Chapter 2  Systems

2.1
2.2  What is a system?

As depicted above, a system is modeled as a black box with at least one input terminal and one output terminal. A system is called a continuous-time system if its input/output are both continuous-time signals. Similarly, a discrete-time system has discrete-time signals as its input/output.

X SISO system: a system with one input and one output terminals.
X MIMO system: a system with multiple input and multiple output terminals.
X memoryless system: a system's output $y(t_0)$ depends only on the input applied at time $t_0$ but independent of the input applied before or after $t_0$. 
Example: Voltage divider

\[ y(t) = \frac{R_2}{R_1 + R_2} u(t), \quad u(t) \]

is a memoryless system.

Example: \( y(t) = u(t-1) + 2u(t) - 3u(t+2) \)

is a system with memory.

Causal or non-anticipatory system: a system's present output depends on past and present inputs but no future input.

Example: Integrator

\[ y(t_0) = \int_{-\infty}^{t_0} u(x) \, dx \]

is a causal system.

Example: Unit-time delay system:

\[ y(t) = u(t-1) \]

is a causal system.
Example

Unit-time advance system

\[ y(t) = u(t+1) \]

is a noncausal system.

2.3 State — set of initial conditions

\[ x(t) \rightarrow y(t) \rightarrow t = t_0 \]

\[ c \]

\[ u(t) \rightarrow v(t) \rightarrow \]

\[ C \]

\[ i(t) = \frac{d}{dt} \left[ CV(t) \right] = C \frac{dv(t)}{dt} \]

\[ U(t) = \int_{t_0}^{t} \frac{i(t')}{C} dt' + U(t_0) \]

We see that \( U(t) \) depends on \( U(t_0) \) and \( i(t') \) for \( t' \in [t_0, t] \). To solve \( U(t) \) for any time instant \( t \), we need information of \( U(t_0) \). Thus

\[ \text{State variable} \rightarrow U(t_0) \quad \rightarrow \quad U(t), \quad t \geq t_0 \]

\[ i(t), \quad t \geq t_0 \]

The input \( i(t) \) applied before \( t = t_0 \) is not needed.
The state summarizes the effect of past input on future output. We can define the state $x(t)$ of a system at time $t$ as the information at $t$, together with the input $u(t)$ to determine uniquely the output $y(t)$ for all $t \geq t_0$. A system is

an **lumped system** if its state has a finite number of state variables. A system is a **distributed system** if its state has infinitely many state variables.

Example: a system is described as

$$y(t) = a \int_0^t u(\tau) \, d\tau.$$ Is this a lumped or distributed system?

Answer: The output $y(t)$ simply depends on input $u(t)$ only. Hence no state is in this system. It is lumped.
Example: A unit delay system described as 
\[ y(t) = u(t-1) \] is lumped or distributed?

Answer: In order to determine \( y(t) \) for \( t \geq t_0 \) from \( u(t) \), for \( t \geq t_0 \), we need information \( u(t) \) for all \( t \) in \([t_0-1, t_0)\). Therefore the state of the system at time \( t_0 \) is \( u(t) \) for all \( t \) in \([t_0-1, t_0)\). There are infinitely many state variables since there are infinitely many points in the time interval \([t_0-1, t_0)\).

It is a distributed system.

2.3.1 Zero-input and zero-state response.

The output of a system with memory depends on the input and initial states.

* Zero-input response of the system: The output \( y(t) \) for \( t \geq t_0 \) is excited exclusively by the initial states.
$X(t_0)$ without any input
zero-state response. The output $Y(x)$, for
$x \geq t_0$ is excited exclusively by the input
$u(x)$, for $x \geq t_0$ without any initial state (initial state values are zero).

We can express these two responses as

\[
\begin{align*}
X(t_0) &\neq 0 \\
u(x) &\equiv 0, \text{ for } x \geq t_0
\end{align*}
\]

\(\rightarrow Y_{zi}(t), \text{ for } t \geq t_0\) (zero-input response)

\[
\begin{align*}
X(t_0) &\equiv 0 \\
u(x) &\neq 0, \text{ for } x \geq t_0
\end{align*}
\]

\(\rightarrow Y_{zs}(t), \text{ for } t \geq t_0\) (zero-state response)

2.4 Linearity of a System

A memoryless system is said to be linear, if, for any permissible pairs
\(\{u_1(x)\} \rightarrow \{y_1(x)\}\) and \(\{u_2(x)\} \rightarrow \{y_2(x)\}\),
the following pairs:

\[
\begin{align*}
\{a_1 u_1(x) + a_2 u_2(x)\} &\rightarrow a_1 y_1(x) + a_2 y_2(x)
\end{align*}
\]
\[ \{ u_1(t) + u_2(t) \} \rightarrow \{ y_1(t) + y_2(t) \} \quad \text{(additivity)} \]

and \[ \{ 2u_1(t) \} \rightarrow \{ 2y_1(t) \} \quad \text{(homogeneity)} \]

for any \( \alpha \) are also permissible.

On combine these two relationships for any \( \alpha_1 \) and \( \alpha_2 \), we obtain

\[ \{ \alpha_1 u_1(t) + \alpha_2 u_2(t) \} \rightarrow \{ \alpha_1 y_1(t) + \alpha_2 y_2(t) \} \]

are also permissible.

Note: If an input \( u(t) \) generates an output \( y(t) \) through a system, we call \( \{ u(t) \} \rightarrow \{ y(t) \} \) is a permissible pair.

Example: Are the following systems linear?

(a) \( y(t) = \sin(u(t)) \)

(b) \( y(t) = \sin(u(t)) \cdot u(t) \)

Answer: (a) \( \sin(u_1(t) + u_2(t)) = \sin(u_1(t)) \cos(u_2(t)) + \cos(u_1(t)) \sin(u_2(t)) \neq \sin(u_1(t)) + \sin(u_2(t)) \)
It is NOT linear or it is nonlinear.

(b) Check if \[ \{ d, u_1(t) + d_2 u_2(t) \} \rightarrow \{ d_1 y_1(t) + d_2 y_2(t) \} \]

\[ y_1(t) = \sin(t) \left[ d_1 u_1(t) + d_2 u_2(t) \right] \]

\[ = d_1 \sin(t) u_1(t) + d_2 \sin(t) u_2(t) \]

\[ = d_1 y_1(t) + d_2 y_2(t) \]

where \( y_1(t) = \sin(t) u_1(t) \) and \( y_2(t) = \sin(t) u_2(t) \).

The definition of linearity for systems is slightly more complicated. The system with memory is defined to be linear if for any two permissible state-input-output pairs, with \( i = 1,2 \),

\[ X_i(t_0), u_i(t), t \geq t_0 \]

the following pairs

\[ X_1(t_0) + X_2(t_0) \]

\[ U_1(t) + U_2(t), t \geq t_0 \]

\( \rightarrow \) \( y_1(t) + y_2(t), t \geq t_0 \)

(additivity)
and

\[ \frac{dx_1(t_0)}{dx_{1(t)}, x \geq t_0} \rightarrow dy_1(t), \ x \geq t_0 \]

(homogeneity)

for any \( d \), one also permissible.

Example:

\[ \text{Consider the network shown above. The input is a current source } i(t) \text{ and the output is the voltage across the resistor with resistance } R \text{ and the capacitor with capacitance } C. \text{ Check if this is a linear system.} \]

Answer: This system is a system with memory since a capacitor exists.
Thus, output $y(t) = R i(t) + \frac{1}{C} \int_{t_0}^{t} v(t) \, dt + u(t_0)$
where $u(t_0)$ is the only initial state.

Given $u_1(t_0) + u_2(t_0)$ as well as $i_1(t) + i_2(t)$,
$t \geq t_0$ as the initial state and the input as depicted below:

![Diagram](image)

We have $y'(t) = R \left( i_1(t) + i_2(t) \right)$

$$+ \frac{1}{C} \int_{t_0}^{t} (i_1(\tau) + i_2(\tau)) \, d\tau + u_1(t_0) + u_2(t_0)$$

$$= y_1(t) + y_2(t)$$

Where $u_1(t_0) \rightarrow y_1(t)$, $t \geq t_0$

$u_1(t)$, $t \geq t_0 \rightarrow y_1(t)$, $t \geq t_0$

$u_2(t_0) \rightarrow y_2(t)$, $t \geq t_0$

$u_2(t)$, $t \geq t_0 \rightarrow y_2(t)$, $t \geq t_0$

Hence additivity property exists.
Given $dU(t_0)$ as well as $di(t)$, $t \geq t_0$ as the initial state and the input as depicted below:

\[ \begin{align*}
    dU(t_0) + \frac{1}{C} \int_{t_0}^{t} (di(t)) \, dt + dU(t_0) & = 0, \\
    U(t) & \rightarrow Y(t), \quad t \geq t_0
\end{align*} \]

We have $y''(t) = R \left( \frac{d}{dt}i(t) \right) + \frac{1}{C} \int_{t_0}^{t} \left( \frac{d}{dt}i(t) \right) \, dt + dU(t_0) = dy(t)$, where

\[ U(t_0) \quad \Rightarrow \quad Y(t), \quad t \geq t_0 \]

Hence, homogeneity property exists since both properties exist for this system, this network is linear.
If a system is linear, its output or response can always be decomposed as

\[
\begin{align*}
\text{total response} & = \, X(t_0) + \text{response due to } u(t), \, t \geq t_0 \\
& = \, X(t_0) + \text{response due to } u(t) = 0, \, t \geq t_0 \\
& + \text{response due to } X(t_0) = 0, \, \text{zero-state response}
\end{align*}
\]

Example: Consider the previous example. What are the zero-input and zero-state responses?

Answer: \( Y_{zi}(t) = U(t_0) \) since input \( i(t) = 0 \) depicted as below:
\[ y_{2S}(t) = R \dot{i}(t) + \frac{1}{C} \int_{t_0}^{t} i(\tau) \, d\tau \]

Since \( U(t_0) = 0 \), it is depicted as below:

\[ i(t) \]

\[ U(t_0) = 0 \]

\[ \phi(t) \]

\[ y_{2S}(t) \]

\[ y(t) = y_{2I}(t) + y_{2S}(t) \]

\[ = R \dot{i}(t) + \frac{1}{C} \int_{t_0}^{t} i(\tau) \, d\tau + U(t_0) \]

2.5 Time invariance of a system

If the characteristics or properties of a system do not change with time, then the system is said to be time invariant. Otherwise, it is time varying.

A memoryless system is linear if and only if its input and output can be described by \( y(t) = a(t) \cdot u(t) \). It is time-invariant if and only if \( a(t) \) is constant, independent of time.
A system with memory is time invariant if for any permissible state-input-output pair

\[ X(t_0) = X_0 \quad \int \rightarrow \quad y(t), \ t \geq t_0 \]
\[ u(t), \ t \geq t_0 \]

and any \( T \), the pair

\[ X(t_0+T) = X_0 \quad \int \rightarrow \quad y(t-T), \ t \geq t_0+T \]
\[ u(t-T), \ t \geq t_0+T \]

is also permissible. Otherwise it is time varying. This means that, for time-invariant systems, if the initial state and the input are the same, no matter at what time they are applied, the output waveform will always be the same.

Example: Which of the following systems are time-invariant?

(a) \( y(t) = 5u(t) \)
(b) \( y(t) = (\sin(t))u(t) \)
Answer:

(a) \[ U(t) \rightarrow y(t) = 5U(t) \]
\[ U(t-T) \rightarrow y(t) = 5U(t-T) = y(t-T) \]

(b) \[ U(t) \rightarrow y(t) = (\sin(t))U(t) \]
\[ U(t-T) \rightarrow y(t) = (\sin(t))U(t-T) \neq y(t-T) \]

It is time-varying.

2.6 Linear Time-invariant Lumped Systems

The response of a linear time-invariant lumped (LTI) system can always be decomposed into zero-state and zero-input responses.
Example: Consider a linear time-invariant (LTI) system. The zero-state response of the system excited by the unit step function is shown in part (a) as below. This output is called the unit step response.

Verify the input-output pairs shown in part (b) and (c)...

\[(a)\:
\begin{align*}
& u(t) = g(t) \\
& y_f(t) \\
& 0 \quad 1 \quad 2 \quad x
\end{align*}
\]

\[(b)\:
\begin{align*}
& u_b(t) \\
& y_b(t) \\
& 0 \quad 1 \quad 2 \quad 3 \quad x
\end{align*}
\]
Answer

(a) \[ \{ g(x) \} \rightarrow \{ y_g(x) \} \]

(b) \[ u_{b/g}(x) = \begin{cases} g(x) - g(x-1) & \text{for } x \geq 1 \\ y_g(x) - y_g(x-1) & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ y_{b/g}(x) = y_g(x) - y_g(x-1) \]
(c) \[ y_c(x) = -y_b(x) + 2y_b(x-2) \]

\[ y_c(x) = -y_b(x) + 2y_b(x-2) \]
2.7 Linear Time-invariant Lumped Discrete-time Systems.

A discrete-time system is called a memoryless system if its output $y[k]$ depends only on $u[k]$. Otherwise, it is a system with memory.

A discrete-time system is said to be causal or nonanticipatory if its output at time instant $k$ does not depend on the input applied after $k$. In other words, for a causal discrete-time system, we have $y[k] = f(u[j], \text{ for } j \leq k)$ where $f$ is a function and $k, j$ are integers.

For a discrete-time system with memory, we define the state $x(k_0)$ as the information that, together with $u[k], k \geq k_0$, uniquely determines $y[k]$ for $k \geq k_0$.

$$x(k_0) \rightarrow u(k), k \geq k_0 \rightarrow y[k], k \geq k_0$$
If the number of components or the state variables in \( x[k] \) is finite, the system is a lumped system. Otherwise, it is a distributed system.

A discrete-time system is linear if for any permissible pairs,

\[
X_0[k_0], u_i[k], k \geq k_0 \rightarrow y_i[k], k \geq k_0,
\]

with \( i = 1, 2 \), the following pairs

\[
X_1[k_0] + X_2[k_0], u_1[k] + u_2[k], k \geq k_0 \rightarrow y_1[k] + y_2[k], k \geq k_0
\] (additivity)

and with any constant \( d \),

\[
d X_1[k_0], \alpha u_1[k], k \geq k_0 \rightarrow \alpha y_1[k], k \geq k_0
\] (homogeneity)

are also permissible.

The response of every linear discrete-time system can be decomposed as

\[
\text{total response} = \text{zero-state response} + \text{zero-input response}
\]
A discrete-time system is time-invariant if for any permissible state-input-output pair,

\[ X[k_0] = x_0 \quad \rightarrow \quad y[k], \quad k \geq k_0 \]
\[ u[k], \quad k \geq k_0 \]

and any integer \( k_1 \), the pair

\[ X[k_0 + k_1] = x_0 \quad \rightarrow \quad y[k-k_1], \quad u[k-k_1], \quad k \geq k_0 + k_1 \]

(time invariance)

is also permissible.

Otherwise, it is time varying.

Example: consider an LTI system with input-output pair shown below. What is the output of the system excited by the impulse sequence \( \delta[k] \)?
A zero-state input-output pair

\[ u[k] = s[k] + s[k-1] + s[k-2] \]

Assume \( \{s[k]\} \rightarrow \{y[k]\} \) is a permissible pair.

\[ y[k] = y[k] + y[k-1] + y[k-2] \]

Since it is zero-state response,

\[ y[k] = 0, \quad k < 0. \quad (no \ initial \ value) \]

\[ y[k] = y[k] - y[k-1] - y[k-2], \quad k \geq 0 \]

\[ k=0 \Rightarrow y[0] = y[0] - y[-1] - y[-2] = 0 \]

\[ k=1 \Rightarrow y[1] = y[1] - y[0] - y[-1] \]

\[ = \frac{1}{3} - 0 - 0 = \frac{1}{3} \]


\[ = \frac{2}{3} - \frac{1}{3} - 0 = \frac{1}{3} \]
\[ k = 3 \Rightarrow y_{[3]} = y_{[2]} - y_{[1]} \]
\[ = 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3} \]

\[ k = 4 \Rightarrow y_{[4]} = y_{[3]} - y_{[2]} - y_{[1]} \]
\[ = \frac{2}{3} - \frac{1}{3} - \frac{1}{3} = 0 \]

\[ k = 5 \Rightarrow y_{[5]} = y_{[4]} - y_{[3]} - y_{[2]} \]
\[ = \frac{1}{3} - 0 - \frac{1}{3} = 0 \]

\[ y_{[k]} = 0, \quad k = 6, 7, 8, \ldots \]