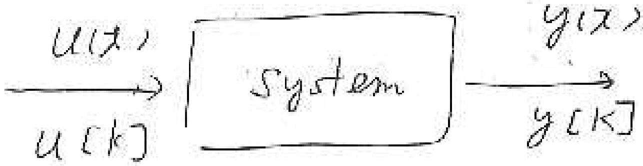
Chapter 2 Systems

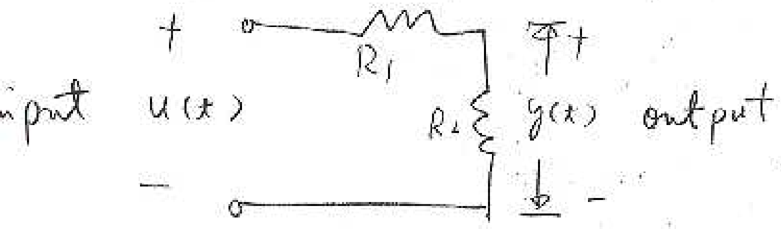
2-1
2.2 What is a system?



As depicted above, a system is modeled as a black box with at least one input terminal and one output terminal. A system is called a continuous-time system if its input/output are both continuous-time signals. Similarly, a discrete-time system has discrete-time signals as its input/output.

- ※ SISO system: a system with one input and one output terminals.
- ※ MIMO system: a system with multiple input and multiple output terminals.
- ※ memoryless system: a system's output $y(t_0)$ depends only on the input applied at time t_0 but independent of the input applied before or after t_0.

Example: Voltage divider



$$y(t) = \frac{R_2}{R_1+R_2}\, u(t), \quad \forall t$$

is a memoryless system

Example: $y(t) = u(t-1) + 2u(t) - 3u(t+2)$

is a system with memory

× Causal or non anticipatory system: a system's present output depends on past and present inputs but no future input.

Example: Integrator

$$y(t_0) = \int_{-\infty}^{t_0} u(t)\, dt$$

is a causal system

Example:

Unit-time delay system:

$$y(t) = u(t-1)$$

is a causal system

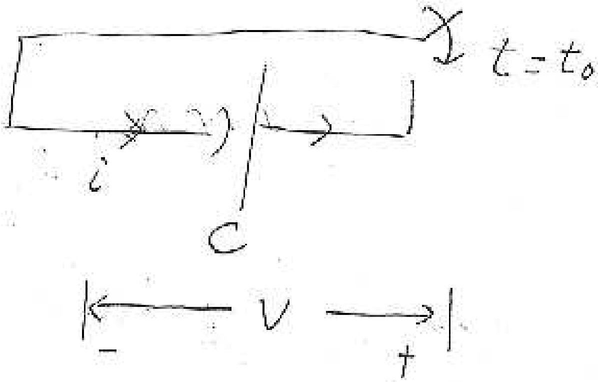
Example:

unit-time advance system:

$$y(t) = u(t+1)$$

is a noncausal system

2.3 State — set of initial conditions



$$i(t) = \frac{d}{dt}[C v(t)] = C\frac{dv(t)}{dt}$$

$$v(t) = \int_{t_0}^{t} \frac{i(t)}{C}\,dt + v(t_0)$$

We see that $v(t)$ depends on $v(t_0)$ and $i(t)$ for t in $[t_0, t]$. To solve $v(t)$ for any time instant t, we need information of $v(t_0)$. Thus

$$\left.\begin{array}{l} \text{state variable} \rightarrow v(t_0) \\ i(t),\ t \ge t_0 \end{array}\right\} \longrightarrow v(t),\ t \ge t_0$$

The input $i(t)$ applied before $t = t_0$ is not needed.

The state summarizes the effect of past input on future output. We can define the state $X(t_0)$ of a system at time t_0 is the information at $t=t_0$ that, together with the input, $t \geq t_0$, determines uniquely the output $y(t)$ for all $t \geq t_0$. A system is

a lumped system if its state has a finite number of state variables. A system is a distributed system if its state has infinitely many state variables.

Example: a system is described as

$y(t) = a_1 u(t)$. Is this a lumped or distributed system?

Answer: The output $y(t)$ simply depends on input $u(t)$ only. Hence no state is in this system.

It is lumped.

Example: A unit delay system described as $y(t) = u(t-1)$ is lumped or distributed?

Answer: In order to determine $y(t)$, for $t \geq t_0$ from $u(t)$, for $t \geq t_0$, we need information $u(t)$ for all t in $[t_0-1, t_0)$. Therefore the state of the system at time t_0 is $u(t)$ for all t in $[t_0-1, t_0)$. There are infinitely many state variables since there are infinitely many points in the time interval $[t_0-1, t_0)$.

It is a distributed System!

2.3.1 Zero-input and zero-state response.

The output of a system with memory depends on the input and initial states.

※ Zero-input response of the system:
The output $y(t)$ for $t \geq t_0$ is excited exclusively by the initial states

$X(t_0)$ without any input.

zero-state response: The output $y(t)$, for $t \geq t_0$ is excited exclusively by the input $u(t)$, for $t \geq t_0$ without any initial state (initial state values are zero)

We can express these two responses as

$$\left.\begin{array}{l} X(t_0) \neq 0 \\ u(t) \equiv 0, \text{ for } t \geq t_0 \end{array}\right\} \longrightarrow y_{zi}(t), \text{ for } t \geq t_0$$

(zero-input response)

$$\left.\begin{array}{l} X(t_0) = 0 \\ u(t) \not\equiv 0, \text{ for } t \geq t_0 \end{array}\right\} \longrightarrow y_{zs}(t), \text{ for } t \geq t_0$$

(zero-state response).

2.4 Linearity of a System

A memoryless system is said to be linear, if, for any permissible pairs $\{u_1(t)\} \rightarrow \{y_1(t)\}$ and $\{u_2(t)\} \rightarrow \{y_2(t)\}$, the following pairs:

$\{u_1(t) + u_2(t)\} \rightarrow \{y_1(t) + y_2(t)\}$ (additivity)

and $\{\alpha u_1(t)\} \rightarrow \{\alpha y_1(t)\}$ (homogeneity)

for any α are also permissible.

Or combine these two relationships for any α_1 and α_2, we obtain

$$\{\alpha_1 u_1(t) + \alpha_2 u_2(t)\} \rightarrow \{\alpha_1 y_1(t) + \alpha_2 y_2(t)\}$$

are also permissible

Note: If an input $u(t)$ generates an output $y(t)$ through a system, we call $\{u(t)\} \rightarrow \{y(t)\}$ is a permissible pair.

Example: Are the following systems linear? (a) $y(t) = \sin(u(t))$

(b) $y(t) = (\sin(t))\, u(t)$

Answer: (a) $\sin(u_1(t) + u_2(t)) = \sin(u_1(t))\cos(u_2(t)) + \cos(u_1(t))\sin(u_2(t)) \neq \sin(u_1(t)) + \sin(u_2(t))$

$\therefore$ It is NOT linear or it is nonlinear.

(b) Check if $\{\alpha_1 u_1(t) + \alpha_2 u_2(t)\} \xrightarrow{?} \{\alpha_1 y_1(t) + \alpha_2 y_2(t)\}$

$$y'(t) = \sin(t)\,[\alpha_1 u_1(t) + \alpha_2 u_2(t)]$$

$$= \alpha_1 \sin(t)\, u_1(t) + \alpha_2 \sin(t)\, u_2(t)$$

$$= \alpha_1 y_1(t) + \alpha_2 y_2(t),$$

where $y_1(t) = \sin(t)\, u_1(t)$ and $y_2(t) = \sin(t)\, u_2(t)$.

The definition of linearity for systems is slightly more complicated. The system with memory is defined to be linear if for any two permissible state-input-output pairs, with $i = 1, 2$,

$$\left.\begin{matrix} X_i(t_0) \\ u_i(t),\ t \ge t_0 \end{matrix}\right\} \longrightarrow y_i(t),\ t \ge t_0,$$

the following pairs

$$\left.\begin{matrix} X_1(t_0) + X_2(t_0) \\ u_1(t) + u_2(t),\ t \ge t_0 \end{matrix}\right\} \longrightarrow y_1(t) + y_2(t),\ t \ge t_0$$

(additivity)

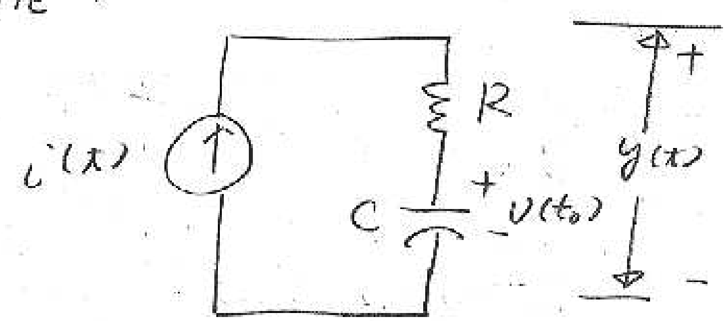
and

$$\left.\begin{array}{l}\alpha x_1(t_0)\\ \alpha u_1(t),\ t \ge t_0\end{array}\right\} \longrightarrow \alpha y_1(t),\ t \ge t_0$$

(homogeneity)

for any α, are also permissible.

Example :



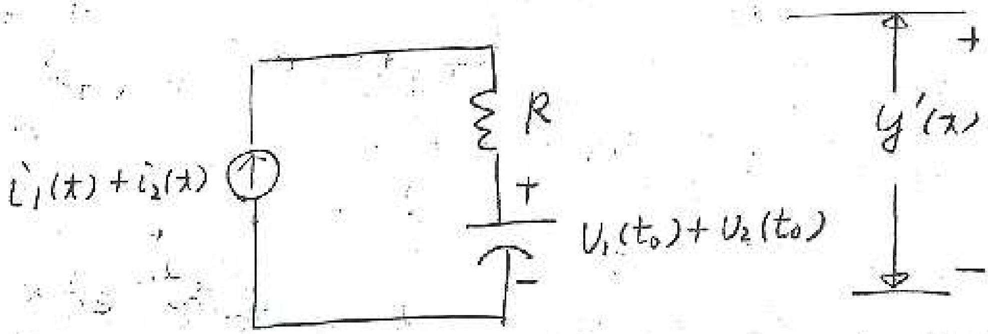
Consider the network shown above. The input is a current source $i(t)$ and the output is the voltage across the resistor with resistance R and the capacitor with capacitance C. Check if this is a linear system.

Answer : This system is a system with memory since a capacitor exists.

Thus, output $y(t) = Ri(t) + \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau + v(t_0)$

where $v(t_0)$ is the only initial state

Given $v_1(t_0) + v_2(t_0)$ as well as $i_1(t) + i_2(t)$, $t \ge t_0$ as the initial state and the input as depicted below:



We have $y'(t) = R\left(i_1(t) + i_2(t)\right)$

$$+ \frac{1}{C}\int_{t_0}^{t}\left(i_1(\tau) + i_2(\tau)\right)d\tau + v_1(t_0) + v_2(t_0)$$

$$= y_1(t) + y_2(t)$$

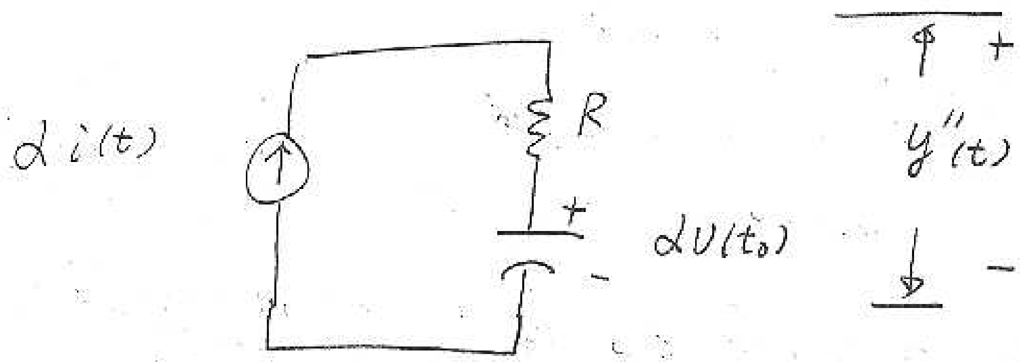
where

$$\left.\begin{matrix} v_1(t_0) \\ u_1(t),\ t \ge t_0 \end{matrix}\right\} \rightarrow y_1(t),\ t \ge t_0$$

$$\left.\begin{matrix} v_2(t_0) \\ u_2(t),\ t \ge t_0 \end{matrix}\right\} \rightarrow y_2(t),\ t \ge t_0$$

Hence additivity property exists.

Given $\alpha U(t_0)$ as well as $\alpha i(t)$, $t \geq t_0$ as the initial state and the input as depicted below:



We have $y''(t) = R\left(\alpha i(t)\right)$

$$+ \frac{1}{C}\int_{t_0}^{t} \left(\alpha i(\tau)\right) d\tau + \alpha U(t_0)$$

$$= \alpha\, y(t) \quad , \quad \text{where}$$

$$\left.\begin{matrix} U(t_0) \\ u(t),\ t \geq t_0 \end{matrix}\right\} \longrightarrow y(t),\ t \geq t_0$$

Hence homogeneity property exists

Since both properties exist for this system, this network is linear.

If a system is linear, its output or response can always be decomposed as

response due to $\begin{cases} x(t_0) \\ u(t), t \ge t_0 \end{cases}$ (total response)

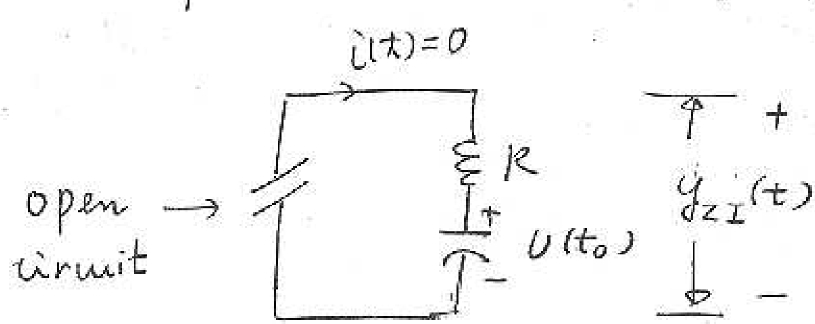
$=$ response due to $\begin{cases} x(t_0) \\ u(t) \equiv 0, t \ge t_0 \end{cases}$ (zero-input response)

$+$ response due to $\begin{cases} x(t_0) = 0 \\ u(t), t \ge t_0 \end{cases}$ (zero-state response)

Example: Consider the previous example. What are the zero-input and zero state responses?

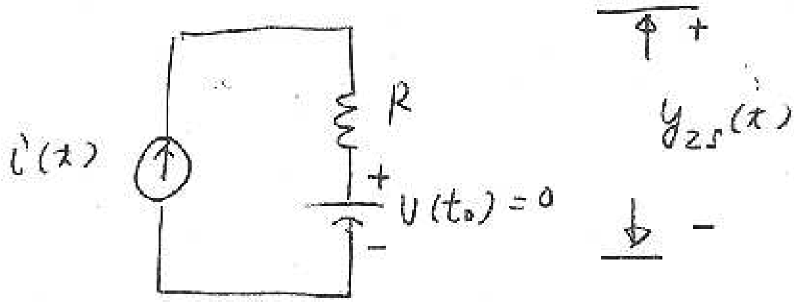
Answer: $y_{zI}(t) = v(t_0)$ since input $i(t) = 0$

depicted as below:



$$y_{zs}(t) = R\,i(t) + \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau \quad \text{since } v(t_0)=0$$

depicted as below:



$$y(t) = y_{zI}(t) + y_{zs}(t)$$

$$= R\,i(t) + \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau + v(t_0)$$

2-5 Time invariance of a system

If the characteristics or properties of a system do not change with time, then the system is said to be time invariant. Otherwise, it is time varying.

A memoryless system is linear if and only if its input and output can be described by $y(t) = a(t)\,u(t)$. It is time-invariant if and only if $a(t)$ is constant, independent of time.

A system with memory is time invariant if for any permissible state-input-output pair

$$\left.\begin{array}{l} x(t_0) = x_0 \\ u(t),\ t \geq t_0 \end{array}\right\} \longrightarrow y(t),\ t \geq t_0$$

and any T, the pair

$$\left.\begin{array}{l} x(t_0+T) = x_0 \\ u(t-T),\ t \geq t_0+T \end{array}\right\} \longrightarrow y(t-T),\ t \geq t_0+T$$

is also permissible. Otherwise it is time varying. This means that, for time-invariant systems, if the initial state and the input are the same, no matter at what time they are applied, the output waveform will always be the same.

Example: Which of the following systems are time-invariant?

(a) $y(t) = 5u(t)$

(b) $y(t) = (\sin(t))\,u(t)$

Answer:

(a) $$u(t) \longrightarrow y(t) = 5u(t)$$

$$u(t-T) \rightarrow y'(t) = 5u(t-T)$$
$$= y(t-T)$$

$$\therefore \quad u(t-T) \longrightarrow y(t-T)$$

it is time-invariant

(b) $$u(t) \longrightarrow y(t) = (\sin(t))\, u(t)$$

$$u(t-T) \rightarrow y'(t) = (\sin(t))\, u(t-T)$$
$$\neq y(t-T)$$

$$\therefore \quad u(t-T) \not\longrightarrow y(t-T)$$

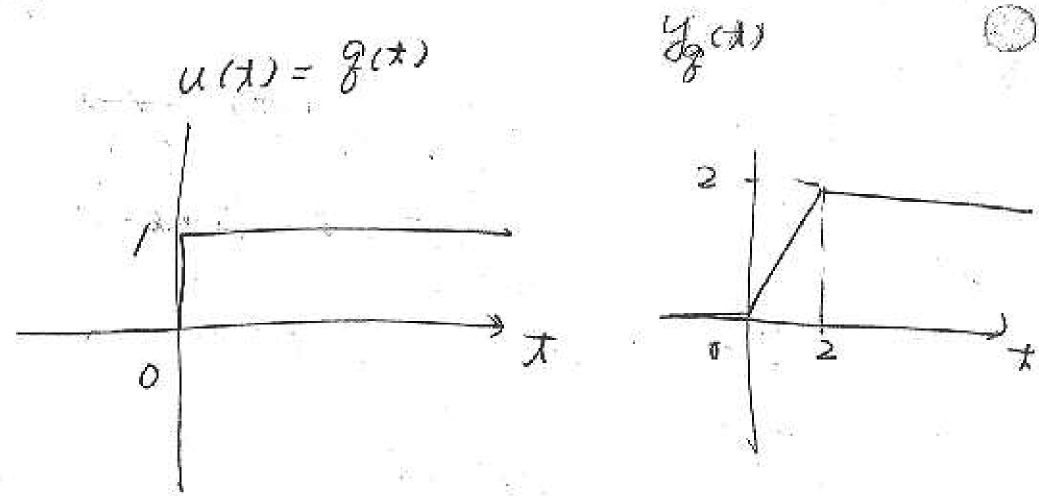
it is time-varying

2.6 Linear Time-invariant Lumped Systems

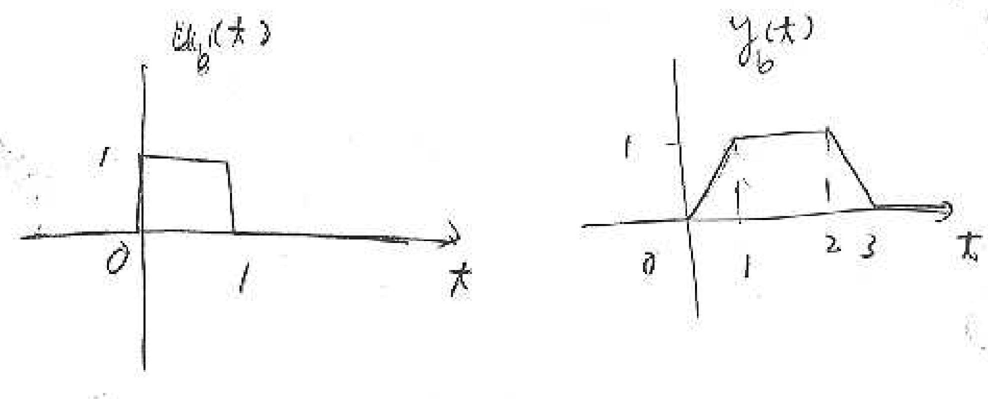
The response of a linear time-invariant lumped (LTIL) system can always be decomposed into zero-state and zero-input responses.

Example: Consider a linear time-invariant (LTI) system. The zero-state response of the system excited by the unit step function is shown in part (a) as below. This output is called the unit step response. Verify the input-output pairs shown in part (b) and (c).

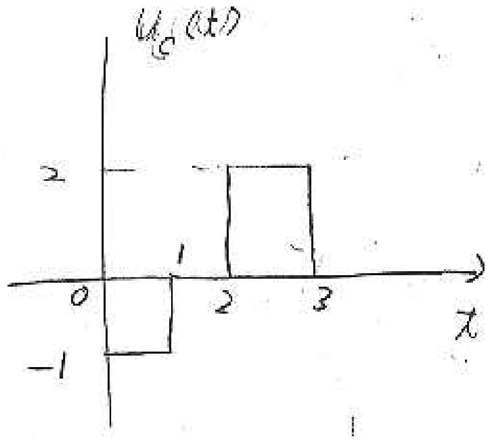
(a)

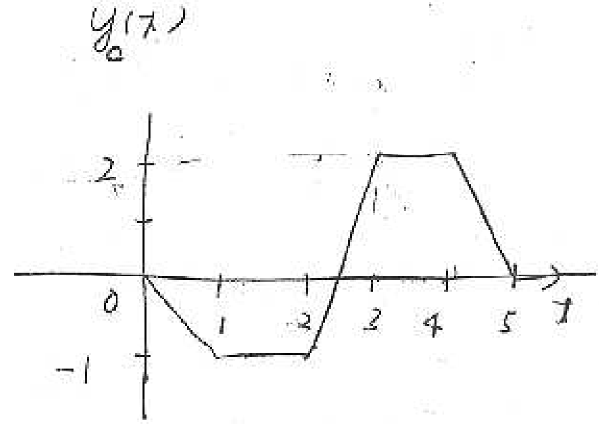


(b)



(c)



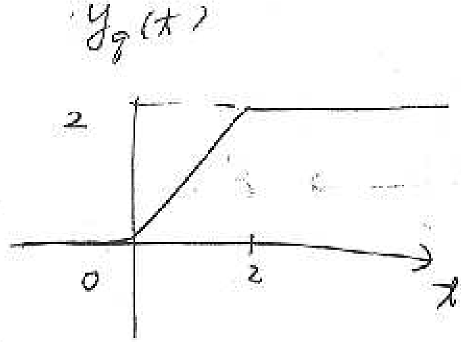


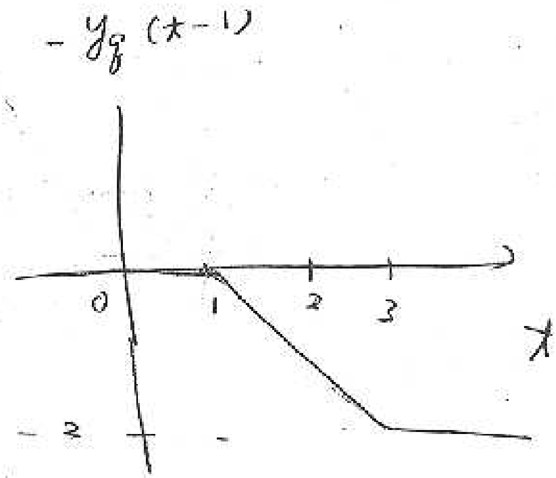
Answer

(a) $\{g(t)\} \longrightarrow \{y_g(t)\}$

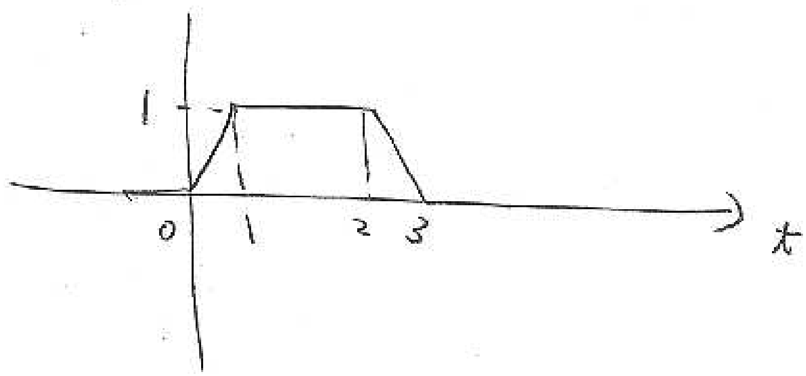
(b) $u_b(t) = g(t) - g(t-1)$

$\therefore\ y_b(t) = y_g(t) - y_g(t-1)$



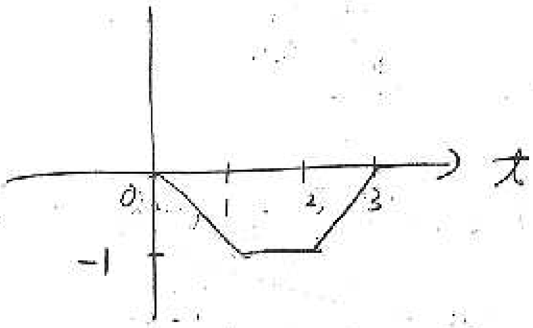


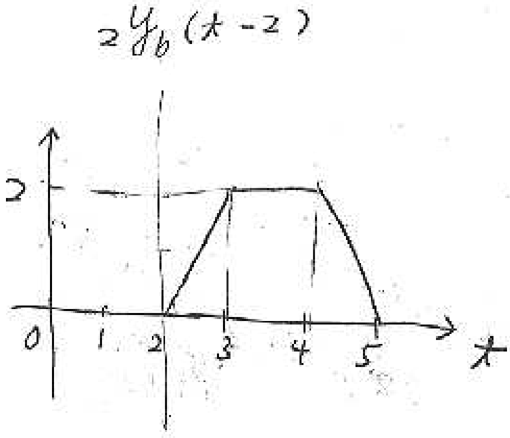
$y_b(t) = y_g(t) - y_g(t-1)$



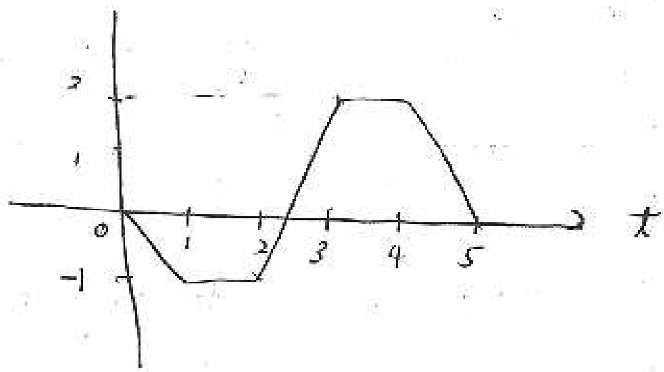
(c) $u_c(t) = -u_b(t) + 2u_b(t-2)$

$y_c(t) = -y_b(t) + 2y_b(t-2)$





$y_c(t) = -y_b(t) + 2y_b(t-2)$



2.7 Linear Time-invariant Lumped Discrete-time Systems.

A discrete-time system is called a memoryless system if its output $y[k]$ depends only on $u[k]$. Otherwise, it is a system with memory.

A discrete-time system is said to be causal or nonantipatory if its output at time instant k does not depend on the input applied after k. In other words, for a causal discrete-time system, we have $y[k] = f(u[j], \text{ for } j \le k)$ where f is a function and k, j are integers.

For a discrete-time system with memory, we define the state $x(k_0)$ as the information that, together with $u[k]$, $k \ge k_0$, uniquely determines $y[k]$ for $k \ge k_0$.

$$\left.\begin{matrix} x[k_0] \\ u[k],\ k \ge k_0 \end{matrix}\right\} \rightarrow y[k],\ k \ge k_0$$

If the number of components or the state variables in $x[k]$ is finite, the system is a lumped system. Otherwise, it is a distributed system.

A discrete-time system is linear if for any permissible pairs,

$$\left.\begin{array}{l} x_i[k_0] \\ u_i[k],\ k \geq k_0 \end{array}\right\} \rightarrow y_i[k],\ k \geq k_0,$$

with $i = 1, 2$, the following pairs

$$\left.\begin{array}{l} x_1[k_0] + x_2[k_0] \\ u_1[k] + u_2[k],\ k \geq k_0 \end{array}\right\} \rightarrow y_1[k] + y_2[k],\ k \geq k_0$$

(additivity)

and with any constant α,

$$\left.\begin{array}{l} \alpha x_1[k_0] \\ \alpha u_1[k],\ k \geq k_0 \end{array}\right\} \rightarrow \alpha y_1[k],\ k \geq k_0$$

(homogeneity)

are also permissible.

The response of every linear discrete-time system can be decomposed as

total response = zero-state response + zero-input response.

A discrete-time system is time-invariant if for any permissible state-input-output pair,

$$\left.\begin{array}{l} x[k_0] = x_0 \\ u[k],\ k \geq k_0 \end{array}\right\} \longrightarrow y[k],\ k \geq k_0$$

and any integer k_1, the pair

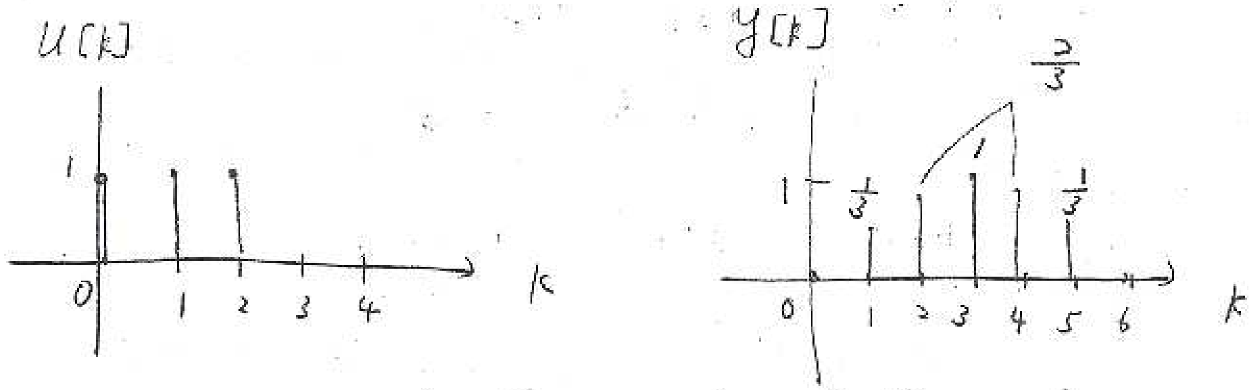
$$\left.\begin{array}{l} x[k_0+k_1] = x_0 \\ u[k-k_1],\ k \geq k_0+k_1 \end{array}\right\} \rightarrow \begin{array}{l} y[k-k_1], \\ k \geq k_0+k_1 \end{array}$$

(time invariance)

is also permissible.

Otherwise it is time varying.

Example: consider an LTI system with input-output pair shown below. What is the output of the system excited by the impulse sequence $\delta[k]$?



A zero-state input-output pair

$u[k] = \delta[k] + \delta[k-1] + \delta[k-2]$

Assume $\{\delta[k]\} \longrightarrow \{y'[k]\}$ is a permissible pair.

$\therefore \quad y[k] = y'[k] + y'[k-1] + y'[k-2]$

Since it is zero-state response,

$y'[k] = 0 \;, \quad k < 0.$ (no initial value)

$y'[k] = y[k] - y'[k-1] - y'[k-2] \;, \quad k \geq 0$

$k=0 \Rightarrow y'[0] = y[0] - y'[-1] - y'[-2] = 0$

$k=1 \Rightarrow y'[1] = y[1] - y'[0] - y'[-1]$

$= \frac{1}{3} - 0 - 0 = \frac{1}{3}$

$k=2 \Rightarrow y'[2] = y[2] - y'[1] - y'[0]$

$= \frac{2}{3} - \frac{1}{3} - 0 = \frac{1}{3}$

$k=3 \Rightarrow \quad y'[3] = y[3] - y'[2] - y'[1]$

$= 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}$

$k=4 \Rightarrow \quad y'[4] = y[4] - y'[3] - y'[2]$

$= \frac{2}{3} - \frac{1}{3} - \frac{1}{3} = 0$

$k=5 \Rightarrow \quad y'[5] = y[5] - y'[4] - y'[3]$

$= \frac{1}{3} - 0 - \frac{1}{3} = 0$

$y'[k] = 0, \quad k = 6, 7, 8, \ldots$