Chapter 1 Signals.

1.1

Signals are waveforms corresponding to a variety of physical quantities or mearures, such as electric voltage, accoustic waves, the light intensity or electromagnetic waves.

Example:

※ natural signals: speech signals, optical images, brain waves.

MRI, EEG, etc.

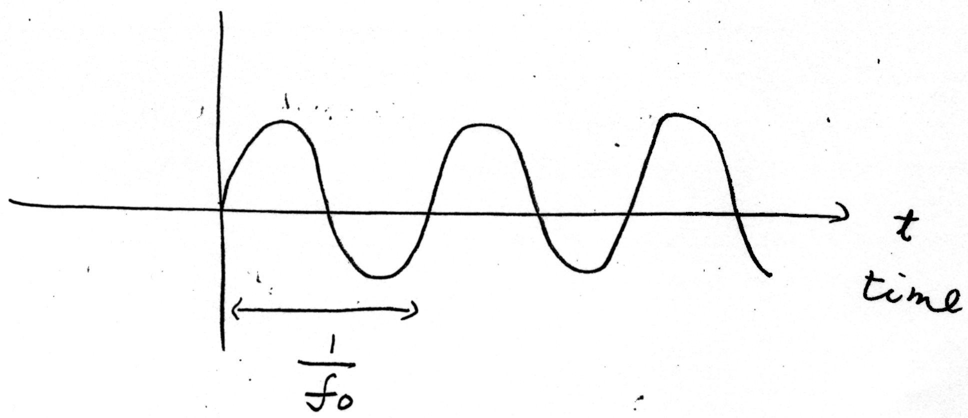
※ artificial signals: digital voltages in the computer data bus, electromagnetic fields for wireless communications, radar imaging signals, etc.

Classification of signals

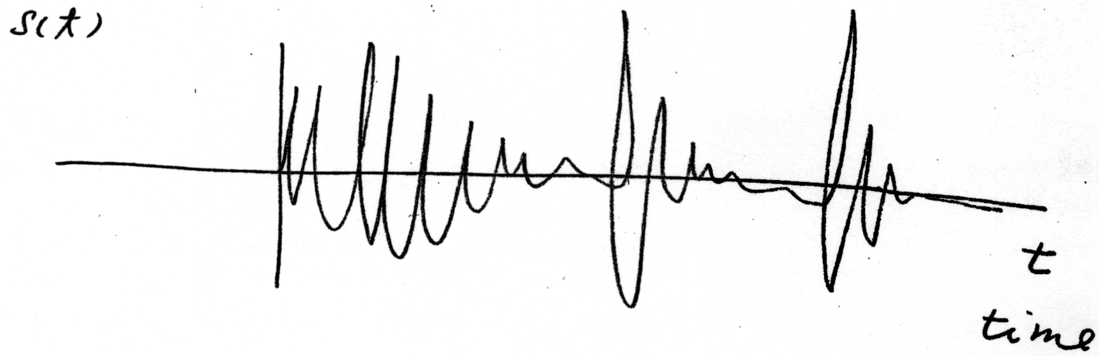
※ one-dimensional signal: signal waveforms are one-dimensional, i.e; physical quantities versus time, space or frequency, etc.

Example 1. Single-tone sinusoid at $f_0 = 100\,Hz$

$$s(t) = 7\sin(\underbrace{200\pi t}_{2\pi f_0 t})$$



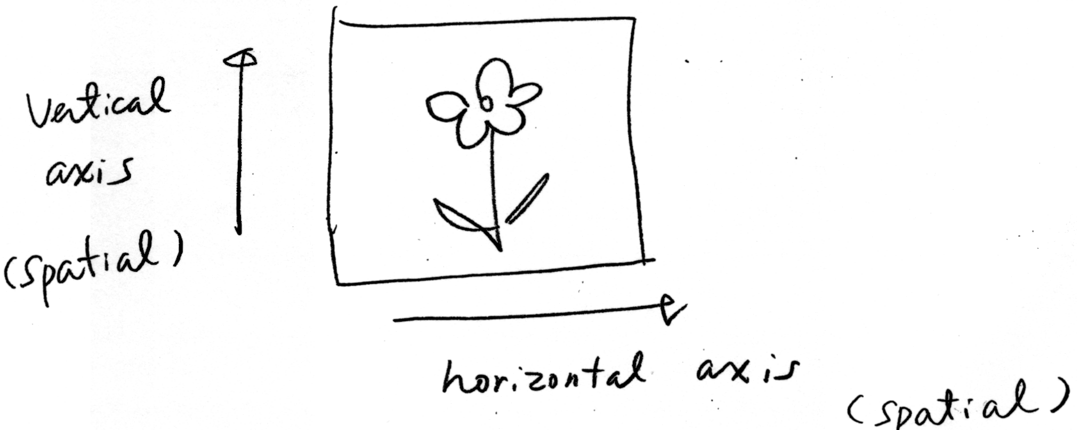
Example 2. Speech signal



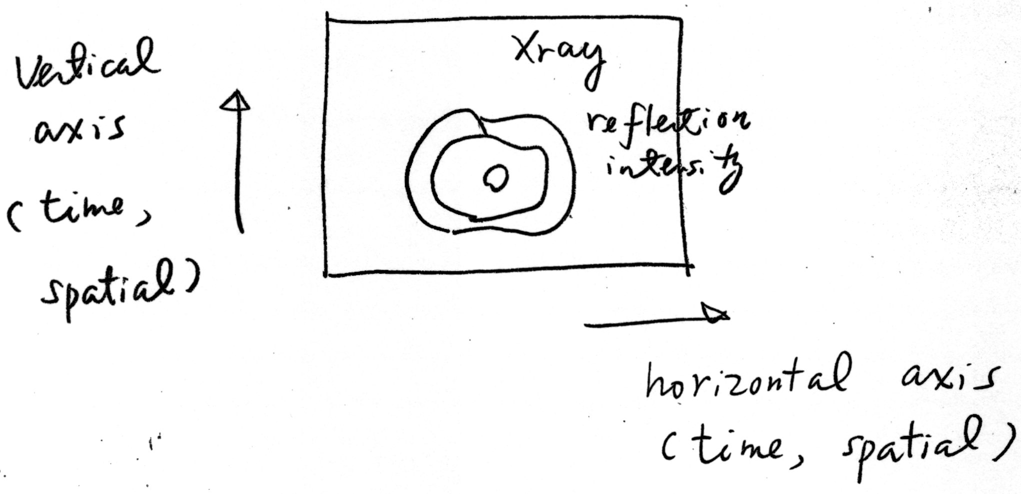
One-dimensional signal can be characterized as a function of one independent variable such as $s(t)$; thus, signals can be considered as single-value functions of time that carry information.

※ Multidimensional signal: signal waveforms are multi-dimensional or in a vector space; i.e., a coordinate of physical quantities versus time, space, frequency, etc.

Example 1. Optical Image



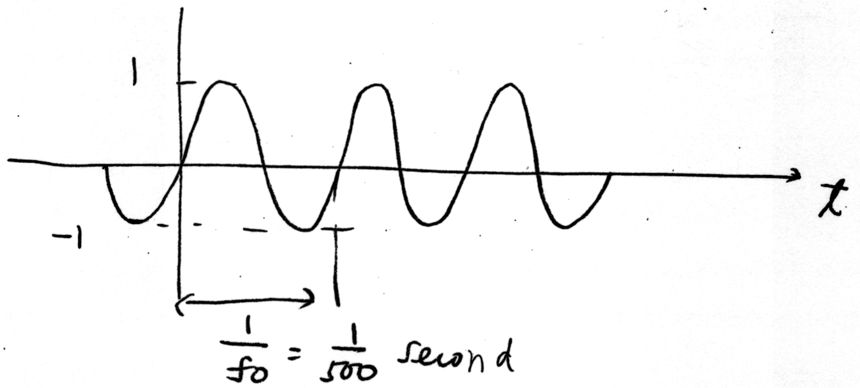
Example 2. MRI Image



※ Continuous-time signals: signals that are defined at every instant of time. It is also called analog signal.

Example: single-tone continuous waveform of a sinusoid, $f_0 = 250$ Hz

$$S(t) = \sin(2\pi f_0 t) = \sin(500\pi t)$$



※ discrete-time signals: signals are sampled from continuous-time signals at certain specific instants of time.

Example: discrete-time samples of a single-tone sinusoid, $f_0 = 250$ Hz.

$s(t) = \sin(500\pi t)$

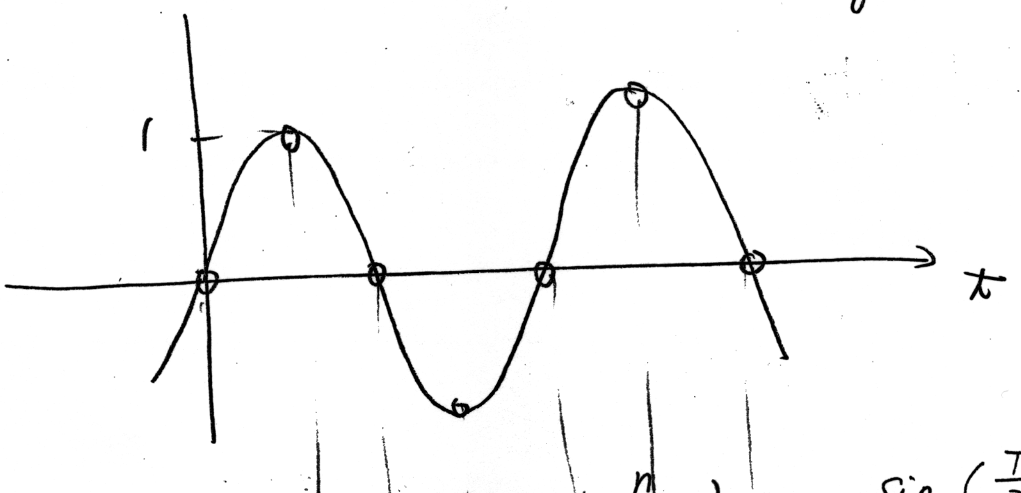
sampled at $t = \frac{n}{f_s}$, f_s: sampling frequency

$T_s = \frac{1}{f_s}$: sampling period

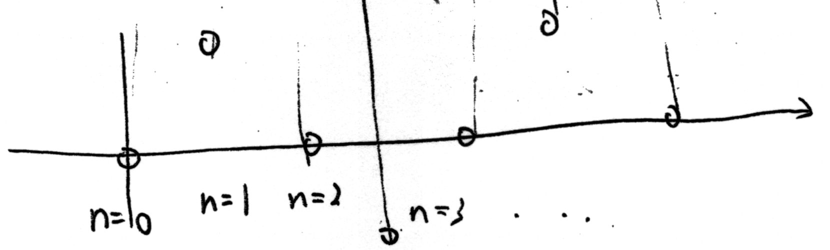
$f_s \geq 2f_0$, n is integer

$f_s = 1000$ Hz

$s(t) = \sin(500\pi t)$ ... original analog signal



$s(n) = \sin(500\pi \frac{n}{1000}) = \sin(\frac{\pi}{2} n)$, $n \in \mathbb{Z}$



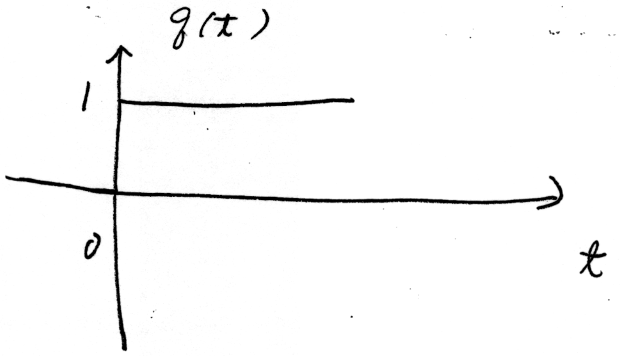
1.2 Elementary continuous-time signals

A continuous-time signal will be denoted by $f(t)$ with t ranging from minus infinity to positive infinity, written as $f(t)$ for all t in $(-\infty, \infty)$. Because $t=0$ is mainly used for reference, we call it the reference time.

Typical Elementary continuous-time signals

※ Unit step function

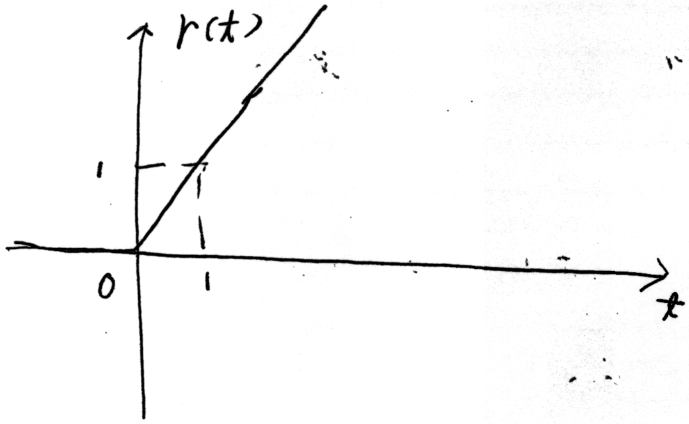
$$g(t) := \begin{cases} 1, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases}$$



※ Ramp function

$$r(t) := \begin{cases} t, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases}$$

$$= t\, g(t)$$

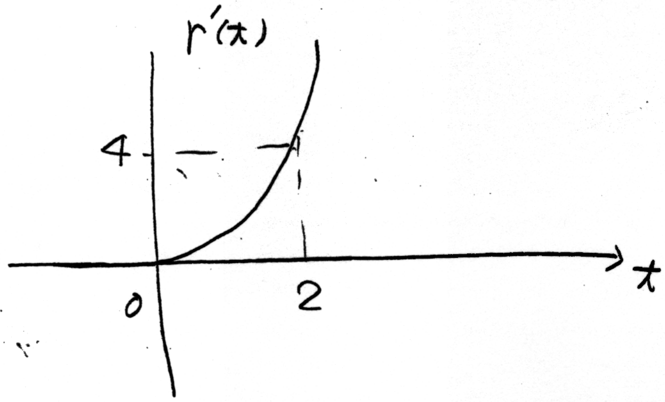


Variation of $r(t)$:

① acceleration function:

$$\gamma'(t) = \begin{cases} t^2, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

$$= t\, r(t) = t^2 g(t)$$



② polynomial function:

$$\gamma''(t) = \begin{cases} b_0 + b_1 t + \cdots + b_n t^n, & \text{for } t \ge 0 \\ 0, & \text{for } t < 0 \end{cases}$$

$$= \sum_{k=0}^{n} b_k t^k g(t)$$

※ periodic function: a continuous-time signal $f(t)$ is said to be periodic with period P if $f(t+P) = f(t)$ for all t.

If $f(t)$ is periodic, then

$$f(t) = f(t+P) = f(t+2P) = \cdots = f(t+nP)$$

n is any arbitrary positive integer.

※ sinusoidal function:

$$f(t) = A\sin(\omega t + \theta)$$

A: amplitude

ω: frequency (unit rad/sec)

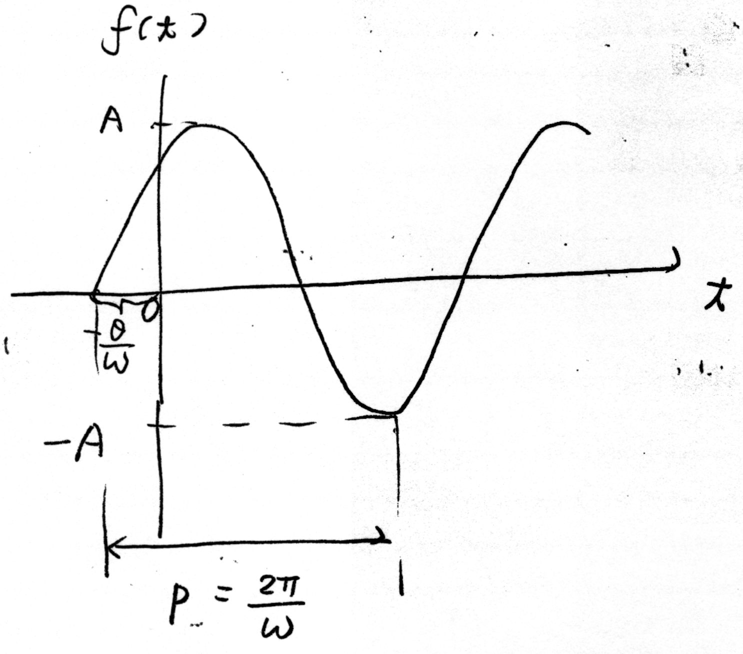
θ: phase (unit radians)

$f(t)$ is periodic with fundamental period:

$$P := \frac{2\pi}{\omega} \quad \text{(unit seconds)}$$

$$\omega = 2\pi f \Rightarrow f = \frac{\omega}{2\pi}$$

unit cycles/second or Hz



※ real exponential function – time constant

$f(t) = e^{at}$, where a is real constant.

The rate of increase or decrease depends on the magnitude of a. If a is negative, then

$$\frac{f\left(t+\frac{1}{|a|}\right)}{f(t)} = \frac{e^{a\left(t+\frac{1}{|a|}\right)}}{e^{at}} = e^{\frac{a}{|a|}} = e^{-1}$$

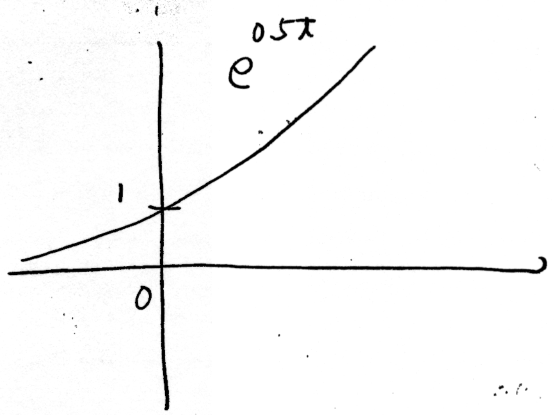
$$= \frac{1}{2.7} = 0.37 = 37\%,$$

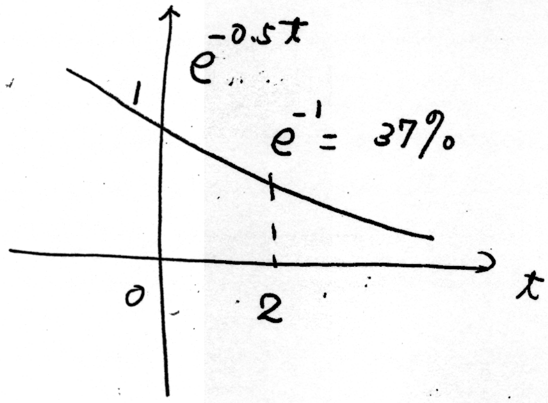
which means that the value of e^{at} decreases to 37% of its original value whenever the time increases by $1/|a|$. $1/|a|$ is called the time constant.

· Complex exponential functions – positive and negative frequencies

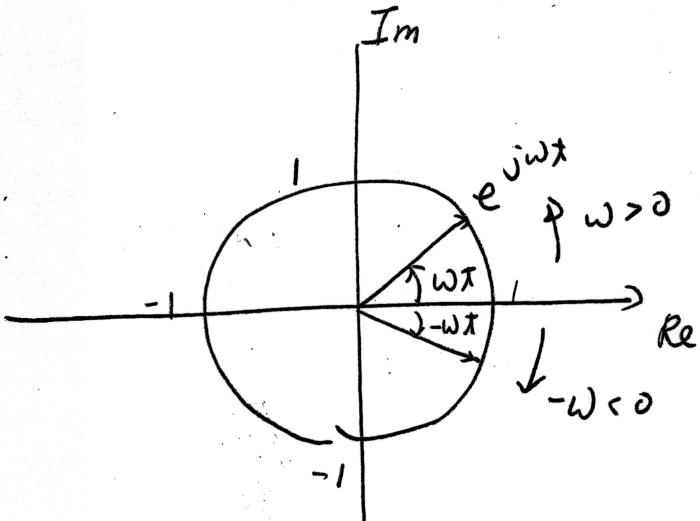
$$f(t) = e^{j\omega t} = \cos(\omega t) + j\sin(\omega t), \quad \text{where } j = \sqrt{-1}.$$

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}, \quad \sin(\omega t) = \frac{e^{j\omega} - e^{-j\omega t}}{2j}$$





Graphic representation of a complex exponential



1.2.1 Boundedness in magnitude and infinity time

A continuous - time signal $f(t)$ is said to be bounded in a time interval if there exists a constant M such that $|f(t)| \le M$, where M is called the upper bound of $|f(t)|$.

Example:

What is the upper bound for $e^{-0.5t}$?

Answer:

$e^{-0.5t}$ is not bounded in $(-\infty, \infty)$ but it is bounded in $[0, \infty)$

The upper bound is 1.

For an exponential function e^{ax}, $a<0$, the function decreases to less than 1% of its original value after five time constants and may be considered to have reach 0, since $(0.37)^5 = 0.007$

We may consider the infinite time

$$\infty \approx 5 \times (\underbrace{\text{time constant}}_{\frac{1}{|a|}})$$ in many applications.

Example: If we consider e^{-bx} for $b>0$, to have reached 0 in five time constants, what are infinite times for $b = 0.01$, 1 and 100?

Answer:

$$\infty \approx \frac{1}{|b|} \times 5 = \frac{5}{|b|}$$

$$b = 0.01 \Rightarrow \infty \approx \frac{5}{0.01} = 500 \text{ sec}$$

$$b = 1 \Rightarrow \infty \approx \frac{5}{1} = 5 \text{ sec}$$

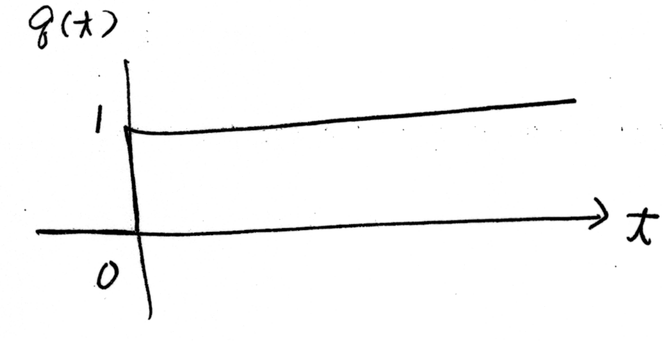
$$b = 100 \Rightarrow \infty \approx \frac{5}{100} = 0.05 \text{ sec}$$

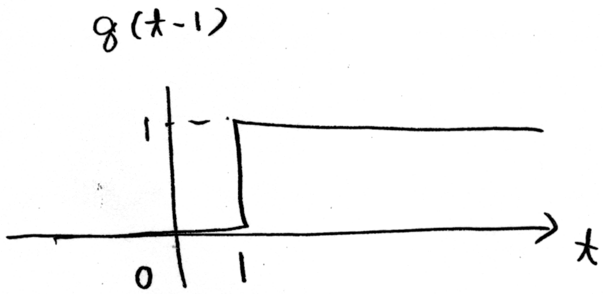
1.3 Manipulation of Continuous-time signals.

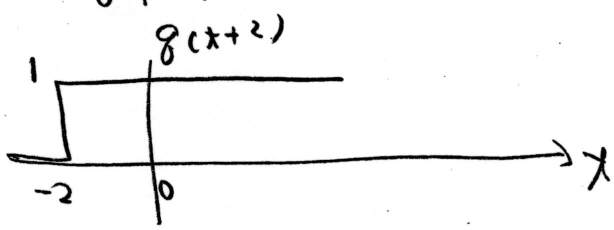
※ shifting: Let $f(t)$ be a function and T be a positive number. Then $f(t-T)$ shifts $f(t)$ to the right or advances $f(t)$ by T seconds.

Example: If $g(t)$ denotes a unit step function, plot $g(t-1)$ and $g(t+2)$.

Answer:

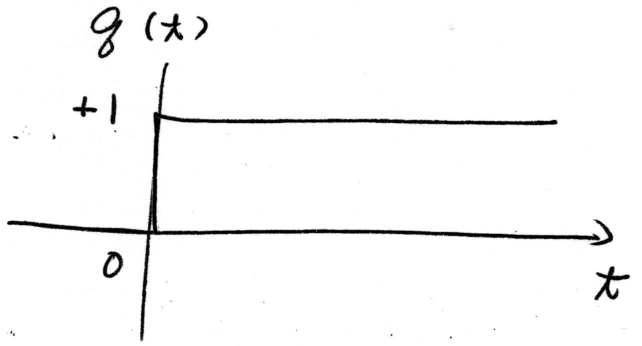


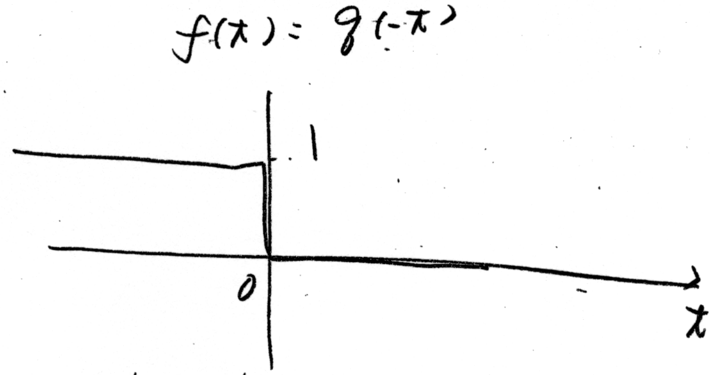




- Flipping: Let $f(t)$ be a function. Then $f(-t)$ is the flipping of $f(t)$ with respect to $t=0$, to the negative time.

Example: $f(t) = g(-t)$. plot $f(t)$.

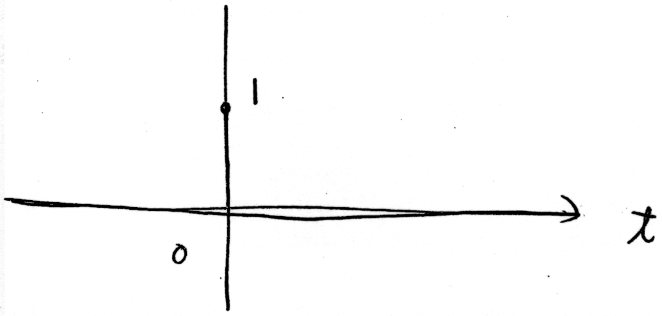




- multiplication: Consider two signals $f(t)$ and $h(t)$ defined for all t in $(-\infty, \infty)$. Then their product $g(t) := f(t)h(t)$ forms a new signal.

Example: plot $f(t) = g(t)\, g(-t)$.

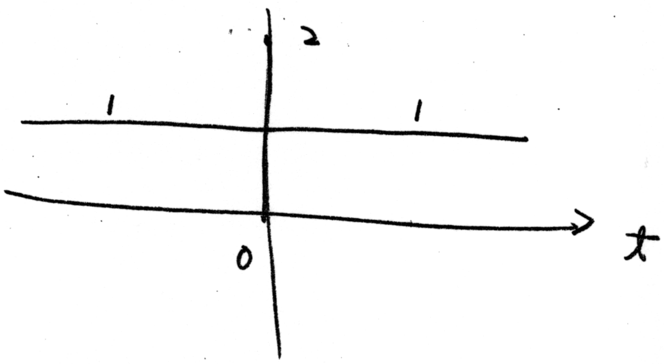
$$f(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{elsewhere} \end{cases}$$



※ Addition: Consider two signals $h(t)$ and $f(t)$ defined for all t. Their sum $g(t) = f(t) + h(t)$, forms a new signal.

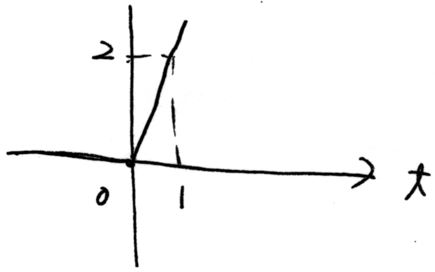
Example: plot $f(t) = g(t) + g(-t)$

$$f(t) = \begin{cases} 2, & t = 0 \\ 1, & \text{elsewhere} \end{cases}$$

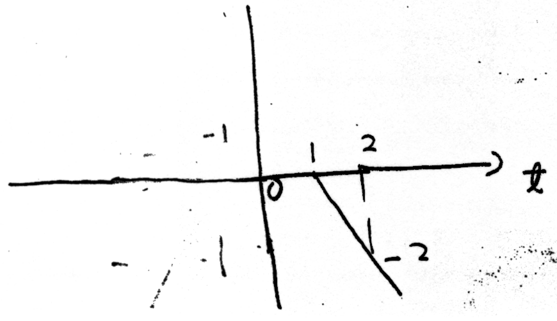


Example: plot $f(t) = 2\gamma(t) - 2\gamma(t-1) - 2g(t-3)$

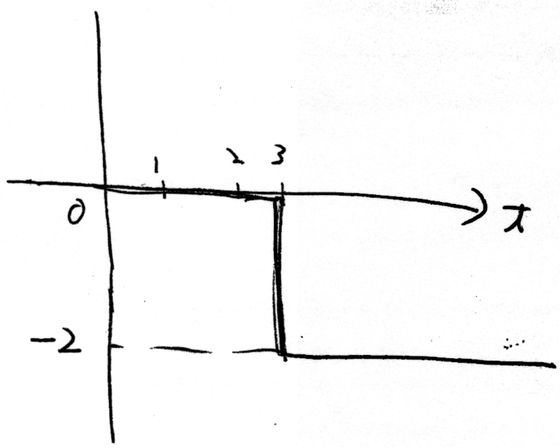
$2\gamma(t)$



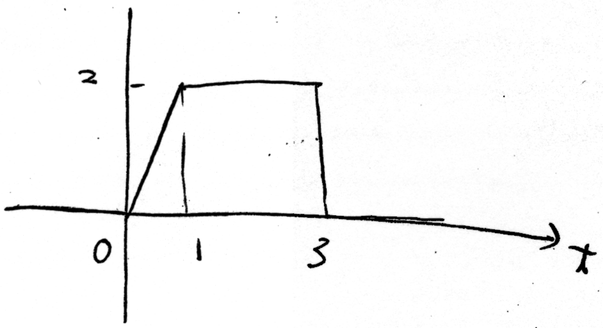
$-2\gamma(t-1)$



$-2\, g(t-3)$



$f(t)$

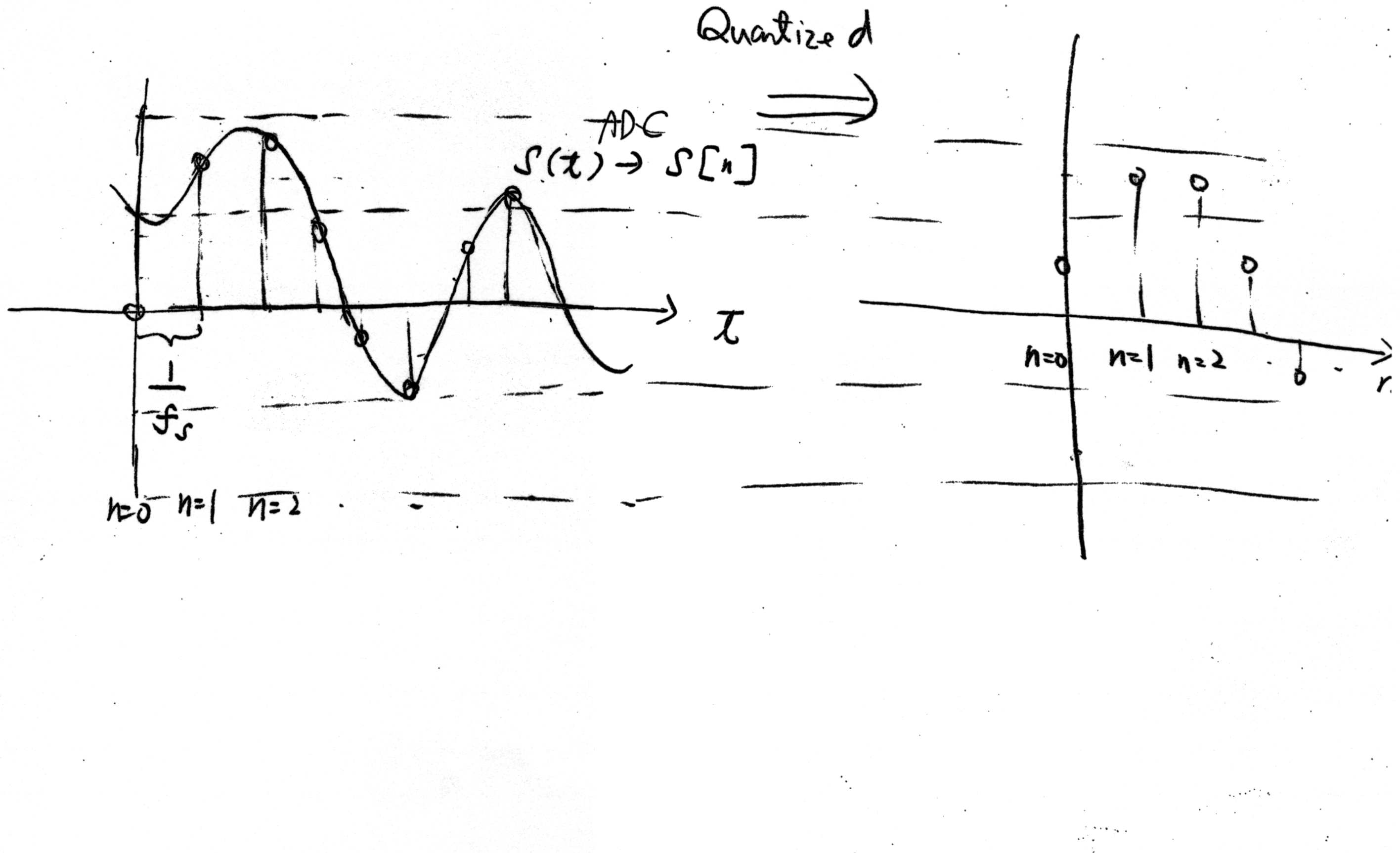


Since $f(t) = \begin{cases} 2t, & t \le 1 \\ 2t - 2(t-1) = 2, & t \in [1, 3] \\ 2t - 2(t-1) - 2 = 0, & t \ge 3 \end{cases}$

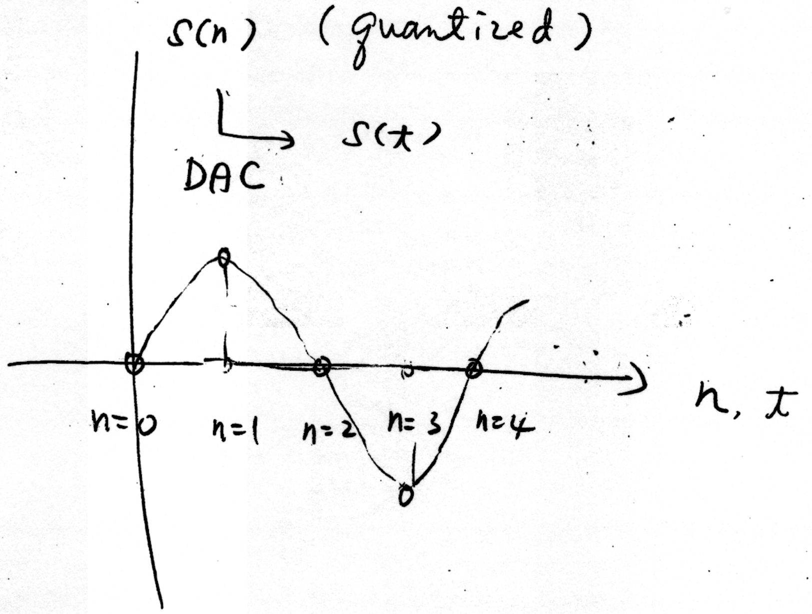
1.4 Discrete-time

1.4 Discrete-time and digital signals

A continuous-time (analog) signal is sampled to generate a discrete-time signal through an Analog-to-digital converter (ADC). Then the amplitude of a discrete-time signal can be quantized to a finite set of values, so the waveforms of the quantized discrete-time signals are called digital signals.



On the contrary, a digital signal can be converted to an analog signal through a DAC (digital-to-analog converter). Basically DAC interpolates a continuous-time function in between any pair of successive discrete-time samples along the waveform.

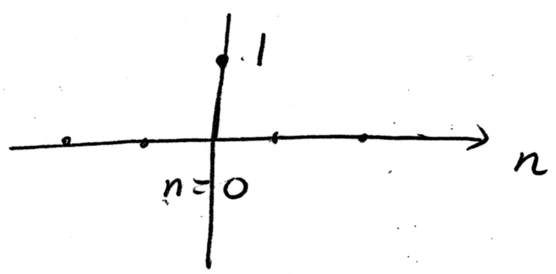


1.5 Elementary Discrete-time Signals and Their Manipulation

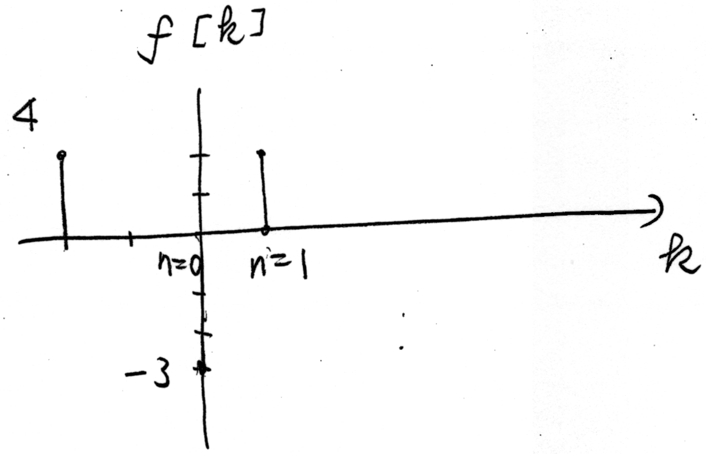
$$f[k] := f(kT) = f\left(\frac{k}{f_s}\right), \quad \text{where } T = \frac{1}{f_s}$$

※ Impulse sequence:

$$\delta[k] = \begin{cases} 0, & \text{for } k \neq 0 \\ 1, & \text{for } k = 0 \end{cases}$$



※ Example: Plot $f[k] = 4\delta[k+2] - 3\delta[k] + 2\delta[k-1]$

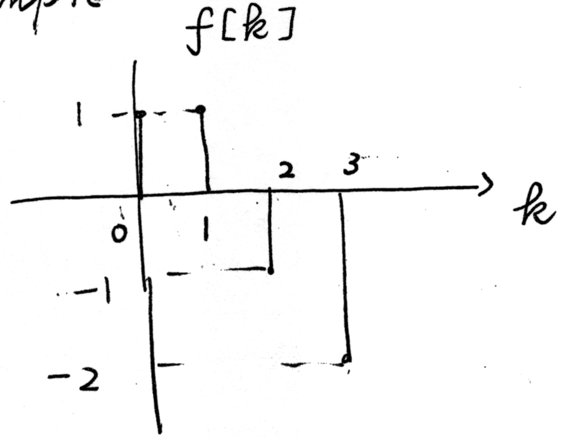


For any discrete-time sequence (signal) $f[k]$,

it can be written as

$$f[k] = \sum_{i=-\infty}^{\infty} f[i]\, \delta[k-i]$$

Example :



$$f[k] = f[0]\,\delta[k] + f[1]\,\delta[k-1] + f[2]\,\delta[k-2] + f[3]\,\delta[k-3]$$

$$= \delta[k] + \delta[k-1] - \delta[k-2] - 2\delta[k-3]$$

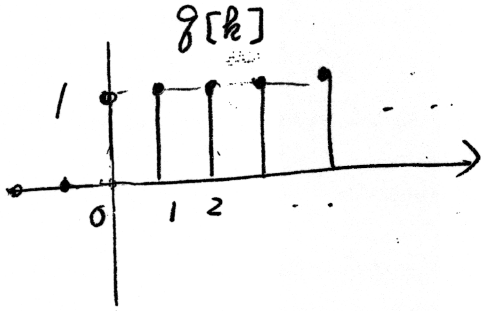
※ Unit step sequence :

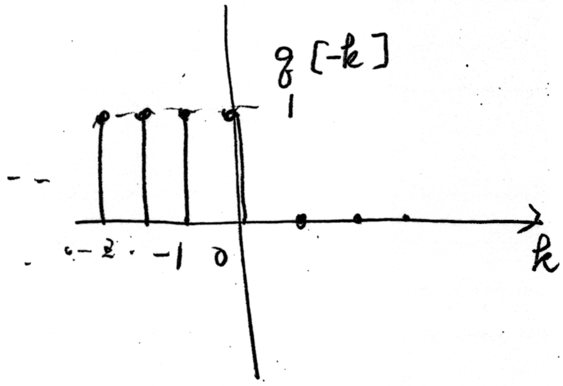
$$q[k] := \begin{cases} 1, & \text{for } k \geq 0 \\ 0, & \text{for } k < 0 \end{cases} \quad \text{or}$$

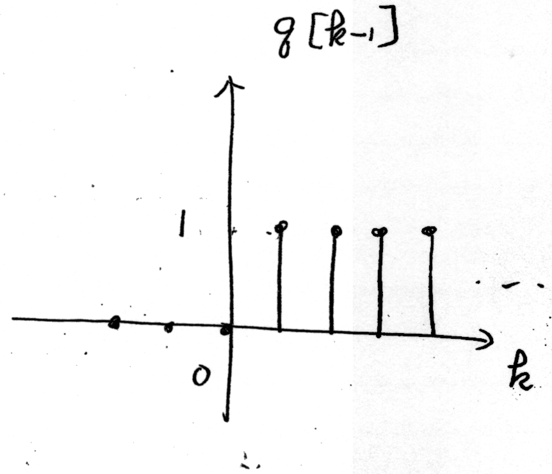
$$q[k-i] := \begin{cases} 1, & \text{for } k \geq i \\ 0, & \text{for } k < i \end{cases}$$

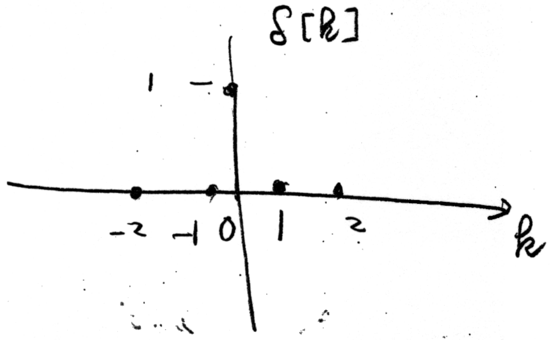
$$\delta[k] = q[k] - q[k-1] = q[k]\, q[-k]$$

$$q[k] + q[-k] - \delta[k] = 1$$





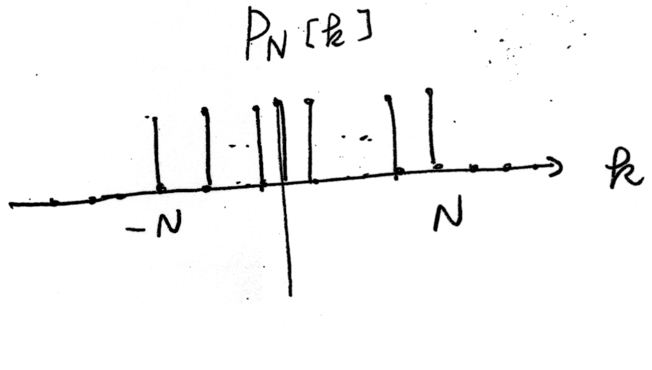




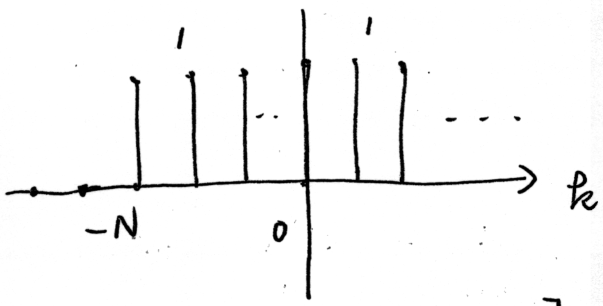
Example: Express a window function $P_N[k]$ by a linear combination of unit-step functions, where

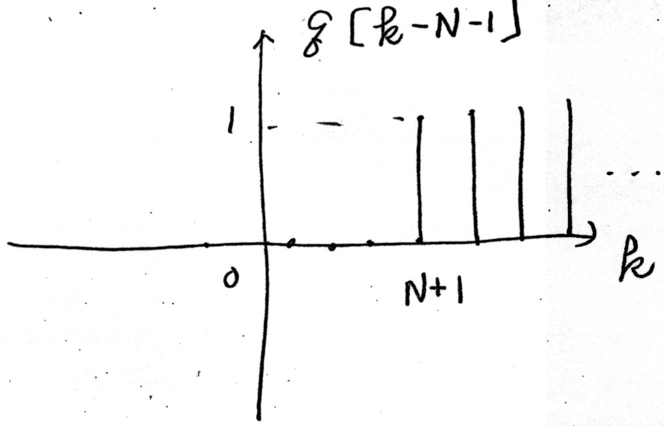
$$P_N[k] := \begin{cases} 1, & \text{for } -N \le k \le N \\ 0, & \text{for } k < -N \text{ and } k > N \end{cases}$$

Answer:



$g[k+N]$





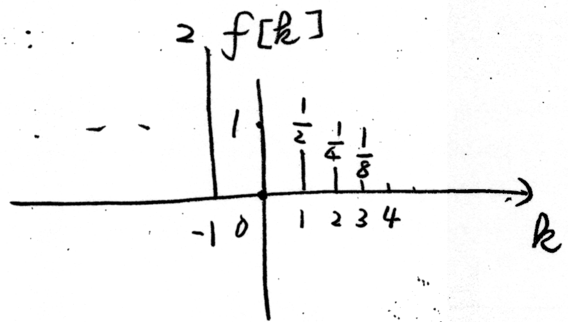
Hence $P_N[k] = g[k+N] - g[k-N-1]$

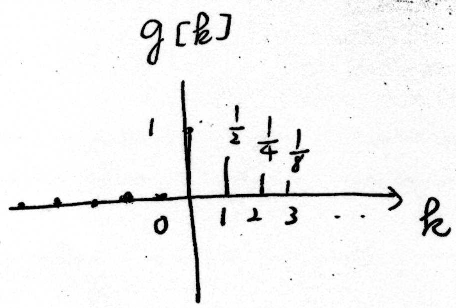
X. exponential function

$$f[k] = b^k$$

Example: Plot $f[k] = (\frac{1}{2})^k$ and $g[k] = (\frac{1}{2})^k g[k]$

Answer:





1.6 Sinusoidal sequence and its frequency

Consider an analog sinusoid $f(t) = \sin(2\pi f_0 t) = \sin(\omega_0 t)$ with its fundamental period $P = \frac{2\pi}{\omega_0} = \frac{1}{f_0}$ where $\omega_0 = 2\pi f_0$. We sample it with sampling period $T > 0$ yield

$$f[k] := f(kT) = \sin(\omega_0 kT), \quad k = 0, \pm 1, \pm 2$$

A discrete-time signal $f[k]$ is said to be periodic with period N, where N is a positive integer if $f[k] = f[k+N]$. If $f[k]$ is periodic then $f[k] = f[k+2N] = f[k+3N] \cdots$

$= f[k+nN]$ for any k and any integer n.

$$\sin(\omega_0 kT) = \sin(\omega_0 (k+N)T)$$

$$\Rightarrow \quad \omega_0 NT = 2\pi n \quad \text{or} \quad N = \frac{2n\pi}{\omega_0 T} \longrightarrow \text{period}$$

The smallest such N is called fundamental period.

Example :

Which of the following sequences are periodic? If any, find their periods and specify fundamental periods.

(a) $\sin(0.1k)$

(b) $\cos(\frac{3}{7}\pi k)$

Answer : (a) $N = \frac{2n\pi}{0.1}$ cannot be integer no matter what kind of n is chosen

Hence it is not periodic.

(b) $N = \frac{2n\pi}{\frac{3}{7}\pi} = \frac{14n}{3}$ can be integer

whenever $n = 3l$, l is any positive integer

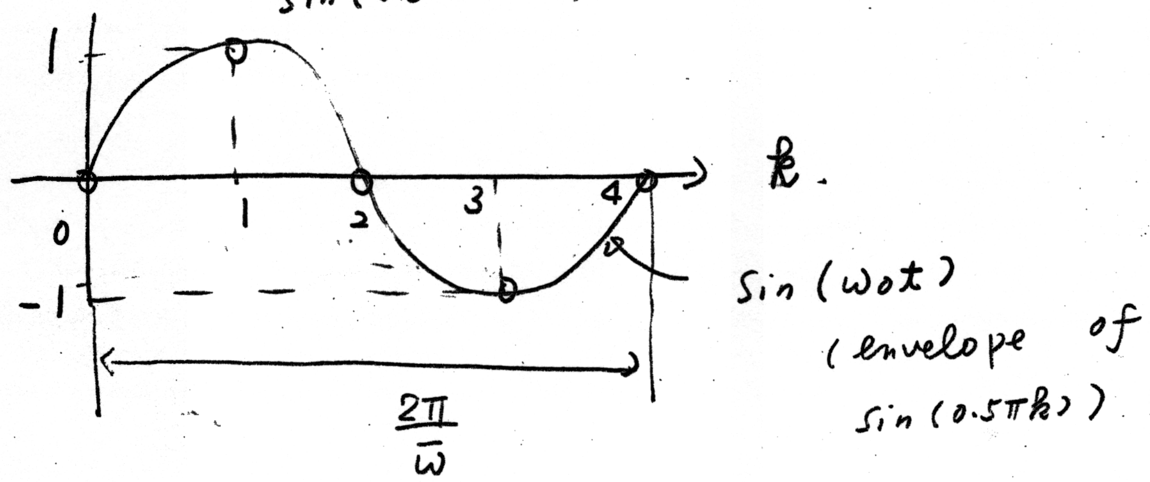
The fundament period is $N = 14$.

(the least integer N)

How can we visualize the continuous-time and discrete-time sinusoids.

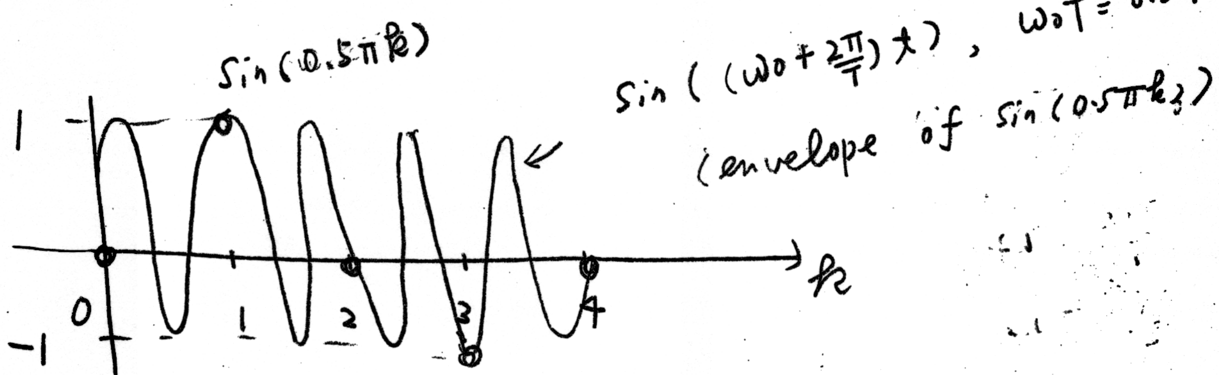
If we sample a continuous-time sinusoid $\sin(\bar{\omega} t)$ at $t = kT$ to generate a discrete-time sinusoid $\sin(\omega_0 kT)$, $N = \frac{2n\pi}{\omega_0 T}$

$\sin(0.5\pi k)$, $\omega_0 T = 0.5\pi$, $\bar{\omega} = \omega_0$



However, if we sample another continuous-time sinusoid $\sin((\omega_0 + \frac{2n\pi}{T})t)$ at $t = kT$, we have $\sin(\omega_0 kT + 2\pi nk) = \sin(\omega_0 kT)$, where n is integer.

We still have the same sequence!!



Hence we can not specify $\bar{\omega}$ according to $\sin(\omega_0 kT)$. We need to specify a primary envelope of $\sin(\omega_0 kT)$ with a unique frequency in the range of $-\pi < \omega T \le \pi$ or $-\frac{\pi}{T} < \omega \le \frac{\pi}{T}$ such that we can have a unique relationship $\sin(\omega t)\Big|_{t=kT} = \sin(\omega_0 kT)$

Example: What is the frequency ω of a primary envelope for $\sin(1.1\pi k)$? What are all possible frequencies $\bar{\omega}$ of envelopes for $\sin(1.1\pi k)$? Both sampled at $T = 0.5$

Answer: Since $\omega_0 = \frac{1.1\pi}{0.5} = 2.2\pi > \frac{\pi}{T} = 2\pi$

we need to carry ω_0 down one period $\frac{2\pi}{T}$. $\omega = \omega_0 - \frac{2\pi}{T} = -1.8\pi > -2\pi$

$\bar{\omega} = \omega + \frac{2n\pi}{T} = -1.8\pi + 4n\pi, \quad n \in \mathbb{Z}$

Definition: The frequency of a digital $\sin(\omega_0 kT)$, is defined as the frequency of an analog primary envelope $\sin(\omega t)$ with $-\frac{\pi}{T} < \omega \le \frac{\pi}{T}$

Example: Find the primary frequencies of the following sequences for $T = 1$.

(a) $\sin(4.2\pi k)$ (b) $\sin(-2.1k)$

Answer: (a) $\omega_0 = 4.2\pi > \pi$, we need to carry ω_0 down 4π such that

$$\omega = \omega_0 - 4\pi = 0.2\pi < \pi$$

$$\omega = 0.2\pi \text{ rad/sec}$$

(b) $\omega_0 = -2.1 > -\pi$

Hence $\omega = \omega_0 = -2.1$ rad/sec.

Sampling Theorem: If the sampling period T is less than half of the fundamental period ω_0 of an analog signal such as $\sin(\omega_0 t)$, then the frequency of $\sin(\omega_0 t)$ can be determined from its sampled sequence.

$$s(\omega_0 kT) \xrightarrow[\text{sampling Theorem}]{\text{complied with}} \sin(\omega t)$$

(primary envelope)

Complex exponential sequence:

$$f[k] = e^{j\omega_0 kT} = \cos(\omega_0 kT) + j\sin(\omega_0 kT)$$

$$f[k+N] = e^{j\omega_0 kT} \, e^{j\omega_0 NT}$$

$f[k]$ is periodic if and only if $\omega_0 NT = 2n\pi$

or $\frac{\omega_0 T}{\pi}$ is a rational number.

Example: Check if the following sequences are periodic. If so, figure out its fundamental period. (a) $e^{j0.7\pi k}$

(b) $e^{-0.3k}$

Answer:

(a) $\frac{\omega_0 T}{\pi} = \frac{0.7\pi}{\pi} = 0.7 = \frac{7}{10}$ .. rational

$e^{j0.7\pi k}$ is periodic,

$$N = \frac{2n\pi}{\omega_0 T} = \frac{2n}{0.7} = \frac{20}{7} n$$

$N = 20$ is the fundamental period.

(b) $\frac{\omega_0 T}{\pi} = \frac{-0.3}{\pi}$ ... irrational

$e^{-0.3k}$ is not periodic.