 3)$

Example: Discrete Convolution also arises in the product of two polynomials.

Let $a(x) = a_0 + a_1 x + a_2 x^2$

and $b(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$

Show that the coefficients of

$c(x) = a(x)\,b(x) = \sum_{k=0}^{5} c_k x^k$ are

the convolution of the coefficients of $a(x)$ and $b(x)$, this is

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}, \quad \text{where } a_3 = a_4 = a_5 = 0$$

and $b_4 = b_5 = 0$.

Answer:

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$$
\begin{array}{r}
a_0 + a_1 x + a_2 x^2 \\
b_0 + b_1 x + b_2 x^2 + b_3 x^3 \\
\hline
a_0 b_3 x^3 + a_1 b_3 x^4 + a_2 b_3 x^5 \\
a_0 b_2 x^2 + a_1 b_2 x^3 + a_2 b_2 x^4 \qquad\qquad \\
a_0 b_1 x + a_1 b_1 x^2 + a_2 b_1 x^3 \qquad\qquad\qquad\qquad \\
a_0 b_0 + a_1 b_0 x + a_2 b_0 x^2 \qquad\qquad\qquad\qquad\qquad\qquad \\
\hline
\end{array}
$$

$$a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + (a_0 b_3 + a_1 b_2 + a_2 b_1) x^3$$

$$+ (a_1 b_3 + a_2 b_2) x^4 + a_2 b_3 x^5 = \sum_{k=0}^{5} C_k x^k \text{, where}$$

$$C_k = \sum_{i=0}^{k} a_i b_{k-i}$$

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In general, given any two polynomials $a(x) = \sum_{i=0}^{M} a_i x^i$, $b(x) = \sum_{i=0}^{N} b_i x^i$, the product polynomial $c(x)$ can be described as $c(x) = a(x)\, b(x)$

$$= \sum_{k=0}^{M+N} C_k x^k, \quad \text{where } C_k = \sum_{i=0}^{k} a_i b_{k-i}$$

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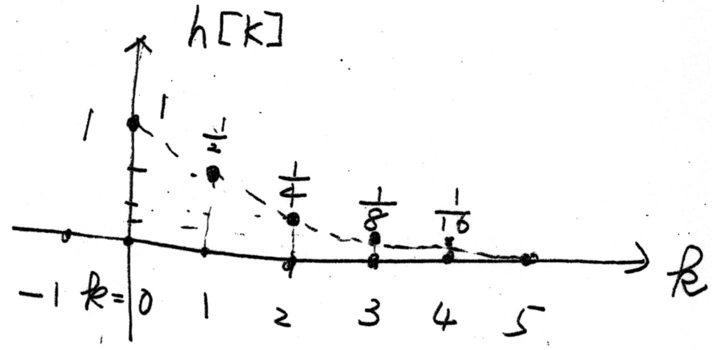
3.2.3

Finite Impulse Response and Infinite Impulse Response Discrete-Time Systems.

※ FIR system: A discrete-time system is said to be a finite impulse response (FIR) system if its impulse response has only a finite number of nonzero elements.

Example: $h[k] = (\frac{1}{2})^k (\delta[k] - \delta[k-5])$

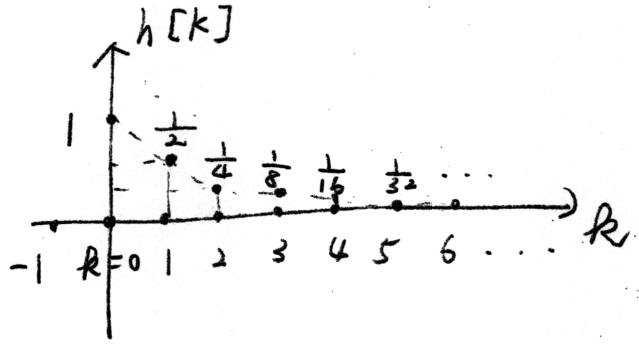
is an impulse response of FIR system.



※ IIR system: A discrete-time system is said to be a infinite impulse response (IIR) system if its impulse response has infinite number of nonzero elements.

Example: $h[k] = \left(\frac{1}{2}\right)^k q[k]$

is an impulse response of IIR system



For an FIR system of length N, i.e., $h[k] = 0$, for $k < 0$ and $k \geq N$, the convolution reduces to

$$y[k] = \sum_{i=0}^{k} h[i]\, u[k-i] = \sum_{i=0}^{k} h[k-i]\, u[i]$$

for $0 \leq k < N$

$$y[k] = \sum_{i=0}^{N-1} h[i]\, u[k-i] = \sum_{i=k-N+1}^{k} h[k-i]\, u[i]$$

for all $k \geq N$

Example: A moving-average (MA) filter of length 3 is described as

$$h[k] = \begin{cases} \frac{1}{3}, & k = 0, 1, 2 \\ 0, & \text{for } k < 0 \text{ and } k \geq 3 \end{cases}$$

What is the output $y[k]$ excited by the input $u[k] = \left(\frac{1}{6}\right)^k q[k]$?

Answer: $N = 3$

$$y[k] = \sum_{i=0}^{2} h[i]\, u[k-i] \quad , \quad 0 \le k < 3$$

$$k = 0 \Rightarrow \quad y[0] = \sum_{i=0}^{2} \frac{1}{3} \cdot \left(\frac{1}{6}\right)^{-i} q[-i]$$

$$= \frac{1}{3}$$

$$k = 1 \Rightarrow \quad y[1] = \sum_{i=0}^{2} \frac{1}{3} \left(\frac{1}{6}\right)^{1-i} q[1-i]$$

$$= \sum_{i=0}^{1} \frac{1}{3} \left(\frac{1}{6}\right)^{1-i}$$

$$= \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times 1$$

$$= \frac{1+6}{18} = \frac{7}{18}$$

$$k = 2 \Rightarrow \quad y[2] = \sum_{i=0}^{2} \left(\frac{1}{3}\right) \left(\frac{1}{6}\right)^{2-i} q[2-i]$$

$$= \frac{1}{3} \times \frac{1}{36} + \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times 1$$

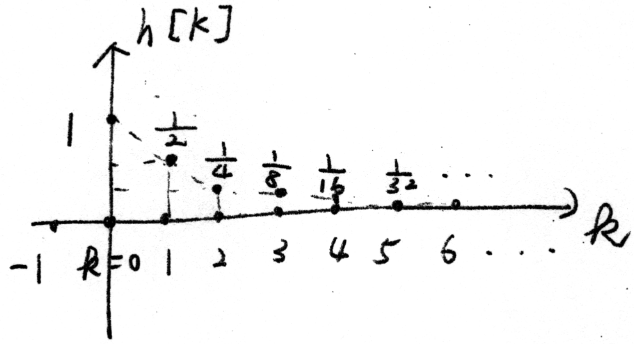
$$= \frac{1+6+36}{108} = \frac{43}{108}$$

$$y[k] = \sum_{i=0}^{2} h[i]\, u[k-i] \quad , \quad k \ge 3$$

$$= \sum_{i=0}^{2} \frac{1}{3} \left(\frac{1}{6}\right)^{k-i} \underset{1}{\underset{\parallel}{q[k-i]}}$$

Example: $h[k] = \left(\frac{1}{2}\right)^k q[k]$

is an impulse response of IIR system



For an FIR system of length N, i.e., $h[k] = 0$, for $k < 0$ and $k \geq N$, the convolution reduces to

$$y[k] = \sum_{i=0}^{k} h[i]\, u[k-i] = \sum_{i=0}^{k} h[k-i]\, u[i]$$

for $0 \leq k < N$

$$y[k] = \sum_{i=0}^{N-1} h[i]\, u[k-i] = \sum_{i=k-N+1}^{k} h[k-i]\, u[i]$$

for all $k \geq N$

Example: A moving-average (MA) filter of length 3 is described as

$$h[k] = \begin{cases} \frac{1}{3}, & k = 0, 1, 2 \\ 0, & \text{for } k < 0 \text{ and } k \geq 3 \end{cases}$$

$$= \sum_{i=0}^{2} \frac{1}{3}\left(\frac{1}{6}\right)^{k-i}, \quad k \geq 3$$

$$= \frac{\frac{1}{3}\left(\frac{1}{6}\right)^{k}\left[1-6^{3}\right]}{1-6}$$

$$= \frac{-215}{-5} \cdot \frac{1}{3}\left(\frac{1}{6}\right)^{k}$$

$$= \frac{43}{3}\left(\frac{1}{6}\right)^{k} \quad k \geq 3,$$

$$\therefore \ y[k] = \begin{cases} \frac{1}{3}, & k = 0 \\ \frac{7}{18}, & k = 1 \\ \frac{43}{108}, & k = 2 \\ \frac{43}{3}\left(\frac{1}{6}\right)^{k}, & k \geq 3 \end{cases}$$

3.3 Difference Equations for LTIL Discrete-Time Systems

For any causal and zero-state system, the output can be described as

$$y[k] = \sum_{i=0}^{k} h[k-i]\,u[i], \quad k = 0, 1, 2, \ldots$$

Some convolutions can be transformed into difference equations.

Example :

A LTI zero-state system is described as

$$y[k] = \sum_{i=0}^{k} \alpha^{k-i} u[i] \ , \quad k \geq 0$$

Then,
$$\alpha y[k] = \sum_{i=0}^{k} \alpha^{k+1-i} u[i] \ , \quad k \geq 0$$

$$y[k+1] = \sum_{i=0}^{k+1} \alpha^{k+1-i} u[i] \ , \quad k \geq 0$$

$$= \alpha y[k] + \alpha^{k+1-(k+1)} u[k+1]$$

$$= \alpha y[k] + u[k+1]$$

$$\Rightarrow y[k+1] - \alpha y[k] = u[k+1]$$

This is called a first-order linear difference equation with constant coefficients or an LTIL difference equation.

It is important to mention that not every discrete convolution can be transformed into an LTIL difference equation.

3.3.1 From Difference Equation to impulse response

Not every convolution can be transformed into a difference equation. However, if the difference equation description of a system is known, the impulse response of the system can be readily obtained.

Example: Given a difference equation of the system $y[k+1] = \alpha y[k] + u[k+1]$ or $y[k] = \alpha y[k-1] + u[k]$, what is the impulse response of this system?

Answer:

※ The impulse response is computed when the system is excited by an impulse sequence $\delta[k]$ with ZERO INITIAL CONDITIONS (that is, $y[-1] = y[-2] = y[-3] = \cdots = 0$ or $y[k] = 0, \forall k < 0$)

$$\therefore\ h[k] = \alpha h[k-1] + \delta[k]$$

$$h[0] = \alpha \underset{\substack{\| \\ 0}}{h[-1]} + \underset{\substack{\| \\ 1}}{\delta[0]} = 1$$

$$h[1] = \alpha \underset{\substack{\| \\ 1}}{h[0]} + \underset{\substack{\| \\ 0}}{\delta[1]} = \alpha$$

$$h[2] = \alpha \underset{\substack{\| \\ \alpha}}{h[1]} + \underset{\substack{\| \\ 0}}{\delta[2]} = \alpha^2 .$$

$$\begin{aligned} h[k] &= \alpha h[k-1] + \delta[k] \\ &= \alpha h[k-1] = \alpha^k, \quad \forall k \geq 0 \\ &= \alpha^k g[k] \end{aligned}$$

3.3.2 Comparison of Discrete Convolution and Difference Equations

Since the impulse response is defined as the output of a system excited by an impulse $\delta[k]$ without any initial conditions, the discrete convolution discribes only the zero-state response of a system.

Although the difference equation is developed from the convolution, it can also be derived from sampling physical quantities of a real system and it describes not only the zero-state but zero-input responses as well.

Example: For the difference equation of a system described as

$$y[k] = \alpha y[k-1] + u[k],$$

① the zero-state response is

$$y[k] = \alpha y[k-1] + u[k], \quad \text{with}$$

$$y[-1] = y[-2] = \cdots = 0 \quad \text{or} \quad y[k] = 0 \quad \forall k < 0.$$

② the zero-input response is

$$y[k] = \alpha y[k-1], \quad \text{with at least}$$

$$y[-1] \neq 0.$$

Example : Given the difference equation of a system such as

$y[k] = \alpha y[k-1] + u[k]$ with initial condition $y[-1] = C$, what is the total response excited by an impulse $\delta[k]$?

Answer :

① zero-state response :

reset $y[-1] = y[-2] = \cdots = 0$

$y[k] = \alpha^k g[k]$ from previous example

② zero-input response

$y[k] = \alpha y[k-1]$, with $y[-1] = C$.

$y[0] = \alpha y[-1] = \alpha C$

$y[1] = \alpha y[0] = \alpha^2 C$

$\vdots$

$y[k] = \alpha y[k-1] = \alpha^{k+1} C, \quad \forall k \geq 0$

$= \alpha^{k+1} C \, g[k]$

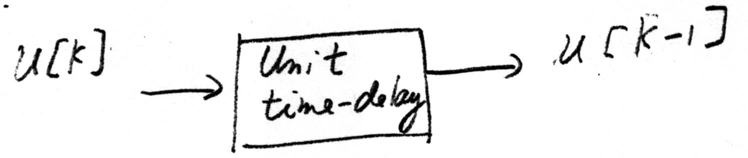
$\therefore$ total response = ① + ②

$= \alpha^k g[k] + \alpha^{k+1} C \, g[k]$

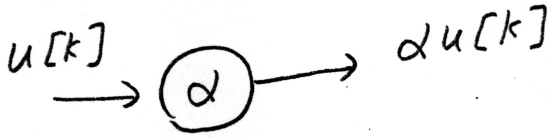
Based on the reason above, the difference equation is preferable to the convolution in describing an LTIL system.

3.4 Setting up Difference Equations

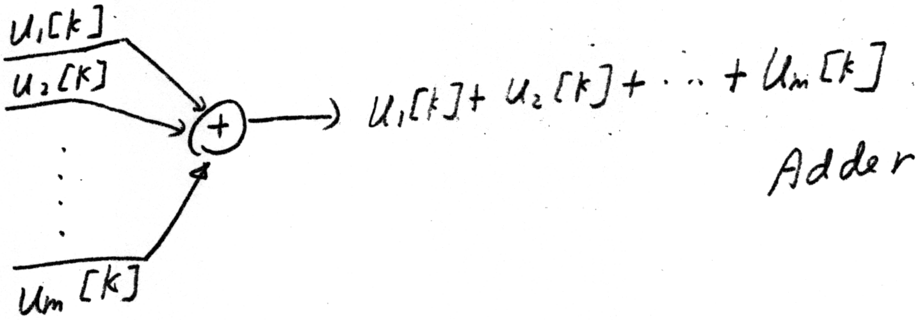
※ Block Diagram



Unit-delay element



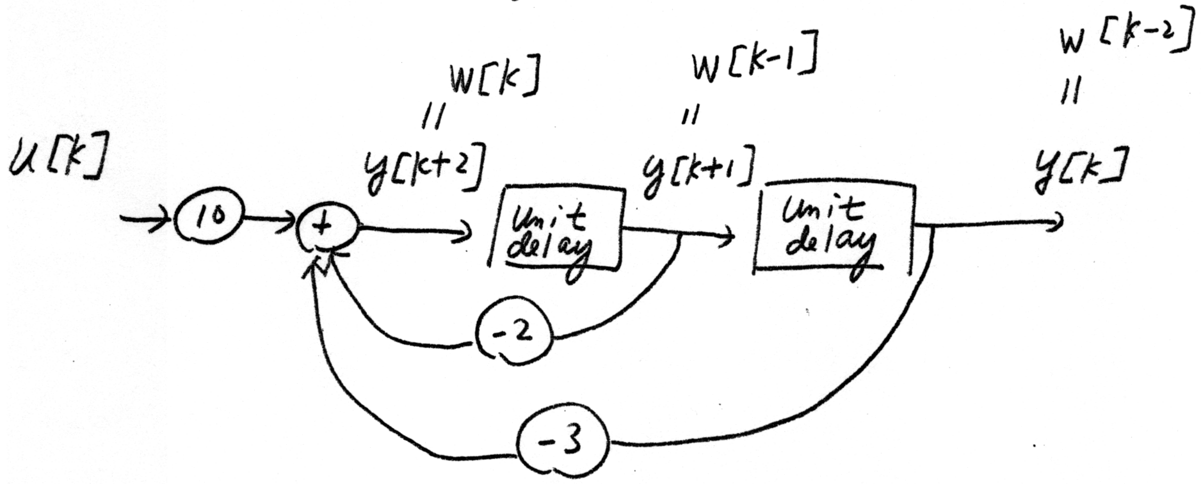
Multiplier



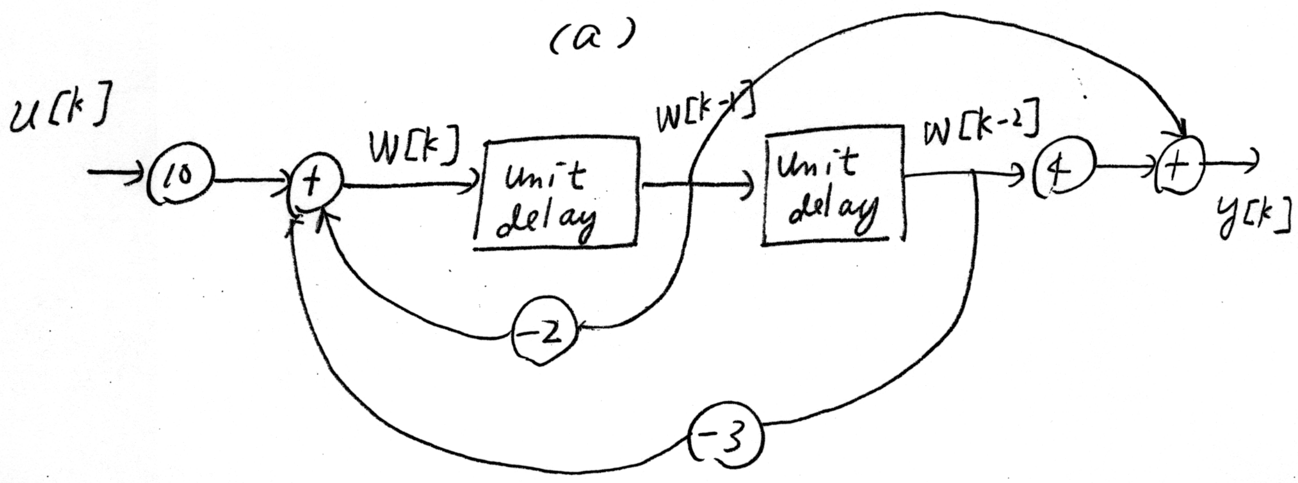
Adder

The three basic elements above will construct discrete-time block diagrams for LTIL systems. A difference equation can be developed from any discrete-time block diagram.

Example: Find the difference equations from the block diagrams depicted in the following figures.



(a)



Answer: (a)

$$y[k+2] = 10u[k] - 2y[k+1] - 3y[k]$$

is a second-order difference equation

or $y[k] = -2y[k-1] - 3y[k-2] + 10u[k-2]$

(b)

$$W[k] = 10u[k] - 2W[k-1] - 3W[k-2]$$

$$y[k] = 4W[k-2] + W[k-1]$$

How can you simplify this?

Two Solutions !! (NOT STRAIGHT FOWARD AS STATED IN TEXT !!)

(i) Z-Transform (talk about this later)

(ii)

$$w[k] + 2w[k-1] + 3w[k-2] = 10u[k]$$

$$1 + 2q^{-1} + 3q^{-2} = A(q)$$

$$\sum_{n=0}^{L_1} a_n w[k-n] \longrightarrow A(q) = \sum_{n=0}^{L_1} a_n q^{-n}$$

($L_1 = 2$, $a_0 = 1$, $a_1 = 2$, $a_2 = 3$ in this case)

$$y[k] = w[k-1] + 4w[k-2]$$

$$1 \cdot q^{-1} + 4q^{-2}$$

$$\sum_{m=0}^{L_2} b_m w[k-m] \longrightarrow B(q) = \sum_{m=0}^{L_2} b_m q^{-m}$$

($L_2 = 2$, $b_0 = 0$, $b_1 = 1$, $b_2 = 4$, in this case)

LCD (least Common divisor polynomial)

of $A(q)$ and $B(q) = C(q)$

$$C(q) := \sum_{l=0}^{L_3} c_l q^{-l}$$

$$G(q) := \frac{C(q)}{A(q)} = \sum_{p=0}^{L_4} g_p q^{-p}$$

$$H(q) := \frac{C(q)}{B(q)} = \sum_{r=0}^{L_5} h_r q^{-r}$$

The simplified difference equation is

$$\sum_{r=0}^{L5} h_r \, y[k-r] = \sum_{p=0}^{L4} 10\, g_p \, u[k-p]$$

In this example, $A(q) = 1 + 2q^{-1} + 3q^{-2}$ and $B(q) = q^{-1} + 4q^{-2}$ are coprime. Hence

$$C(q) = A(q)\, B(q) \Rightarrow G(q) = B(q), \; H(q) = A(q)$$

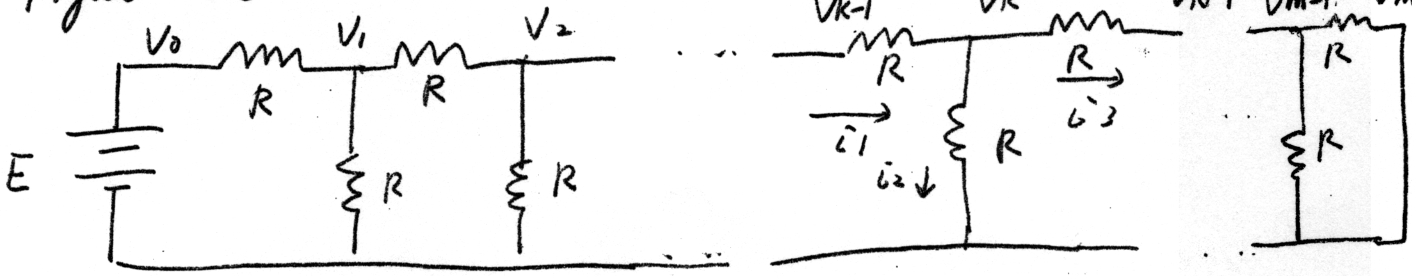
$\therefore \; g_0 = b_0 = 0, \; g_1 = b_1 = 1, \; g_2 = b_2 = 4$

$h_0 = a_0 = 1, \; h_1 = a_1 = 2, \; h_2 = a_2 = 3$

$$\Rightarrow \; y[k] + 2y[k-1] + 3y[k-2] = 10u[k-1] + 40u[k-2]$$

※ Circuit Diagram

Consider the resistive network shown in Figure below.



$$\frac{V_{k-1} - V_k}{R} = \frac{V_k}{R} + \frac{V_k - V_{k+1}}{R}$$

$$\Rightarrow \quad V_{k-1} - 3V_k + V_{k+1} = 0$$

is a difference equation for an LTIL system.

3.4.1 General Forms of Difference Equations

In general, any LTIL system can be described as the following difference equation

$$y[k+n] + a_{n-1}\, y[k+n-1] + \cdots + a_1 y[k+1] + a_0 y[k] = b_m\, u[k+m] + b_{m-1}\, u[k+m-1] + \cdots + b_1\, u[k+1] + b_0\, u[k] \quad \cdots \text{advanced form}$$

Since we need a recursive formula for $y[k]$, usually we rewrite the difference equation as

$$y[k] = -a_{n-1}\, y[k-1] - \cdots - a_1\, y[k-n+1] - a_0\, y[k-n] + b_m\, u[k+m-n] + b_{m-1}\, u[k+m-n-1] + \cdots + b_1\, u[k-n+1] + b_0\, u[k-n] \quad \cdots \text{delayed form}$$

3.4.2 Recursive and nonrecursive difference Equations

If $a_i = 0$, $i = 0, 1, 2, \cdots, n-1$,

the reduced difference equation

$$y[k] = b_m u[k-n+m] + b_{m-1} u[k-n+m-1] + \cdots + b_1 u[k-n+1] + b_0 u[k-n]$$

is called nonrecursive difference equation;

Otherwise, if there is at least one coefficient $a_i \neq 0$, the difference equation

$$y[k] = -a_{n-1} y[k-1] - \cdots - a_1 y[k-n+1] - a_0 y[k-n] + b_m u[k+m-n] + b_{m-1} u[k+m-n-1] + \cdots + b_1 u[k-n+1] + b_0 u[k-n]$$

is called recursive difference equation.

Nonrecursive difference equations will generate FIR but recursive difference equations will not necessarily generate IIR

Example:

Classify the following difference equations as recursive or nonrecursive and compute their impulse responses. Classify the systems they describe as IIR or FIR

(a) $y[k+2] + 2y[k+1] - 3y[k] = u[k+1]$

It is a recursive system

$y[k] + 2y[k-1] - 3y[k-2] = u[k-1]$

$h[k] + 2h[k-1] - 3h[k-2] = \delta[k-1]$

$h[k] = \delta[k-1] - 2h[k-1] + 3h[k-2]$

$h[0] = 0 - 2\times 0 + 3\times 0 = 0$

$h[1] = 1 - 2\times 0 + 3\times 0 = 1$

$h[2] = 0 - 2\times 1 + 3\times 0 = -2$

$h[3] = 0 - 2\times(-2) + 3\times 1 = 7$

$h[4] = 0 - 2(7) + 3(-2) = -20$

$\vdots$

$h[k] \neq 0, \quad \forall k > 0$

It is an IIR

(b) $y[k] = 4u[k] - 5u[k-1]$

It is non-recursive system.

$h[k] = 4\delta[k] - 5\delta[k-1]$

$h[0] = 4$

$h[1] = -5$

$h[k] = 0, \quad \text{for } k \geq 2$

It is FIR.

(c) $y[k+1] + y[k] = u[k+1] + 2u[k] + u[k-1]$

It is recursive system.

$y[k] = -y[k-1] + u[k] + 2u[k-1] + u[k-2]$

$h[k] = -h[k-1] + \delta[k] + 2\delta[k-1] + \delta[k-2]$

$h[0] = -0 + 1 + 0 + 0 = 1$

$h[1] = -1 + 0 + 2 + 0 = 1$

$h[2] = -1 + 0 + 0 + 1 = 0$

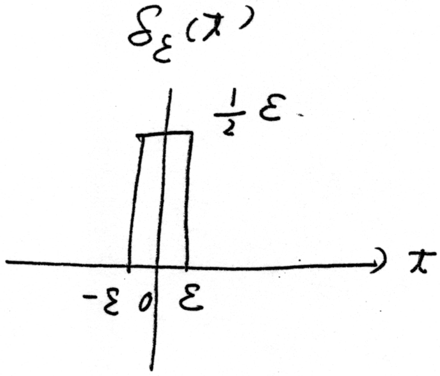
$h[3] = 0 + 0 + 0 + 0 = 0$

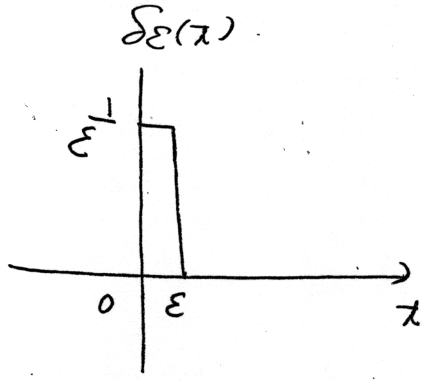
$h[k] = 0, \quad \forall k \geq 2$

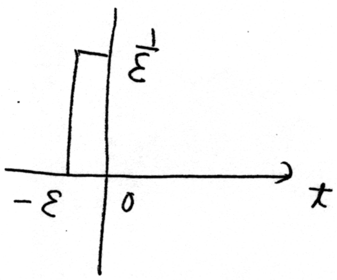
It is FIR.

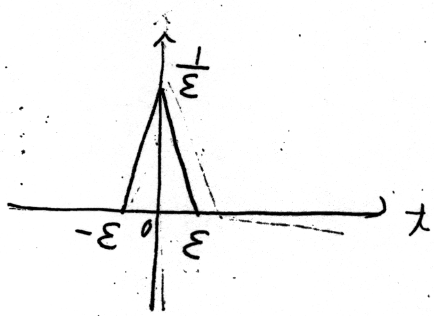
3.5 LTI Continuous-time Systems — Convolutions

3.5.1 The impulse









$\delta(t) := \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$ is called the

Dirac delta function, δ- function, impulse function.

Properties of impulse function

(i) $\delta(t) = 0$, for $t \neq 0$

(ii) $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

(iii) $\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1$

(iv) $f(t)\, \delta(t - t_0) = f(t_0)\, \delta(t - t_0)$

(v) $\int_{-\infty}^{\infty} f(t)\, \delta(t - t_0)\, dt = f(t_0)$

Example: Compute the following:

(a) $\int_{-\infty}^{\infty} \sin(t)\,\delta(t-\frac{\pi}{2})\,dt$

$= \sin(\frac{\pi}{2}) = 1$

(b) $\int_{0}^{1} \cos(t)\,\delta(t)\,dt$ not defined

(c) $\int_{0^-}^{1} \cos(t)\,\delta(t)\,dt = \cos(0) = 1$

(d) $\int_{0^+}^{1} \cos(t)\,\delta(t)\,dt = 0$

(e) $\int_{0}^{3^+} \cos(t)\,\delta(t-3)\,dt = \cos(3)$

$$a^+ = \lim_{\varepsilon \to 0} a+\varepsilon$$

$$a^- = \lim_{\varepsilon \to 0} a-\varepsilon$$

3.5.2 Convolution Integrals

Consider an LTI continuous-time system. The system is assumed to be initially relaxed at $t=0$ (no initial condition).

A particular output excited by the input as impulse $\delta(t)$ is denoted by $h(t)$ and called impulse response.

Impulse Response $= h(t) :=$ output excited by $\delta(t)$

If the system is causal, no output will appear befor the application of the impulse $\delta(t)$

Therefore, we have $h(t) = 0$, for $t < 0$

In fact, the condition $h(t) = 0$, $t < 0$ is necessary and sufficient for any LTI system to be causal.

Since $\delta(t)$ excites the output $h(t)$,

$\{\delta(t)\} \to \{h(t)\}$ is a permissible pair.

Any zero-state output $y(t)$ for the LTI system can be written as

$$y(t) = u(t) \overset{\text{convolution}}{\circledast} h(t)$$

$$= \int_{-\infty}^{\infty} u(t-\tau)\, h(\tau)\, d\tau$$

$$= h(t) \circledast u(t)$$

$$= \int_{-\infty}^{\infty} h(t-\tau)\, u(\tau)\, d\tau$$

$\dot{X}$. $\quad a(t) \circledast \left[\sum_{i=1}^{N} c_i b_i(t) \right]$

$$= \sum_{i=1}^{N} c_i \left[a(t) \circledast b_i(t) \right] = \sum_{i=1}^{N} c_i \left[b_i(t) \circledast a(t) \right]$$

Example: A LTI system has an impulse response

$$h(t) = e^{-0.5t}[g(t) - g(t-3)].$$

What is the output excited by the input $u(t) = e^{j\omega t}$?

Answer:

$$y(t) = u(t) \circledast h(t)$$

$$= \int_{-\infty}^{\infty} e^{j\omega(t-\tau)} e^{-0.5\tau}[g(\tau) - g(\tau-3)]\,d\tau$$

$$= \int_{0}^{3} e^{j\omega t} e^{-(0.5+j\omega)\tau}\,d\tau$$

$$= e^{j\omega t} \frac{[e^{-1.5-3j\omega} - 1]}{-0.5-j\omega}$$

$$= e^{j\omega t} \frac{1 - e^{-1.5-3j\omega}}{0.5+j\omega}$$

3.5.3 Graphical Computation of Convolution

The continuous-time convolution consists of four steps: $\left(y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, u(\tau)\, d\tau \right)$

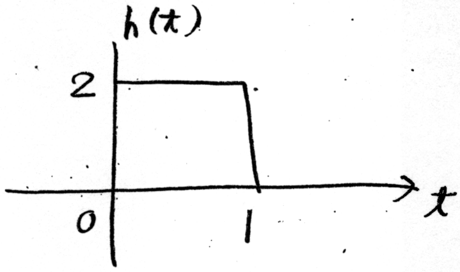
① flipping $h(\tau)$ to $h(-\tau)$

② Shifting $h(-\tau)$ to $h(t-\tau)$ by t

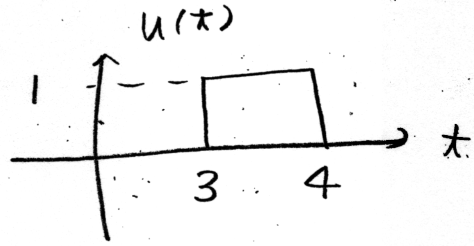
③ multiplying $h(t-\tau)$ with $u(\tau)$

④ Integrating $\int_{-\infty}^{\infty} h(t-\tau)\, u(\tau)\, d\tau$ to generate $y(t)$

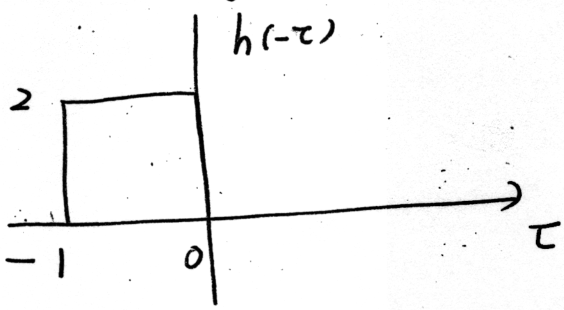
Example: Consider a LTI system with the following impulse response $h(t)$.



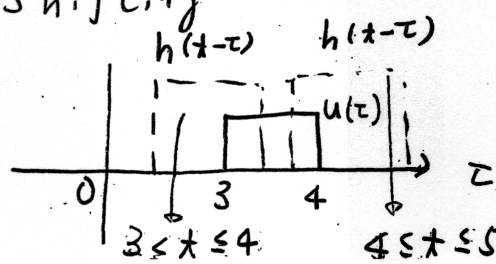
What is the output $y(t)$ excited by the input $u(t)$ depicted as below?



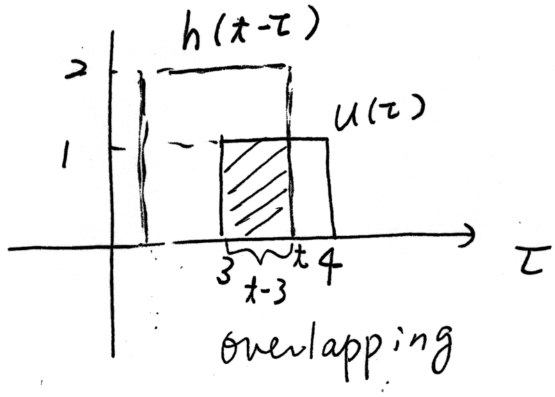
Answer: ① flipping $h(\tau)$ to $h(-\tau)$



② shifting $h(-\tau)$ to $h(t-\tau)$ by t

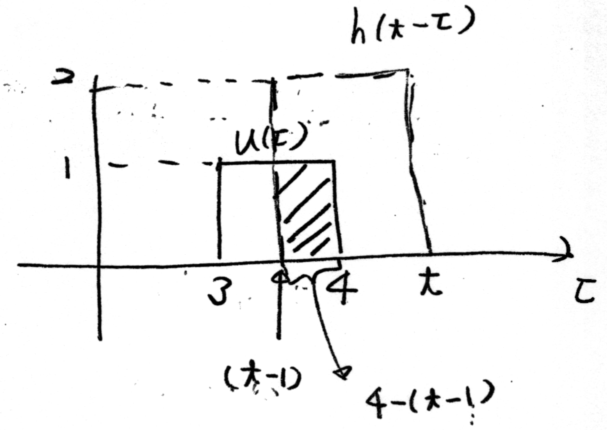


③ If $3 \le t \le 4$



Multiplying $h(t-\tau)$ with $u(\tau) = \begin{cases} 2 \times 1 = 2, & 3 \le \tau \le t \\ 0, & \text{elsewhere} \end{cases}$

If $4 \le t \le 5$

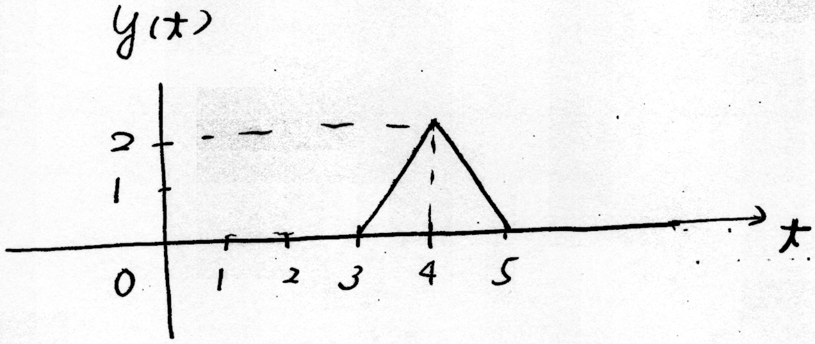


Multiplying $h(t-\tau)$ with $u(\tau) = \begin{cases} 2 \times 1 = 2, & (t-1) \le \tau \le 4 \\ 0, & \text{elsewhere} \end{cases}$

④ Integrating $h(t-\tau)\,u(\tau)$

If $3 \le t \le 4$, $\quad y(t) = \int_3^t 2 \, d\tau = 2(t-3)$

If $4 \le t \le 5$, $\quad y(t) = \int_{t-1}^4 2 \, d\tau = 2(5-t)$



3.5.4 Impulse Response and Unit Step Response

Again, the unit step function is defined

as
$$g(t-t_0) := \begin{cases} 1, & \text{for } t \geq t_0 \\ 0, & \text{for } t < t_0 \end{cases}$$

The impulse and unit step function are related by

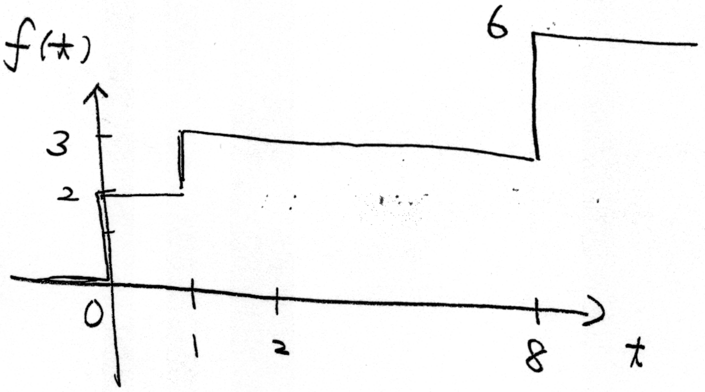
$$\delta(t-t_0) = \frac{d}{dt}[g(t-t_0)]$$

Example: If a function can be described as

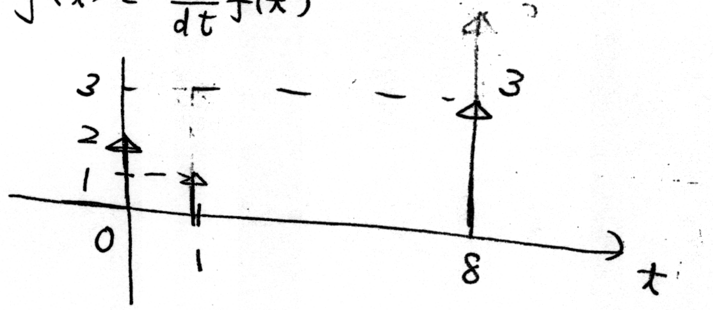
$$f(t) = 2g(t) + g(t-1) + 3g(t-8),$$

then its derivative is

$$\frac{d}{dt}f(t) = 2\delta(t) + \delta(t-1) + 3\delta(t-8)$$



$\dot{f}(t) = \frac{d}{dt} f(t)$



As shown above, the area of each impulse is indicated by the height of the arrow.

※ Unit Step Response

The unit step response for a causal LTI system ($h(t) = 0$, for $t < 0$) is defined as

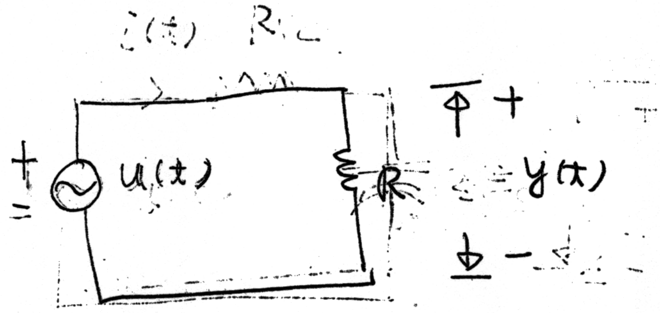
$$y_g(t) = \int_0^t h(\tau)\, g(t-\tau)\, d\tau = \int_0^t h(\tau)\, d\tau, \quad \text{for } t \geq 0$$

The differentiation will yield

$$h(t) = \frac{d}{dt} y_g(t)$$

In practice, the unit step function is easily generated and the impulse response can be obtained from the differentiation of the unit - step response.

Example : Consider a LTI system depicted as below.



zero-state unit-step response

$y_g(t) = R\, g(t)$

The impulse response

$h(t) = \frac{d}{dt} y_g(t) = R\, \delta(t)$

3.6 LTIL Continuous-Time Systems - Differential Equations

The zero-state response of every LTI continuous-time causal system can be described as

$$y(t) = \int_0^t h(t-\tau)\, u(\tau)\, d\tau$$

3.7 Setting Up Differential Equations

Differential Equations can be obtain by the differentration of ~~continuous~~ convolution for an LTIL system.

If we differentiate the output of an LTIL causal system described by $y(t) = \int_0^t h(t-\tau)\, u(\tau)\, d\tau$,

we obtain $\frac{dy(t)}{dt} = \int_0^t \frac{d}{dt}[h(t-\tau)]\, u(\tau)\, d\tau$

$$+ h(t-\tau)\, u(\tau) \Big|_{\tau = t}$$

※ Lebnitz's rule:

Assume $f(t) = \int_{A(t)}^{B(t)} g(t,\tau)\, d\tau$

$$\frac{df(t)}{dt} = \int_{A(t)}^{B(t)} \frac{dg(t,\tau)}{dt}\, d\tau + \left[\frac{d}{dt} B(t)\right] g(t,\tau)\Big|_{\tau = B(t)}$$

$$- \left[\frac{d}{dt} A(t)\right] g(t,\tau)\Big|_{\tau = A(t)}$$

$$\frac{dy(t)}{dt} = \int_0^t \frac{d}{dt}\left[h(t-\tau)\right] u(\tau)\, d\tau + h(t-\tau)\,u(\tau)\Big|_{\tau = t}$$

sometimes leads to a differential equation.

Example : The impulse response for an LTIL system is $h(t) = \beta e^{-ct} q(t)$

The zero-state response is

$$y(t) = \int_0^t h(t-\tau)\, u(\tau)\, d\tau$$

Then,
$$\frac{dy(t)}{dt} = \int_0^t \frac{d}{dt}\left[\beta e^{-c(t-\tau)} q(t-\tau)\right] u(\tau)\, d\tau + \beta e^{-c(t-\tau)} q(t-\tau)\, u(\tau)\Big|_{\tau = t}$$

$$= \int_0^t -\beta c\, e^{-c(t-\tau)} u(\tau)\, d\tau + \beta u(t)$$

$$= -c \int_0^t \beta e^{-c(t-\tau)} u(\tau)\, d\tau + \beta u(t)$$

$$= -c\, y(t) + \beta u(t)$$

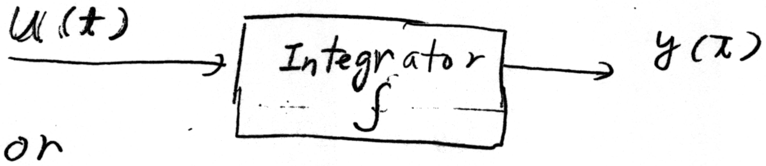
$\therefore$ $\dot{y}(t) + c\,y(t) = \beta u(t)$

is the differential equation for this LTIL system (First-order)

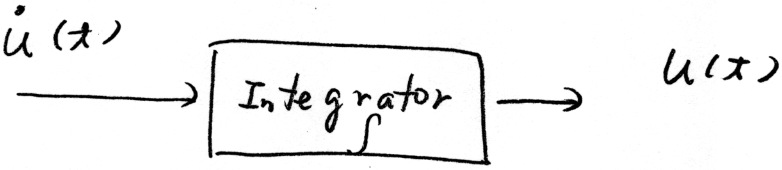
Setting up differential equations from the Block Diagram

※ Integrator

$$y(t) = \int_{t_0}^{t} u(\tau)\,d\tau + u(t_0)$$

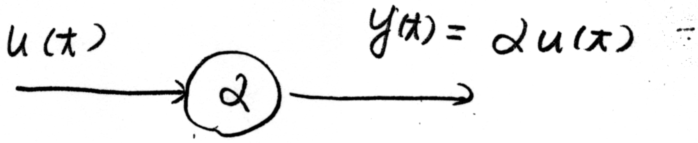


or



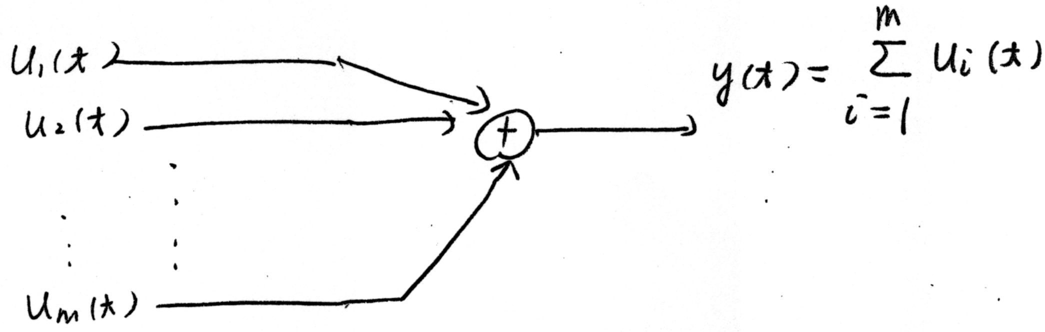
※ Multiplier

$$y(t) = \alpha u(t)$$

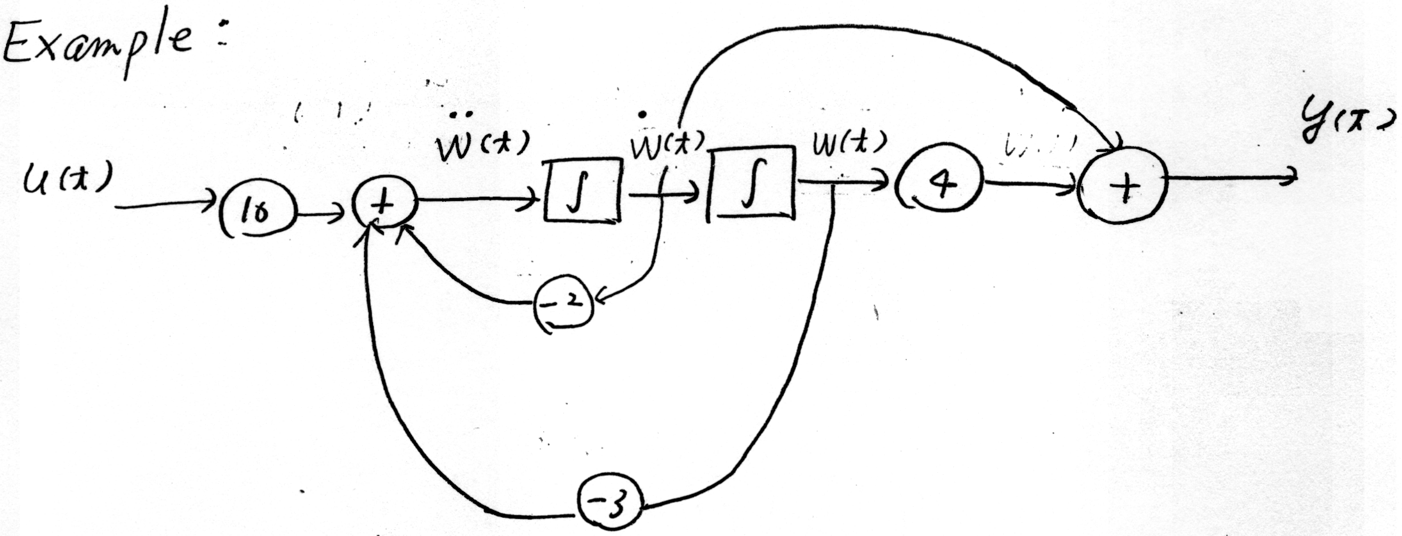


※ Adder

$$y(t) = u_1(t) + u_2(t) + \cdots + u_m(t)$$



Example:



$$y(t) = \dot{w}(t) + 4w(t)$$

$$\ddot{w}(t) = -2\dot{w}(t) - 3w(t) + 10u(t)$$

$$\Rightarrow \ddot{w}(t) + 2\dot{w}(t) + 3w(t) = 10u(t)$$

$$\ddot{y}(t) = \dddot{w}(t) + 4\ddot{w}(t)$$

$$2\dot{y}(t) = \qquad 2\ddot{w}(t) + 8\dot{w}(t)$$

$$3y(t) = \qquad 3\dot{w}(t) + 12w(t)$$

+)

$$\ddot{y}(t) + 2\dot{y}(t) + 3y(t)$$

$$= \dddot{w}(t) + 6\ddot{w}(t) + 11\dot{w}(t) + 12w(t)$$

$$= \underbrace{\dddot{w}(t) + 2\ddot{w}(t) + 3\dot{w}(t)}_{10\dot{u}(t)}$$

$$+ \underbrace{4\ddot{w}(t) + 8\dot{w}(t) + 12w(t)}_{40u(t)}$$

$$= 10\dot{u}(t) + 40u(t)$$

$$\therefore \ddot{y}(t) + 2\dot{y}(t) + 3y(t) = 10\dot{u}(t) + 40u(t)$$

How to simplify this?

(i) Laplace-Transform (talk about this later))

(ii) $$\sum_{n=0}^{L_1} a_n \left[\frac{d^{(n)}}{dt^n} w(t)\right] = 10u(t)$$

$$\rightarrow \quad A(q) = \sum_{n=0}^{L_1} a_{L_1-n}\, q^{-n}$$

$(L_1 = 2,\ a_0 = 3,\ a_1 = 2,\ a_2 = 1)$

$$y(t) = \sum_{m=0}^{L_2} b_m \left[\frac{d^{(m)}}{dt^m} w(t)\right]$$

$$\rightarrow \quad B(q) = \sum_{m=0}^{L_2} b_{L_2-m}\, q^{-m}$$

$(L_2 = 1,\ b_0 = 4,\ b_1 = 1$ in this case))

LCD of $A(q)$ and $B(q) = C(q)$

$$C(q) := \sum_{l=0}^{L_3} C_l\, q^{-l}$$

$$G(q) := \frac{C(q)}{A(q)} = \sum_{p=0}^{L_4} g_p\, q^{-p}$$

$$H(q) := \frac{C(q)}{B(q)} = \sum_{r=0}^{L_5} h_r\, q^{-r}$$

The simplified differential equation is

$$\sum_{r=0}^{L_5} h_{L_5-r} \left[\frac{d^{(r)}}{dt^r} y(t)\right] = \sum_{p=0}^{L_4} 10 g_{L_4-p} \left[\frac{d^{(p)}}{dt^p} u(t)\right],$$

In this example, $A(q) = 1 + 2q^{-1} + 3q^{-2}$ and $B(q) = 1 + 4q^{-1}$ are co-prime. Hence

$C(q) = A(q)B(q) \Rightarrow G(q) = B(q)$, $H(q) = A(q)$

$\therefore$ $g_0 = 1$, $g_1 = 4$, $L_4 = 1$.
$h_0 = 1$, $h_1 = 2$, $h_2 = 3$, $L_5 = 2$.

$$\Rightarrow \quad \ddot{y}(t) + 2\dot{y}(t) + 3y(t) = 10\dot{u}(t) + 40u(t)$$