The Complex Envelope of a Bandlimited OFDM Signal Converges Weakly to a Gaussian Random Process

Shuangqing Wei, Dennis L. Goeckel, and Patrick A. Kelly

Abstract

Orthogonal frequency division multiplexing (OFDM) systems have been used extensively in wireless communications applications in recent years; thus, there is significant interest in analyzing the properties of the transmitted signal in such systems. In particular, a large amount of recent work has focused on analyzing the variation of the complex envelope of the transmitted signal and on designing methods to minimize this variation. In this paper, it is established that the complex envelope of a bandlimited uncoded OFDM signal converges weakly to a Gaussian random process as the number of subcarriers goes to infinity. This establishes that the properties of the OFDM signal will asymptotically approach those of a Gaussian random process over any finite time interval. The symbol length in a bandlimited OFDM system will eventually exceed any finite time interval as the number of subcarriers approaches infinity; however, practical interest is in how asymptotic approximations apply for a finite number of carriers, and, hence, the convergence proof is reasonable motivation for considering how the extremal value theory of Gaussian random processes might provide accurate approximations for the distribution of the peak-to-mean envelope power ratio (PMEPR) of practical OFDM systems. Indeed, numerical results are presented that indicate that the resulting simple expressions are accurate for a wide range of the distribution for moderate numbers of subcarriers. The important extensions of the analytical and numerical results to coded OFDM systems are also presented.

Keywords: Orthogonal frequency division multiplexing, peak-to-mean envelope power ratio, Gaussian random process, extreme value theory.

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1 Introduction

A major goal of modern communication systems is to allow high-speed communication, regardless of the location or mobility of the system users. However, this goal is difficult to achieve due to the multipath fading that affects wireless communication signals. One alternative for achieving high-speed wireless communication in the presence of multipath fading is to employ a multicarrier system, generally implemented as an orthogonal frequency division multiplexing (OFDM) system [1], in conjunction with error control coding. Such coded OFDM systems have emerged recently as a strong competitor to single-carrier systems and have been employed or are being considered for a number of applications, including digital audio broadcast and digital video broadcast in Europe [2], wireless local area networks [3], broadband fixed wireless access [4], and cellular data [5].

One of the challenges to be overcome when employing an OFDM system in low-power peer-to-peer wireless communication systems is that the complex envelope of the transmitted OFDM signal can demonstrate significant variation; in other words, its peak-to-mean envelope power ratio (PMEPR) can be much larger than that of an analogous single-carrier system [1, 6]. This large PMEPR can require significant backoff of the average operating power of the power amplifier in the transmitter if it is to be operated in the linear region, which results in significant power inefficiency [7, 8]. Thus, there has been a large body of work in the analysis of the variation of the complex envelope of the OFDM signal and in methods to reduce this variation. Here, the focus is on the analysis problem. Although not universally adopted (see [9] and [10, 12, 13, 14] as examples of approaches that do not rely on such), many recent papers that have analyzed the PMPER of the transmitted OFDM signal [15, 16, 17, 18] or its effects [19] often assume that the complex envelope of the transmitted OFDM signal converges in some sense to a Gaussian random

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process as the number of subcarriers becomes large. For example, in the work of [17] and [18], the assumption of such convergence is used when studying the PMEPR distribution to justify the use of Rice's level-crossing results for the envelope of a complex Gaussian random process [21].

However, there exists no rigorous investigation into the limiting form of the complex envelope of the transmitted OFDM signal, despite the theoretical and practical importance of such an endeavor. Thus, in this paper, a formal proof that a bandlimited OFDM signal converges weakly to a Gaussian random process is established, and its implications (including what it does *not* imply) are considered. The main result of this paper is Theorem 2:

Theorem 2

Consider the complex signal

$$s_N(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k e^{j\omega_k t}, \tag{1}$$

where $\omega_k = \frac{2\pi k}{NT_c}$, $T_c \in (0, \infty)$, and $\{A_k,\, k = 0, \cdots, N-1\}$ is an independent and identically distributed (IID) sequence of complex random variables, where the real part (A_k^R) and imaginary part (A_k^I) are bounded ($|A_k^R| \leq \bar{A}$ and $|A_k^I| \leq \bar{A}$), with $E\left[A_k^R\right] = E\left[A_k^I\right] = 0$, $E[A_k^R A_k^I] = 0$, and $E\left[\left(A_k^R\right)^2\right] = E\left[\left(A_k^I\right)^2\right] = \sigma^2$. Then, as $N \to \infty$, for any closed and finite interval $T \subseteq R$,

$$\{s_N(t),\, t \in T\} \xrightarrow{\mathcal{D}} \{s(t),\, t \in T\}$$

where $\xrightarrow{\mathcal{D}}$ implies *convergence in distribution* and $s(t)$ is a zero-mean stationary Gaussian random process defined over the interval T, with real part $x(t)$ and imaginary part $y(t)$ such that

$$E\left[x(t_i)x(t_j)\right] = E\left[y(t_i)y(t_j)\right] = \sigma^2 \mathrm{sinc}\left(\frac{2(t_j - t_i)}{T_c}\right),$$

and

$$E\left[x(t_i)y(t_j)\right] = \sigma^2 \frac{\sin^2\left(\frac{(t_j - t_i)\pi}{T_c}\right)}{\frac{\pi(t_j - t_i)}{T_c}}$$

for all t_i and t_j in T. The implied weak convergence of the underlying measures is in the metric space $(C_T \times C_T, \overline{\rho})$, where C_T is the space of continuous functions on the interval T, and

$\overline{\rho}\left((x_1, x_2), (y_1, y_2)\right) = \max\left\{\rho(x_1, y_1), \rho(x_2, y_2)\right\}$, where $x_1,\, x_2,\, y_1,\, y_2$ are in C_T and $\rho(x, y) = \sup_{t\in T} |x(t) - y(t)|$. $\Box$

Theorem 2 can then be used to prove the following analogous result for the complex baseband representation of the transmitted signal in multicarrier systems that are symmetric about the carrier [17].

Theorem 3

Consider the complex signal:

$$V_N(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k e^{j\omega_k t},$$

where $\omega_k = \frac{2\pi}{NT_c}\left(k - \frac{N-1}{2}\right)$ and $\{A_k,\, k = 0, \cdots, N-1\}$ is as defined above. Then, as $N \to \infty$, for any closed and finite interval $T \subseteq R$,

$$\{V_N(t),\, t \in T\} \xrightarrow{\mathcal{D}} \{V(t),\, t \in T\}$$

where $V(t)$ is a zero-mean stationary complex Gaussian random process defined over the interval T with independent real and imaginary parts, each with autocorrelation function

$$\sigma^2 \text{sinc}\left(\frac{(t_j - t_i)}{T_c}\right), \quad \forall\, t_i,\, t_j \in T.$$

The implied weak convergence of the underlying measures is on the metric space $(C_T \times C_T, \overline{\rho})$ as defined above in Theorem 2. $\Box$

Remarks: Note that we only assume uncorrelation between A_K^R and A_k^I in the theorems, not the independence between the real and imaginary parts of each symbol [12]. This assumption holds not only for quadrature amplitude modulation (QAM) constellations, but also for phase-shift keying (PSK) constellations.

Using the assumption that the envelope of the transmitted OFDM signal is asymptotically Gaussian, previous work [17, 18] has relied largely on the work of Rice [21] to develop results for the PMEPR distribution of the OFDM signal. The work of [17] employs [21] in conjunction with a number of approximations and a parameter obtained through simulation to arrive at a final expression for the PMEPR. The work of [18] finds lower and upper bounds for the PMEPR distribution through the use of extensive manipulation on top of the results found in [21]. However, even Theorem 2 and Theorem 3 cannot be used to rigorously justify such exercises. In particular, since the proof of weak convergence consists of demonstrating convergence over any finite interval and the symbol period for a bandlimited OFDM signal approaches infinity as the number of subcarriers goes to infinity, Theorem 2 and Theorem 3 cannot be applied as rigorous justification for the work in [17, 18]. And, unfortunately, the extension of Theorem 2 and Theorem 3 to an infinite interval has proved elusive.

However, since our true interest in practice is how results obtained from asymptotic behavior apply for a finite number of subcarriers, Theorem 2 and Theorem 3 motivate the consideration of the asymptotic properties of a Gaussian random process, and our simulation results will firmly support such an endeavor. Given the Gaussianity of the signal, rather than following the complicated approaches of [17] and [18], the modern theory of extreme values of chi-squared random processes (i.e. those corresponding to the envelope process of the complex Gaussian process) is employed to arrive in a straightforward manner at simple and accurate approximations to the PMEPR distributions for the envelope of the transmitted OFDM signal. It is demonstrated through simulation that these simple and well-justified expressions are extremely accurate for a large part of the distribution, and, like the results in [17] and [18], apply surprisingly well for OFDM systems with only a modest number of subcarriers.

After the presentation of the numerical results for uncoded OFDM systems, attention is turned to coded systems. Because an OFDM system effectively forms a large number of frequency-nonselective subchannels, it is well-known that uncoded OFDM systems will perform poorly on wireless communication channels due to a lack of diversity. Thus, wireless OFDM systems almost always employ some form of error control coding. This introduces statistical dependence among the symbols placed on the subcarriers, and thus Theorem 2 cannot be applied directly. However, by invoking results from modern central limit theory for sums of dependent random variables, it is possible to prove Theorem 4, which generalizes the results of Theorem 2 to most block coded and convolutionally coded systems. The corresponding PMEPR distribution approximation, which relies only on correlation statistics and is identical to the uncoded case for most codes, then follows directly. Numerical results confirm the accuracy of the derived expressions for moderate numbers of subcarriers.

This paper is organized as follows. Section 2 provides the proofs of the main results of the paper. First, Theorem 1, which establishes the appropriate convergence of the real part of $s_N(t)$, is proven. Theorem 2 and Theorem 3 then easily follow, as well as Theorem 4, which extends the results to coded OFDM systems. Section 4 performs the extension of the results of previous sections to uncoded OFDM systems with an unequal power distribution across subcarriers, a situation studied extensively in [18]. Finally, Section 5 presents a discussion of critical issues and the conclusions of this work.

2 Proofs of the Main Theorems

2.1 Proofs of Theorems 2 and 3

The following result can be used in a straightforward manner to prove all of the theorems in this paper. In particular, Lemma 2.3 contains the crux of the proofs for Theorems 2 and 3.

Theorem 1

Consider

$$x_N(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(A_k^R \cos\left(2\pi \frac{k}{NT_c} t\right) - A_k^I \sin\left(2\pi \frac{k}{NT_c} t\right)\right), \quad t \in T \tag{2}$$

for any closed and finite interval $T \subseteq R$, where the complex sequence $\{A_k = A_k^R + jA_k^I,\, k = 0, \cdots, N-1\}$ is an independent and identically distributed (IID) sequence of complex random variables, where the real part (A_k^R) and imaginary part (A_k^I) are bounded ($|A_k^R| \le \bar{A}$ and $|A_k^I| \le \bar{A}$), with $E\left[A_k^R\right] = E\left[A_k^I\right] = 0$, $E[A_k^R A_k^I] = 0$, and $E\left[\left(A_k^R\right)^2\right] = E\left[\left(A_k^I\right)^2\right] = \sigma^2$. Then,

$$\{x_N(t),\, t \in T\} \xrightarrow{\mathcal{D}} \{x(t),\, t \in T\}$$

where $x(t)$ is a zero-mean stationary random process defined over T, with autocorrelation function

$$E\left[x(t_i)x(t_j)\right] = \sigma^2 \mathrm{sinc}\left(\frac{2(t_j - t_i)}{T_c}\right), \ \forall t_i, t_j \in T.$$

The implied weak convergence of the underlying measures is in the metric space (C_T, ρ), where C_T is the space of continuous functions on the interval T, and $\rho(x, y) = \sup_{t \in T} |x(t) - y(t)|$. $\square$

In this paper, all probabilities are defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is the outcome space, $\mathcal{F}$ is the σ-field on Ω, and $\mathcal{P}$ is the probability measure defined on $\mathcal{F}$. Measurability of the appropriate quantities is then easily established [20].

To prove convergence in distribution of a sequence of random functions $\{x_N\}$ to some $\{x(t), t \in T\}$ in C, it is sufficient to show that the sequence $\{x_N\}$ is tight and that each of the finite-dimensional distributions $P_N \pi^{-1}_{t_1, \cdots, t_k}$ of x_N converges weakly to the measure $\mu_{t_1, \cdots, t_k}$ induced by x on $(R^k, \mathcal{R}^k)$, for each $(t_1, \cdots, t_k)$ [22, pg. 47].

The sequence $\{x_N\}$ of random functions of C is tight if and only it satisfies the following two conditions [22, pg. 55]:

<u>Condition 1.</u> For each positive η, there exists an *a* such that

$$\mathcal{P}\{|x_N(0)| > a\} \leq \eta, \quad N \geq 1 \tag{3}$$

<u>Condition 2.</u> For each positive ε and η, there exists a δ, with $0 < \delta < 1$, and an integer N_0 such that

$$\mathcal{P}\left\{ \sup_{\substack{|s-t|<\delta \\ s,t \in [0,1]}} |x_N(s) - x_N(t)| \geq \varepsilon \right\} \leq \eta, \quad N \geq N_0. \tag{4}$$

Condition 1 is easily established via the following lemma. For lemmas in Section 2, proofs that are omitted can be found in [20].

<u>**Lemma 2.1**</u>

Let $\{x_N\}$ be defined as in (2). Then, for each positive η, there exists an *a* such that

$$\mathcal{P}\{|x_N(0)| > a\} \leq \eta, \quad N \geq 1 \tag{5}$$

□

Establishing Condition 2 is the crux of the entire proof. First, a preliminary lemma is presented and then Condition 2 is established. Note that only Lemma 2.2 restricts the class of signals to which the convergence results applies, and the OFDM signals of interest are shown to be part of this class.

Lemma 2.2

$$E\left|x_N(t+h)-x_N(t)\right|^2 \leq \beta\, h^2,\ \beta=\frac{4}{3}\left(\frac{\pi\sigma}{T_c}\right)^2, \quad \forall N\geq 1$$

□

Proof: The proof can be found in [20]; however, the interested reader will have no problem adapting the proof of Lemma 4.1 below to this case. □

Lemma 2.3 Let $\{x_N\}$ be defined as in (2). Then, for each positive ε and η, there exists a δ, with $0<\delta<1$, and an integer N_0 such that

$$\mathcal{P}\left\{\sup_{\substack{|s-t|<\delta\\ s,t\in[0,1]}}|x_N(s)-x_N(t)|\geq\varepsilon\right\}\leq\eta, \quad N\geq N_0. \tag{6}$$

□

Proof:

Based on the proposition in [23, pg. 55-56], since $\{x_N(t), t\in T\}\in C$, then every countable set S dense in T is a separating set, which means, with probability 1:

$$\sup_{\substack{t,s\in S\\ |t-s|<\delta}}|x_N(t)-x_N(s)|=\sup_{\substack{t,s\in T\\ |t-s|<\delta}}|x_N(t)-x_N(s)|\,, \quad 0<\delta<1 \tag{7}$$

Define the set S to be the set of dyadic rationals:

$$S=\left\{\frac{k}{2^n}, k=0,1,\cdots,2^n-1;\, n=0,1,2,\cdots\right\}, \tag{8}$$

Define the random variables

$$Z_v^{(N)}(\omega)=\sup_{0\leq k\leq 2^v-1}\left|x_N\left(\omega,\frac{k+1}{2^v}\right)-x_N\left(\omega,\frac{k}{2^v}\right)\right|, \quad \omega\in\Omega, \tag{9}$$

then [23, pg. 56],

$$\sup_{\substack{t,s\in S\\ |t-s|<2^{-M}}} |x_N(\omega,t)-x_N(\omega,s)| \le 2\sum_{v=M+1}^{\infty} Z_v^{(N)}(\omega), \qquad \omega\in\Omega \tag{10}$$

where M is a positive integer. By employing (7) and (10),

$$\begin{aligned}
\mathcal{P}\left\{\sup_{\substack{s,t\in T\\ |t-s|<2^{-M}}} |x_N(s)-x_N(t)| \ge \varepsilon\right\} &= \mathcal{P}\left\{\sup_{\substack{s,t\in S\\ |t-s|<2^{-M}}} |x_N(s)-x_N(t)| \ge \varepsilon\right\} \\
&\le \mathcal{P}\left\{\sum_{v=M+1}^{\infty} Z_v^{(N)} \ge \frac{\varepsilon}{2}\right\} \\
&\le \mathcal{P}\left\{\bigcup_{v=M+1}^{\infty}\left\{Z_v^{(N)} \ge \frac{D(\epsilon)}{q^v}\right\}\right\} \\
&\le \sum_{v=M+1}^{\infty}\mathcal{P}\left\{Z_v^{(N)} \ge \frac{D(\epsilon)}{q^v}\right\}
\end{aligned} \tag{11}$$

where $D(\epsilon)$ and q are constants. The constant q will be specified later, and the constant $D(\epsilon)$ can be determined by the following equation for $q>1$:

$$\sum_{v=M+1}^{\infty}\frac{D(\epsilon)}{q^v} = D(\epsilon)\frac{1/q^{M+1}}{1-1/q} = \frac{1}{2}\varepsilon \tag{12}$$

From (9),

$$\begin{aligned}
\mathcal{P}\left\{Z_v^{(N)} \ge \frac{D(\epsilon)}{q^v}\right\} &= \mathcal{P}\left\{\sup_{0\le k\le 2^v-1}\left|x_N\left(\frac{k+1}{2^v}\right)-x_N\left(\frac{k}{2^v}\right)\right| \ge \frac{D(\epsilon)}{q^v}\right\} \\
&\le \sum_{k=0}^{2^v-1}\mathcal{P}\left\{\left|x_N\left(\frac{k+1}{2^v}\right)-x_N\left(\frac{k}{2^v}\right)\right| \ge \frac{D(\epsilon)}{q^v}\right\}
\end{aligned} \tag{13}$$

By Lemma 2.2 and Chebyshev's inequality [24],

$$\begin{aligned}
\mathcal{P}\left\{\left|x_N\left(\frac{k+1}{2^v}\right)-x_N\left(\frac{k}{2^v}\right)\right| \ge \frac{D(\epsilon)}{q^v}\right\} &\le \frac{E\left|x_N\left(\frac{k+1}{2^v}\right)-x_N\left(\frac{k}{2^v}\right)\right|^2}{(D(\epsilon)/q^v)^2} \\
&\le \frac{\beta(\frac{1}{2^v})^2}{D(\epsilon)^2\frac{1}{q^{2v}}} = \frac{\beta}{D(\epsilon)^2}\left(\frac{q^2}{4}\right)^v.
\end{aligned} \tag{14}$$

Then, from (13)

$$\mathcal{P}\left\{Z_v^{(N)} \ge \frac{D(\epsilon)}{q^v}\right\} \le \frac{\beta}{D(\epsilon)^2}\left(\frac{q^2}{2}\right)^v. \tag{15}$$

Given (11) and (15), if $1 < q < \sqrt{2}$,

$$\mathcal{P}\left\{\sup_{\substack{s,t\in T\\|t-s|<2^{-M}}} |x_N(s) - x_N(t)| \geq \varepsilon\right\} \leq \sum_{v=M+1}^{\infty} \mathcal{P}\left\{Z_v^{(N)} \geq \frac{D(\epsilon)}{q^v}\right\}$$
$$\leq \frac{\beta}{D(\epsilon)^2} \frac{\left(\frac{q^2}{2}\right)^{M+1}}{1 - \frac{q^2}{2}}, \quad \forall N \geq 1 \tag{16}$$

By substituting in $D(\epsilon)$ from (12),

$$\frac{\beta}{D(\epsilon)^2} \frac{\left(\frac{q^2}{2}\right)^{M+1}}{1 - \frac{q^2}{2}} = \frac{2\beta\, G(q)}{\varepsilon^2 2^M} \tag{17}$$

where $G(q) = \left[(1 - 1/q)^2 \cdot (1 - q^2/2)\right]^{-1}$.

Thus, for any positive ε and η, select $1 < q < \sqrt{2}$ and positive integer M to satisfy

$$M \geq \log_2\left[\frac{2\beta}{\varepsilon^2 \cdot \eta} G(q)\right] = \log_2\left[\frac{8\pi^2\sigma^2}{3T_c^2\varepsilon^2\eta} G(q)\right], \tag{18}$$

and let $\delta = 2^{-M}$. Then, the condition of (4) is satisfied:

$$\mathcal{P}\left\{\sup_{\substack{|s-t|<\delta\\ s,t\in[0,1]}} |x_N(s) - x_N(t)| \geq \varepsilon\right\} \leq \eta, \qquad N \geq N_0 = 1.$$

Since ε and η were arbitrary, this establishes Condition 2. □

Hence, for the sequence $\{x_N\}$ in (2) of random functions of C, both Condition 1 and Condition 2 are satisfied, and thus $\{x_N\}$ is tight [22, pg. 55]. Given Lemma 3, establishing Theorem 1 only requires a demonstration that the finite-dimensional distribution $P_N\pi^{-1}_{t_1,\cdots,t_k}$ of x_N, which is determined by the random vector $(x_N(t_1), \cdots, x_N(t_k))$, converges weakly to the measure $\mu_{t_1,\cdots,t_k}$ induced by x on $(R^k, \mathcal{R}^k)$, for each $(t_1, \cdots, t_k)$ [22, pg. 54]. First, a technical lemma is presented, and then the Cramér-Wold Device [22, pg. 49] is employed in a straightforward manner to establish the result.

Lemma 2.4

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \cos\left(\frac{2\pi k}{NT_c}\tau\right) = \text{sinc}\left(\frac{2\tau}{T_c}\right)$$

$\square$

Lemma 2.5

Let $x_N(t)$ be defined as in (2), and pick any integer $L \geq 1$ and collection of sample times $\{t_1, t_2, \ldots, t_L\}$. Then

$$\underline{\Gamma}_N = (x_N(t_1), x_N(t_2), \ldots, x_N(t_L))^T \xrightarrow{\mathcal{D}} \underline{\Gamma},$$

where $\underline{\Gamma} = (\Gamma_1, \Gamma_2, \ldots, \Gamma_L)^T$ is an L-dimensional vector with jointly Gaussian components, mean vector $\underline{0}$, and covariance matrix Σ, where the $(i,j)^{th}$ element of Σ is given by

$$\Sigma_{i,j} = E[\Gamma_i \Gamma_j] = \sigma^2 \text{sinc}\left(\frac{2(t_i - t_j)}{T_c}\right) \tag{19}$$

$\square$

Proof:

Pick any integer $L \geq 1$ and collection of sample times $\{t_1, t_2, \ldots, t_L\}$. The Cramér-Wold Theorem [22, pg. 49] will be employed below; thus, consider any linear combination:

$$Z_N = \sum_{l=1}^{L} a_l x_N(t_l)$$

where $a_1, a_2, \ldots, a_L$ are real constants. Then,

$$\begin{aligned} Z_N &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k^R \sum_{l=1}^{L} a_l \cos\left(2\pi \frac{k}{NT_c} t_l\right) - \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k^I \sum_{l=1}^{L} a_l \sin\left(2\pi \frac{k}{NT_c} t_l\right) \\ &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k^R r_{k,N} - \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k^I i_{k,N} \\ &= \sum_{k=0}^{N-1} \gamma_{k,N} \end{aligned}$$

where

$$\begin{aligned} r_{k,N} &= \sum_{l=1}^{L} a_l \cos\left(2\pi \frac{k}{NT_c} t_l\right), \\ i_{k,N} &= \sum_{l=1}^{L} a_l \sin\left(2\pi \frac{k}{NT_c} t_l\right), \\ \gamma_{k,N} &= \frac{1}{\sqrt{N}} \left(A_k^R r_{k,N} - A_k^I i_{k,N}\right). \end{aligned}$$

Noting $|r_{k,N}| < \sum_{l=1}^{L} |a_l|$ and $|i_{k,N}| < \sum_{i=1}^{L} |a_l|$, and $\left|A_k^R\right| \le \bar{A}$, $\left|A_k^I\right| \le \bar{A}$, Lindeberg's condition for triangular arrays [24, pg. 116] is satisfied as follows. Since

$$\begin{aligned} \left|A_k^R r_{k,N} - A_k^I i_{k,N}\right| &\le \left|A_k^R\right| |r_{k,N}| + \left|A_k^I\right| |i_{k,N}| \\ &\le 2\bar{A} \sum_{l=0}^{N-1} |a_l| = C_0, \end{aligned}$$

and, for any $\epsilon > 0$, there exists N_0, such that when $N \ge N_0$, $\sqrt{N}\epsilon > C_0$. Therefore, if $N \ge N_0$,

$$\begin{aligned} \sum_{k=0}^{N-1} E\left\{|\gamma_{k,N}|^2 ; |\gamma_{k,N}| > \epsilon\right\} &= \sum_{k=0}^{N-1} E\left\{|\gamma_{k,N}|^2 ; \left|A_k^R r_{k,N} - A_k^I i_{k,N}\right| > \sqrt{N}\epsilon\right\} \\ &= 0 \end{aligned}$$

The limiting value of the variance of Z_N will determine two separate cases. Thus, noting $E[Z_N] = 0$, the variance of Z_N is computed as follows. First, note

$$E[Z_N^2] = \sum_{l=1}^{L} \sum_{m=1}^{L} a_l a_m E[x_N(t_l) x_N(t_m)]$$

Next, note

$$\begin{aligned} E[x_N(t_l)x_N(t_m)] &= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{v=0}^{N-1} E\left[\left(A_k^R \cos\left(\frac{2\pi k}{NT_c} t_l\right) - A_k^I \sin\left(\frac{2\pi k}{NT_c} t_l\right)\right)\right. \\ &\qquad \left.\left(A_v^R \cos\left(\frac{2\pi v}{NT_c} t_m\right) - A_v^I \sin\left(\frac{2\pi v}{NT_c} t_m\right)\right)\right] \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \left(\sigma^2 \cos\left(\frac{2\pi k}{NT_c} t_l\right) \cos\left(\frac{2\pi k}{NT_c} t_m\right)\right. \\ &\qquad \left. + \sigma^2 \sin\left(\frac{2\pi k}{NT_c} t_l\right) \sin\left(\frac{2\pi k}{NT_c} t_m\right)\right) \\ &= \frac{\sigma^2}{N} \sum_{k=0}^{N-1} \cos\left(\frac{2\pi k (t_m - t_l)}{NT_c}\right) \\ &\longrightarrow \sigma^2 \mathrm{sinc}\left(\frac{2(t_m - t_l)}{T_c}\right), \end{aligned}$$

which implies

$$\psi^2 \triangleq \lim_{N\to\infty} E[Z_N^2] = \sum_{l=1}^{L} \sum_{m=1}^{L} \sigma^2 a_l a_m \mathrm{sinc}\left(\frac{2(t_m - t_l)}{T_c}\right).$$

If $\psi^2 > 0$, Lindeberg's conditions for triangular arrays [24, pg. 116] are thus satisfied; therefore, $Z_N \xrightarrow{\mathcal{D}} Z$, where Z is normal, $E[Z] = 0$, and $E[Z^2] = \psi^2$.

If $\psi^2 = 0$: Chebyshev's inequality [24] yields

$$P(|Z_N| \geq \eta) \leq \frac{E[Z_N^2]}{\eta^2} \longrightarrow 0,$$

for any $\eta > 0$, which implies $Z_N \xrightarrow{P} 0$. Thus, Z_N converges in distribution to a Gaussian random variable with mean 0 and variance 0.

The two cases together imply that $Z_N \xrightarrow{\mathcal{D}} Z \sim N(0, \psi^2)$ for any ψ^2. Now, for the same constants $a_1, a_2, \ldots, a_L$, define $U = \sum_{l=1}^{L} a_l \Gamma_l$, where Γ_i denotes the i^{th} element of $\underline{\Gamma}$. U is normal with mean $E[U] = 0$ and variance $E[U^2] = \psi^2$. Thus, $U \stackrel{\mathcal{D}}{=} Z$ for any L and collection of $\{a_1, a_2, \ldots, a_L\}$. By the Cramér-Wold Theorem,

$$\underline{\Gamma}_N = (x_N(t_1), x_N(t_2), \ldots, x_N(t_L))^T \xrightarrow{\mathcal{D}} \underline{\Gamma}$$

$\square$

Thus, Theorem 1 is established. Establishing Theorems 2 and 3 is then a straightforward extension. The reader interested in the detailed proofs is referred to [20].

2.2 Extension to Coded Systems

Per Section 1, one of the guiding tenets of wireless OFDM systems is that the bandwidth of each subcarrier should be less than the coherence bandwidth of the wireless channel, which results in no intersymbol interference (ISI) on a given subcarrier and thus obviates the need for complex equalization at the receiver. However, by definition, this makes the effective channel on each subcarrier a frequency non-selective fading channel, which implies that uncoded OFDM systems will perform very poorly. Thus, it has been widely recognized that some form of error control coding is necessary in wireless OFDM systems. However, when error control coding is applied, the independence assumptions required for the central limit theorem results of Theorem 2 and Theorem 3 are violated. Thus, in this section, the results of the previous sections are extended to systems employing error control coding.

It is clear from the work of other researchers that error control coding can have a significant impact on the distribution of the PMEPR of OFDM systems; in fact, a recent line of research has exploited such a fact to develop error control codes for OFDM systems that greatly reduce the PMEPR (see [8] and references therein). In this section, it is shown that, despite the dependence of the symbols at the output of the error control coder on one another, analogous results to those of Theorem 2 and Theorem 3 hold under very broad conditions. In particular, the results hold well for any system with enough “mixing” of codewords or, perhaps surprisingly, for many standard codes for a number of subcarriers on the order for which the results held in the uncoded case.

To establish an analog to Theorem 2, first consider the type of symbol sequence that is employed in a coded system in place of the IID symbol sequence of the uncoded OFDM system. Clearly, the sequence output from the coded modulation in a system employing some form of error control coding contains dependent symbols, for the introduction of such dependence *is* the role of the error control coder. However, most good codes for random errors do not introduce correlation into the symbol stream [25, pg. 527][26], and thus, although it certainly contains dependence, the coded symbol stream can be modeled as uncorrelated. Also, note that such a symbol stream is only locally dependent for traditional codes (i.e. codes that do not introduce the long-term dependence exemplified by, for example, turbo codes [27]). For block codes, symbols separated in index by more than a block length are independent; for convolutional codes, symbols separated in index by more than the constraint length are independent. Thus, the random process at the output of the coded modulation is a form of random process known as “m-dependent” [22], which will be important to establish the mixing results required in the proof of Theorem 4. Finally, note that most coded OFDM systems employ some form of interleaving between the coded modulator and the IFFT in order to obtain some form of diversity; thus, it is important to allow for the possibility of such, although it should be noted that it is *not* required for the results. These assumptions lead to the statement of Theorem 4, which is a generalization of Theorem 2. The proof of Theorem 4 follows from the work of Section 2.1 and [28]; for details, see [20].

Theorem 4

Consider the complex signal

$$s_N(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} B_{k,N} e^{j\omega_k t}, \tag{20}$$

where $\omega_k = \frac{2\pi k}{NT_c}$, $T_c \in (0, \infty)$, and $\{B_{k,N},\ k = 0, \cdots, N-1\}$ is defined by $\underline{B}_N = (B_{0,N}, B_{1,N}, \ldots, B_{N-1,N})^T = P_N \underline{A}_N$, where P_N is an arbitrary $N \times N$ permutation matrix, which permutes the entries of $\underline{A}_N = (A_0, A_1, \ldots, A_{N-1})^T$, and let $\{A_k, k = 0, 1, \ldots, N-1\}$ be drawn from a stationary sequence of identically distributed (but not necessarily independent) random variables where, for all k and l:

1. $E[A_k^R] = E[A_k^I] = 0$.
2. $E[(A_k^R)^2] = E[(A_k^I)^2] = \sigma^2 < \infty$.
3. $\left|A_k^R\right| < \overline{A}$ and $\left|A_k^I\right| < \overline{A}$.
4. A_l and A_k are uncorrelated, $k \neq l$.
5. A_k^R and A_k^I are uncorrelated.
6. There exists an integer n_0 such that A_k and A_l are independent if $|k - l| \geq n_0$.

Then, as $N \to \infty$, for any closed and finite interval $T \subseteq R$

$$\{s_N(t),\ t \in T\} \xrightarrow{\mathcal{D}} \{s(t),\ t \in T\}$$

where $\xrightarrow{\mathcal{D}}$ implies *convergence in distribution* and $s(t)$ is a zero-mean stationary Gaussian random process defined over the interval T, with real part $x(t)$ and imaginary part $y(t)$ such that

$$E\left[x(t_i)x(t_j)\right] = E\left[y(t_i)y(t_j)\right] = \sigma^2 \text{sinc}\left(\frac{2(t_j - t_i)}{T_c}\right),$$

and

$$E\left[x(t_i)y(t_j)\right] = \sigma^2 \frac{\sin^2\left(\frac{(t_j - t_i)\pi}{T_c}\right)}{\frac{\pi(t_j - t_i)}{T_c}}.$$

for all t_i and t_j in T. The implied weak convergence of the underlying measures is on the metric space $(C_T \times C_T, \overline{\rho})$ as defined above in Theorem 2. $\square$

Thus, a convergence result analogous to that demonstrated in Section 2 for uncoded systems has been established for coded systems.

3 PMEPR Distribution of OFDM Signals by Extremal Theory

As noted in Section 1, the establishment of Theorems 2 and 3 does not rigorously establish the work of [17] and [18]. This is because weak convergence of the OFDM process requires only demonstrating convergence on every finite interval of the real line, and, since the symbol interval of a bandlimited OFDM system will become infinite as $N \to \infty$, the result cannot be strictly applied. However, per above, the convergence result provides reasonable motivation for such an endeavor. Rather than following the work of [17] and [18], we exploit modern extreme value theory for the envelope of a complex Gaussian random process to arrive rather quickly at approximations for the PMEPR distribution of OFDM systems with a finite number of subcarriers.

Consider an OFDM system whose passband is symmetric about its carrier; then, the complex baseband representation of the OFDM signal can be expressed as:

$$\tilde{s}_N(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k e^{j\frac{2\pi}{NT_c}\left(k-\frac{N-1}{2}\right)t}, \tag{21}$$

and the PMEPR of $\tilde{s}_N(t)$ can be defined as [17]

$$\mathcal{P}_N \triangleq \frac{\max_{0\leq t\leq NT_c} |\tilde{s}_N(t)|^2}{P_{av}}, \tag{22}$$

where $P_{av} = 2\sigma^2$. Note that $\mathcal{P}_N$ is a random variable, and it is the distribution of $\mathcal{P}_N$ that is studied in this section.

3.1 Preliminaries

Extreme value theory [29, 30] can be employed to obtain the desired result. The required elements of extreme value theory are briefly summarized here. Let $\zeta(t)$, $\eta(t)$, $t > 0$ be independent stationary Gaussian processes, each with zero mean, unit variance, and autocovariance function $r(t) = \text{Cov}\left(\zeta(s), \zeta(s+t)\right) = \text{Cov}\left(\eta(s), \eta(s+t)\right)$, where $\text{Cov}(x, y)$ denotes the covariance of the random variables x and y. Suppose $r(t)$ admits the expansion

$$r(t) = 1 - \lambda \frac{t^2}{2} + o(t^2), \tag{23}$$

as $t \to 0$, and that $\zeta(t)$ and $\eta(t)$ have continuously differentiable sample paths, with $\text{Var}\left(\zeta'(t)\right) = \text{Var}\left(\eta'(t)\right) = \lambda = -r''(0)$, where $\text{Var}(x)$ denotes the variance of the random variable x. Then

$$\chi^2(t) = \zeta^2(t) + \eta^2(t)$$

is said to be a stationary $\chi^2(2)$-process with continuously differentiable sample paths. Suppose further that

$$r(t) \log(t) \to 0, \ \text{as } t \to \infty,. \tag{24}$$

Then

$$\text{P}\left\{ \sup_{0 < t \leq T} \chi^2(t) \leq u^2 \right\} \to e^{-\tau} \tag{25}$$

if $T\mu(u) \to \tau$ as T, $u \to \infty$, where $\mu(u)$ is termed the upcrossing intensity of level u^2 [30, 31], i.e. the mean number of exits by $\left(\zeta(t), \eta(t)\right)$, $0 \leq t \leq T$, across S_u, where

$$S_u = \left\{ (x_1, x_2) \in R^2, x_1^2 + x_2^2 = u^2 \right\}.$$

The upcrossing intensity is related to T by

$$u^2 = 2 \log T + \log \log T + \log\left(\lambda/\pi\right) - 2 \log T\mu(u) + o(1) \tag{26}$$

Combining (25) and (26) yields

$$\text{P}\left\{ a_T \left(\max_{0 \leq t \leq T} \chi^2(t) - b_T \right) \leq x \right\} \to \exp\left(-e^{-x}\right), \ \text{as } T \to \infty \ , \tag{27}$$

for

$$a_T = 1/2, \ b_T = 2 \log T + \log \log T + \log\left(\lambda/\pi\right). \tag{28}$$

3.2 Application to the PMEPR Distribution of Uncoded OFDM

Per Section 2, as $N \to \infty$, the complex baseband OFDM signal (21) is converging weakly to a complex Gaussian random process $\tilde{s}(t) = X(t) + jY(t)$ in any finite closed interval T, with

$$r(\tau) = E[X(t)X(t+\tau)] = E[Y(t)Y(t+\tau)] = \sigma^2 \text{sinc}\left(\frac{\tau}{T_c}\right), \tag{29}$$

and

$$E[X(t_1)Y(t_2)] = 0, \forall t_1 \text{ and } t_2,\ t_1 \in T, t_2 \in T. \tag{30}$$

It is clear that $\chi^2(t) = \frac{1}{\sigma^2}\left(X^2(t) + Y^2(t)\right)$ is a $\chi^2(2)$-process.

From the definition of $x_N(t)$ in (2), it can be seen that, for each N, $x_N(t)$ has continuously differentiable sample paths. Let $\dot{X}(t)$ be the first derivative function of $X(t)$, which is well defined due to the continuity of the second derivative of $r(\tau)$, $\ddot{r}(\tau)$, at $\tau = 0$ [23, pg. 79]. Then, $E\left[\dot{X}(t+\tau)\dot{X}(t)\right] = -\ddot{r}(\tau)$. With $r(\tau)$ determined as in (29), it can be shown that

$$E\left|\dot{X}(t+h) - \dot{X}(t)\right|^2 \leq 0.2\sigma^2 \left(\frac{\pi}{T_c}\right)^4 h^2,\ \forall t \in T, h < 1. \tag{31}$$

Therefore, almost every sample function of $\dot{X}(t)$ is uniformly continuous on T by the Kolmogorov condition [23, pg. 57]. Hence, $X(t)$ has continuously differentiable sample paths almost surely, as does $Y(t)$.

The conditions stated in (23) and (24) are satisfied by $r(\tau)$ in (29), with $\lambda = \frac{1}{3}\left(\frac{\pi}{T_c}\right)^2$. Hence, as $N \to \infty$, the probability density function (PDF) of the PMEPR of the baseband OFDM signal has the following asymptotic characteristic,

$$\begin{aligned} \mathrm{P}\left\{\max_{0\leq t\leq T} \frac{1}{2\sigma^2}\left[X^2(t) + Y^2(t)\right] \leq y\right\} &= \mathrm{P}\left[\max_{0\leq t\leq T} \chi^2(t) \leq 2y\right\} \\ &\longrightarrow \exp\left(-e^{-x}\right), \text{ as } T \to \infty, \end{aligned} \tag{32}$$

where $x = (2y - b_T)\, a_T$. The variables a_T and b_T are defined the same as in (27), with $\lambda = \frac{1}{3}\left(\frac{\pi}{T_c}\right)^2$.

Whereas (32) gives the strict convergence result from extremal value theory, the normalization required for such is not that which is typically employed in the study of the PMEPR of OFDM systems. In particular, practical interest is in the distribution of the PMEPR of a codeword, whose average grows with N and thus whose distribution does not demonstrate strict convergence. Before moving to the latter, the accuracy of the strict convergence of (32) is demonstrated. Figures 1 and 2 demonstrate that (32) is quite accurate, particularly for the $N = 256$ case.

Next, consider scaling (32) to obtain the distribution of the PMEPR of an OFDM symbol. Let $T = NT_c$. When the time-scale is normalized by T_c, $\lambda = \frac{\pi^2}{3}$ and the PDF of the PMEPR $\mathcal{P}_N$ can be approximated as

$$\mathrm{P}\left\{\mathcal{P}_N \leq y\right\} \approx \exp\left\{-e^{-y} N \sqrt{\frac{\pi}{3}\log N}\right\}, \tag{33}$$

when N is large enough.

Remarks: In [13, pp. 2840] under Remark 2 about our work [11], the authors argue that the above result in (33) implies that the number of codewords that have a constant PMEPR $y = \lambda$ (independent of the number of subcarriers N) is given by

$$q^N \cdot \mathrm{P}\left\{\mathcal{P}_N \leq y\right\} \approx q^{N\left(1 - \frac{e^{-\lambda}}{\log q}\sqrt{\frac{\pi}{3}\log N}\right)}, \tag{34}$$

where a codeword is defined as $C_N \triangleq [A_0, A_1, \cdots, A_{N-1}]$ with A_k selected from a finite alphabet set $\mathcal{Q}$ with q elements. It can be inferred from (34) that $q^N \cdot \mathrm{P}\left\{\mathcal{P}_N \leq y\right\}$ goes to zero as N approaches infinity, which contradicts Corollary 1 in [13], and consequently, as put in [13], indicates the CDF of PMEPR in (33) is not correct. However, this argument does not hold as it is based on a misinterpretation of (33). The main reason is that the approximation of CDF for PMEPR obtained in (33) is only valid for large N and large y, where y is determined from (26):

$$y = u^2/2 = \log T + \frac{1}{2}\log\log T + \frac{1}{2}\log\left(\lambda/\pi\right) - \log T\mu(u) + o(1). \tag{35}$$

Therefore, this approximation cannot hold for a fixed constant $y = \lambda$ while letting $T = NT_c$ grow to infinity, which explains why we cannot assume $y = \lambda < \infty$ and use (33) to approximately compute the average number of codewords whose PMEPR is below λ. In addition, it can be easily seen that the PMEPR is in the order of $\log N + O(\log\log N)$, which was shown in [13] using different approaches.

In [17], the following approximations for large N is derived by employing the method of level-crossing rates [21]:

$$\mathrm{P}\left\{\mathcal{P}_N \leq y\right\} \approx \exp\left\{-e^{-y} N \sqrt{\frac{\pi}{3} y}\right\}, \quad \text{for } \mathrm{P}\left\{\gamma > y | \gamma > \bar{r}\right\} \to 0\,, \tag{36}$$

where γ is an arbitrary peak in one OFDM symbol (within $[0, NT_c]$) and $\bar{r}$ is a proper threshold selection such that each positive crossing of the level $\bar{r}$ has a single positive peak that is above the level $\bar{r}$ [17].

The upper bound of the complementary cumulative distribution function (CDF) of the PMEPR was derived as shown in (40) of [18] for N large as

$$\mathrm{P}\left\{\mathcal{P}_N > y\right\} \leq e^{-y} N \sqrt{\frac{\pi}{3} y}. \tag{37}$$

By comparing (36) with (37), it can be seen that the upper bound in (37) is exactly the first order approximation obtained through a Taylor series expansion of $1 - \exp\left\{-e^{-y} N \sqrt{\frac{\pi}{3} y}\right\}$ in (36), which is expected to be accurate as y becomes large.

The comparison of (33), (36) and (37), in terms of the complementary CDF (i.e. $\mathrm{P}\left\{\mathcal{P}_N \geq y\right\}$) with simulation results is shown in Figure 3 and Figure 4. The continuous signal was approximated by a "32-time oversampling" of the complex baseband signal, which means a sampling at 32 times the Nyquist rate. It can be observed that, although the asymptotic form of (36) and (37) differ from the rigorously justified expression of (33), all three expressions provide good characterizations of the PMEPR of the uncoded OFDM system for a moderate number of subcarriers in the OFDM system.

3.3 Extension to Coded Systems

To consider the distribution of the PMEPR for coded systems, note that the expressions for such in the case of uncoded systems depend only on the second order statistics of the limiting process. Thus, since the second order statistics of the limiting process of the coded system are identical to that for the uncoded system, the result of 3.2 can be applied to coded systems without modification.

Numerical results for coded systems are shown in Figures 5 through 8. Figure 5 and 6 are for the case where a binary convolutional code with rate $1/2$ and constraint length 6 is employed without interleaving. Figure 7 and 8 are for the same convolutional code, but now with a block symbol-wise 8×8 interleaver. It can be seen from these figures that as N is increased from 100 to 256, the simulation results are approaching the analytical ones as expressed in (33), (36) and (37).

4 Extensions to Systems with Unequal Power Distributions

4.1 Issues Regarding Convergence

In previous sections, it has been assumed that the power allocated on each subcarrier of the OFDM system is identical, i.e. $E\left|\frac{1}{\sqrt{N}}A_k^2\right| = \frac{2\sigma^2}{N},\ k = 0, \cdots, N-1$. However, since OFDM systems are usually used in channels with nonflat frequency response, it is often desirable to allocate different amounts of power to different subcarriers [18], particularly if some sort of channel state information is available at the transmitter. Let $s_N(t)$ be a complex OFDM symbol, which is redefined as

$$s_N(t) = \sum_{k=0}^{N-1} s_{N,k} e^{j\omega_k t}, \tag{38}$$

where $s_{N,k},\ k = 0, \cdots, N-1$, are independent complex random variables, and $\omega_k = \omega_0 + \frac{2\pi}{NT_c}k$. Let $s_{N,k}^R$ and $s_{N,k}^I$ be the real and imaginary parts of $s_{N,k}$, which have the following statistical characteristics: $E\left[s_{N,k}^R\right] = E\left[s_{N,k}^I\right] = 0$, $E\left[\left(s_{N,k}^R\right)^2\right] = E\left[\left(s_{N,k}^I\right)^2\right] = g_N(k)$, and $E\left[s_{N,k}^R s_{N,k}^I\right] = 0$. Assume there exists a finite constant D_0, such that

$$P\left[\frac{s_{N,k}^R}{\sqrt{g_N(k)}} \le D_0\right] = P\left[\frac{s_{N,k}^I}{\sqrt{g_N(k)}} \le D_0\right] = 1. \tag{39}$$

The function $g_N(k)$ gives the amount of power allocated to the kth subcarrier. Here, it is assumed that the OFDM system is designed to approximate some given power spectral density $G(\omega)$ [18]. The function $G(\omega)$ is assumed to be Riemann-integrable in the interval $[\omega_0, \omega_0 + 2\pi/T_c]$, and bounded by some constant M_G, with

$$\int_{\omega_0}^{\omega_0 + 2\pi/T_c} G(\omega)\, d\omega = \sigma^2. \tag{40}$$

This power distribution may be approximated if the power allocated to the kth subcarrier is

$$g_N(k) = \sigma^2 \frac{G(\omega_k)}{\sum_{m=0}^{N-1} G(\omega_m)}. \tag{41}$$

in which case the average power of the baseband OFDM signal is $P_{av} = 2\sum_{k=0}^{N-1} g_N(k) = 2\sigma^2$. Let λ_1 and λ_2 be the first and second normalized moment of $\frac{1}{\sigma^2}G(\omega)$, respectively, as defined in [18],

$$\begin{aligned}
\lambda_1 &= \lim_{N\to\infty} \frac{1}{\sigma^2} \sum_{k=0}^{N-1} g_N(k)\omega_k = \frac{1}{\sigma^2}\int_{\omega_0}^{\omega_0+BW} \omega G(\omega)d\omega \\
\lambda_2 &= \lim_{N\to\infty} \frac{1}{\sigma^2} \sum_{k=0}^{N-1} g_N(k)\omega_k^2 = \frac{1}{\sigma^2}\int_{\omega_0}^{\omega_0+BW} \omega^2 G(\omega)d\omega
\end{aligned} \tag{42}$$

where $BW = \frac{2\pi}{T_c}$.

As before, let $s_N(t) = x_N(t) + jy_N(t)$ and $R_{s_N}(\tau) = E\left[s_N^*(t)s_N(t+\tau)\right]$; then,

$$\begin{aligned}
E\left[s_N^*(t)s_N(t+\tau)\right] &= \sum_{k=0}^{N-1} E\left|s_{N,k}\right|^2 e^{j\omega_k\tau} \\
&= 2\sigma^2 \frac{\sum_{k=0}^{N-1} \frac{BW}{N} G(\omega_k)e^{j\omega_k\tau}}{\sum_{k=0}^{N-1} \frac{BW}{N} G(\omega_k)} \\
&\to 2\int_{\omega_0}^{\omega_0+BW} G(\omega)e^{j\omega\tau}d\omega = R_s(\tau), \ \text{as } N\to\infty,
\end{aligned} \tag{43}$$

where $R_{x_N}(\tau) = E\left[x_N(t)x_N(t+\tau)\right]$ and $R_{(y_N,x_N)}(\tau) = E\left[x_N(t)y_N(t+\tau)\right]$. It can be shown that

$$\begin{aligned}
R_{x_N}(\tau) &= R_{y_N}(\tau) = \sum_{k=0}^{N-1} g_N(k)\cos(\omega_k\tau) \\
R_{(y_N,x_N)}(\tau) &= \sum_{k=0}^{N-1} g_N(k)\sin(\omega_k\tau).
\end{aligned}$$

Therefore, the autocorrelation functions of the random processes $x_N(t)$ and $y_N(t)$ and their cross-correlation function have the following relationships,

$$\begin{aligned}
R_{s_N}(\tau) &= 2\left(R_{x_N}(\tau) + jR_{(y_N,x_N)}(\tau)\right) \\
R_{(y_N,x_N)}(\tau) &= -R_{(x_N,y_N)}(\tau) = -R_{(y_N,x_N)}(-\tau).
\end{aligned} \tag{44}$$

Let $s(t)$, $x(t)$ and $y(t)$ be the random processes to which $s_N(t)$, $x_N(t)$ and $y_N(t)$ are converging in distribution, respectively. The convergence of these random processes will be proved in the coming paragraphs. Hence, as $N \to \infty$, $R_{x_N}(\tau) \to R_x(\tau) = \text{Re}\left\{\int_{\omega_0}^{\omega_0+BW} G(\omega)e^{j\omega\tau}d\omega\right\}$, and $R_{(y_N,x_N)}(\tau) \to R_{(y,x)}(\tau) = \text{Im}\left\{\int_{\omega_0}^{\omega_0+BW} G(\omega)e^{j\omega\tau}d\omega\right\}$. Since $R_{(y_N,x_N)}(\tau) = -R_{(y_N,x_N)}(-\tau)$, $R_{(y_N,x_N)}(0) = 0$; in other words, $x_N(t)$ and $y_N(t)$ are uncorrelated for each t, as are $x(t)$ and $y(t)$.

To prove that $x_N(t)$ is converging to a Gaussian random process $x(t)$ with autocorrelation function $R_x(\tau) = \text{Re}\left\{\int_{\omega_0}^{\omega_0+BW} G(\omega)e^{j\omega\tau}d\omega\right\}$, it is sufficient to show the tightness of $\{x_N(t)\}$ and convergence of the finite distributions of arbitrary finite samplings of $x_N(t)$ as has been done in the previous sections for the equal power case. Using (39) and the fact that $G(\omega)$ is upper bounded by M_G, it is trivial to prove the convergence of finite distributions, as well as the conditions for tightness. Thus, all that is required is to show a counterpart to Lemma 2.2.

Lemma 4.1: $\forall\epsilon > 0$, there exists $N_0(\epsilon)$, such that

$$E\left|x_N(t+h) - x_N(t)\right|^2 \leq \beta h^2,\ \forall N \geq N_0.$$

Proof:

$$\begin{aligned} E\left|x_N(t+h) - x_N(t)\right|^2 &= 4\sum_{k=0}^{N-1} g_N(k)\text{sin}^2\left(\frac{\omega_k h}{2}\right) \\ &\leq \sum_{k=0}^{N-1} g_N(k)\omega_k^2 h^2 \\ &= h^2\sigma^2\frac{\sum_{k=0}^{N-1} G(\omega_k)\omega_k^2\frac{BW}{N}}{\sum_{k=0}^{N-1} G(\omega_k)\frac{BW}{N}} \\ &\to h^2\int_{\omega_0}^{\omega_0+BW} \omega^2 G(\omega)\,d\omega = h^2\lambda_2,\ \text{as } N \to \infty \end{aligned}$$

□

Then, $\forall\epsilon > 0$, there exists N_0 such that $E\left|x_N(t+h) - x_N(t)\right|^2 \leq \beta h^2$, if $N \geq N_0$, where $\beta = \lambda_2 + \epsilon$. As a result, in (18) of the proof of Lemma 2.3, the corresponding lower bound of M will be

$$M \geq \log_2\left[\frac{2\left(\lambda_2+\epsilon\right)}{\varepsilon^2\cdot\eta}G(q)\right], \tag{45}$$

and the rest follows in an identical fashion to the proof of Lemma 2.3.

For $\{s_N(t)\}$, it can be shown in a straightforward manner that tightness and the convergence of the finite-dimensional distributions is assured. Therefore, it can be concluded that as $N \to \infty$, the sequence of complex random processes $\{s_N(t)\}$ is converging in distribution to a complex Gaussian random process $s(t) = x(t) + jy(t)$, with zero mean and autocorrelation function $R_s(\tau) = \int_{\omega_0}^{\omega_0+BW} G(\omega)e^{j\omega\tau}d\tau$. Since $R_{(y,x)}(0) = 0$, $x(t)$ and $y(t)$ are independent of each other at each t. Thus $|s(t)|^2 = x^2(t) + y^2(t)$ is a $\chi^2(2)$-process.

4.2 Distribution of the PMEPR

Let the PMEPR of a baseband OFDM signal defined in (38) be as that defined in (22) with $P_{av} = 2\sigma^2$, and $\tilde{s}_N(t) = s_N(t)$. To derive the probability density function of the PMEPR of a baseband OFDM signal in the case of unequal power allocations on each subcarrier, extreme value theory will be employed again. In [30], it was pointed out that the results in [30] still hold, even if the imaginary part $\xi(t)$ and real part $\eta(t)$ of a complex Gaussian random process are correlated, if the following conditions are satisfied:

1. $\xi(t)$ and $\eta(t)$ have continuously differentiable paths.
2. $\xi(t)$ and $\eta(t)$ are independent for each t.
3. $\sup_\tau |\text{Cov}\,(\xi(t), \eta(t+\tau))| < 1$.
4. $\text{Cov}\,(\xi(t), \eta(t+\tau)) \log \tau \to 0$, as $\tau \to \infty$.
5. $r(\tau) = \text{Cov}\,(\xi(t), \xi(t+\tau)) = \text{Cov}\,(\eta(t), \eta(t+\tau))$
6. $r(\tau) \log \tau \to 0$, as $\tau \to \infty$.

Define $\eta(t) = \frac{1}{\sigma}x(t)$, $\xi(t) = \frac{1}{\sigma}y(t)$, and $\chi^2(t) = \eta^2(t) + \xi^2(t) = 1/\sigma^2\,|s(t)|^2$. By employing the same arguments as those in Section 3.2, it can be shown that $s(t)$ has continuously differentiable paths. Since $R_{y,x}(0) = 0$, $\xi(t)$ and $\eta(t)$ are independent for each t. Since

$$R_\eta(\tau) + jR_{\xi,\eta}(\tau) = \frac{1}{\sigma^2} \int_{\omega_0}^{\omega_0 + 2\pi/T_c} G(\omega) e^{j\omega\tau} \, d\omega, \tag{46}$$

and

$$\begin{aligned} |R_\eta(\tau) + jR_{\xi,\eta}(\tau)| &= \frac{1}{\sigma^2} \left| \int_{\omega_0}^{\omega_0 + 2\pi/T_c} G(\omega) e^{j\omega\tau} \, d\omega \right| \\ &\leq \frac{1}{\sigma^2} \int_{\omega_0}^{\omega_0 + 2\pi/T_c} \left| G(\omega) e^{j\omega\tau} \right| \, d\omega \\ &= 1, \end{aligned} \tag{47}$$

condition (3) is satisfied. Conditions 4 and 5, require that

$$\log(\tau) \int_{\omega_0}^{\omega_0 + 2\pi/T_c} G(\omega) e^{j\omega\tau} \, d\omega \to 0, \text{as } \tau \to \infty.$$

Thus, the cross-correlation and autocorrelation functions must decrease faster than $1/\log(\tau)$, as $\tau \to \infty$, which is true in most cases. Assuming conditions 4 and 5 are satisfied, let $\mu(u)$ be the upcrossing intensity of a level u^2 for a $\chi^2(2)$-process $\chi^2(t) = \eta^2(t) + \xi^2(t)$ with dependent components, which can be derived to be

$$\mu(u) = (2\pi)^{-1/2} \, u e^{-\frac{1}{2}u^2} \left(b - a^2 \right)^{1/2}, \tag{48}$$

where $b = -\ddot{R}_\eta(\tau)|_{\tau=0}$ and $a = \dot{R}_{\xi,\eta}(\tau)|_{\tau=0}$. The functions $\ddot{R}_\eta(\tau)$ and $\dot{R}_{\xi,\eta}(\tau)$ are defined as the second and first derivative of $R_\eta(\tau)$ and $R_{\xi,\eta}(\tau)$ with respect to τ, respectively.

Based on (46), it can be shown that

$$a = \frac{1}{\sigma^2} \int_{\omega_0}^{\omega_0 + 2\pi/T_c} \omega G(\omega) \, d\omega = \lambda_1, \tag{49}$$

and

$$b = \frac{1}{\sigma^2} \int_{\omega_0}^{\omega_0 + 2\pi/T_c} \omega^2 G(\omega) \, d\omega = \lambda_2. \tag{50}$$

Let $\tilde{\lambda} = b - a^2 = \lambda_2 - \lambda_1^2$. Then, $\mu(u) = \left(\frac{\tilde{\lambda}}{2\pi}\right)^{1/2} u e^{-u^2/2}$, which is exactly the same as for the case when $\xi(t)$ and $\eta(t)$ are independent Gaussian random processes [30, Theorem 1.1]. Therefore, the

PMEPR of a OFDM signal with unequal power allocated on each subcarrier in terms of the power spectral density $G(\omega)$ has the same form as that in (32) with λ replaced by $\tilde{\lambda}$:

$$\begin{aligned} \mathrm{P}\left\{\max_{0\le t\le T}\frac{1}{2\sigma^2}\left[x^2(t)+y^2(t)\right]\le y\right\} &= \mathrm{P}\left[\max_{0\le t\le T}\chi^2(t)\le 2y\right\} \\ &\to \exp\left\{-e^{-y}T\sqrt{\frac{\tilde{\lambda}}{\pi}\log T}\right\} \end{aligned} \tag{51}$$

when $T = NT_c \to \infty$, as $N \to \infty$. After the time scale has been normalized by T_c, i.e. $T_c = 1$ as that in (33), the CDF of PMEPR $\mathcal{P}_N$ in the case of unequal power allocation can be approximated as

$$\mathrm{P}\left\{\mathcal{P}_N \le y\right\} \approx \exp\left\{-e^{-y}N\sqrt{\frac{\tilde{\lambda}}{\pi}\log N}\right\}. \tag{52}$$

From (52), it can be seen that the probability distribution function of the PMEPR of an OFDM signal defined as in (38), whose power on each subcarrier is determined by (40) and (41), is only dependent on the first and second moment of the normalized power spectral density $G(\omega)/\sigma^2$, λ_1 and λ_2, respectively, as well as the number of subcarriers N, when N is large.

Equation (39) in [18] is the upper bound of the complementary CDF of the PMEPR as N is large in the case of an unequal power distribution across N subcarriers. After replacing the parameters with the ones employed in this work, this upper bound

$$\mathrm{P}\left\{\mathcal{P}_N > y\right\} \le Ne^{-y}\sqrt{\frac{\tilde{\lambda}}{\pi}y}. \tag{53}$$

As one example, let the power spectral density $G(\omega)$ be

$$G(\omega) = \begin{cases} \frac{T_c}{\pi} & \text{if } -\frac{\pi}{T_c} \le \omega < 0, \\ \frac{T_c}{2\pi}\left(1+\cos\left(\omega T_c\right)\right) & \text{if } 0 \le \omega \le \frac{\pi}{T_c}. \end{cases} \tag{54}$$

and $\sigma^2 = \int_{-\pi/T_c}^{\pi/T_c} G(\omega)\, d\omega = 3/2$. Thus by substituting $G(\omega)$ of (54) into (49) and (50), it can be shown that

$$\tilde{\lambda} = \frac{1}{T_c^2}\left[\frac{\pi^2}{3} - \frac{2}{3} - \frac{4}{9}\left(\frac{\pi}{4}+\frac{1}{\pi}\right)^2\right] \approx \frac{2.0818}{T_c^2}. \tag{55}$$

It can also be shown that $|R_s(\tau)| = \left|\int_{\omega_0}^{\omega_0+2\pi/T_c} G(\omega)e^{j\omega\tau}d\omega\right|$ is in the order of $1/\tau$, as $\tau \to \infty$. Therefore, condition 4 and 5 are satisfied for this example, and the CDF of the PMEPR in (52) can thus be applied for large N.

The comparison of (52) and the corresponding upper bound (53) with simulation results for the complementary cumulative distribution function of the PMEPR of an uncoded OFDM signal is shown in Figures 9 and 10. Once again, it can be seen that the numerical results agree well with Monte-Carlo simulation results for moderate numbers of subcarriers.

5 Conclusions

In this paper, it has been demonstrated that, under a broad range of conditions, the complex envelope of a bandlimited OFDM system converges (in the limit of a large number of subcarriers) weakly to a Gaussian random process. Although this result applies only to analysis over a finite time interval (and thus does not strictly apply to bandlimited OFDM systems in this limit), the result motivates the application of modern extremal theory for Gaussian random processes to the OFDM problem. This leads in a straightforward manner to simple yet accurate expressions for the distribution of the PMEPR in both uncoded and coded OFDM systems for moderate numbers of subcarriers.

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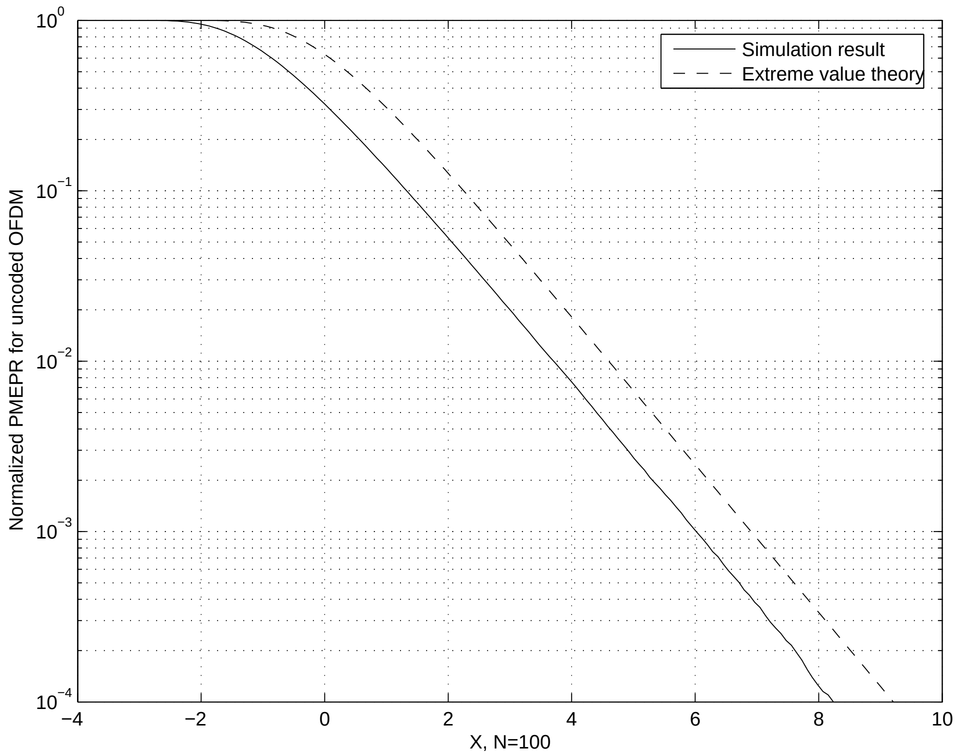

Figure 1: Accuracy of (32) for an uncoded OFDM signal with 100 subcarriers and employing QPSK.

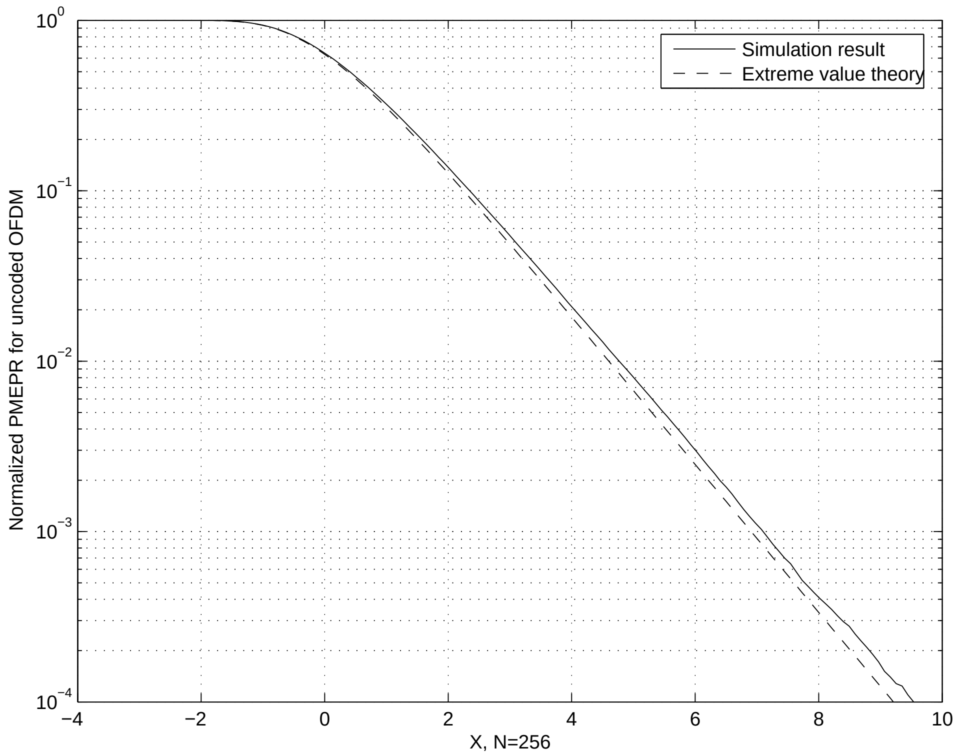

Figure 2: Accuracy of (32) for an uncoded OFDM signal with 256 subcarriers and employing QPSK.

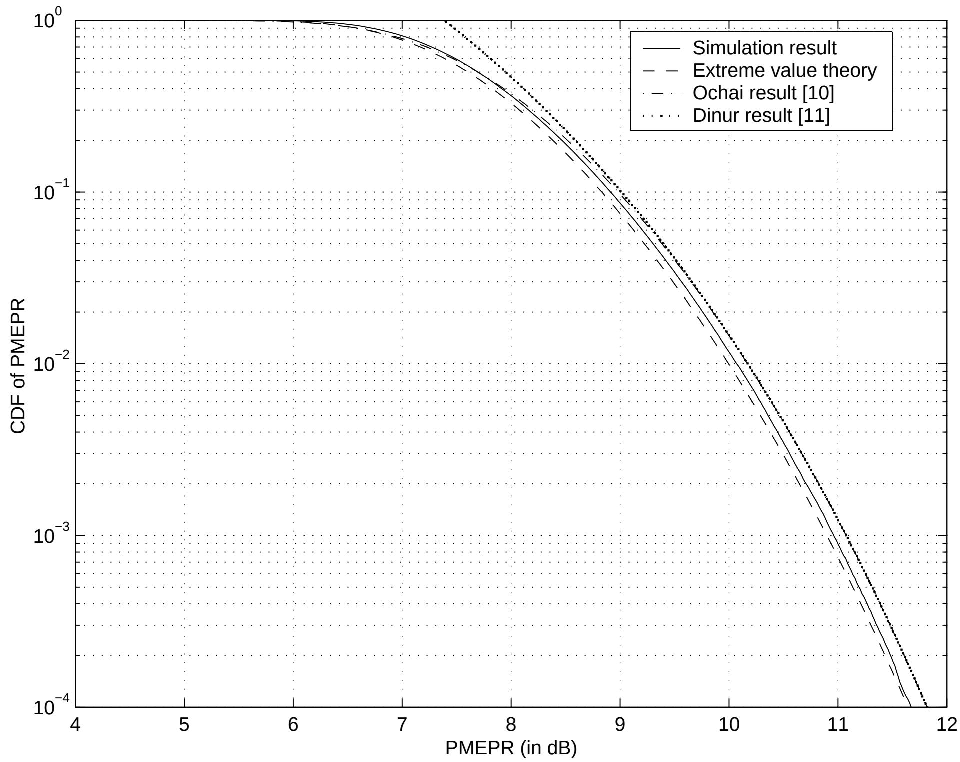

Figure 3: Complementary cumulative distribution function of the peak-to-mean envelope power ratio (PMEPR) of an uncoded OFDM signal with 100 subcarriers and employing QPSK: simulation, the proposed expression (33), and that of (36) and (37), with equal power distributions across subcarriers. Note the close agreement of the proposed expression with the simulated PMEPR for a number of subcarriers as small as 100. The simulation curves are obtained by running 10^6 independent OFDM symbols.

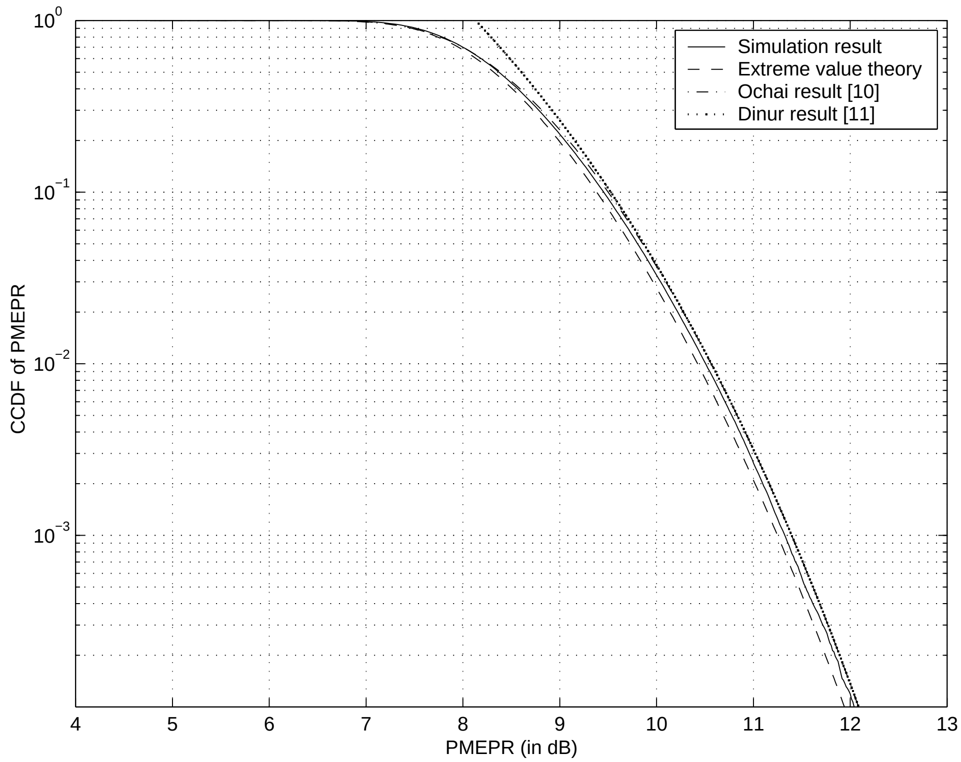

Figure 4: Complementary cumulative distribution function of the peak-to-mean envelope power ratio (PMEPR) of an uncoded OFDM signal with 256 subcarriers and employing QPSK: simulation, the proposed expression (33), and that of (36) and (37), with equal power distributions across subcarriers. Note the close agreement of the proposed expression with the simulated PMEPR. The simulation curves are obtained by running 10^6 independent OFDM symbols.

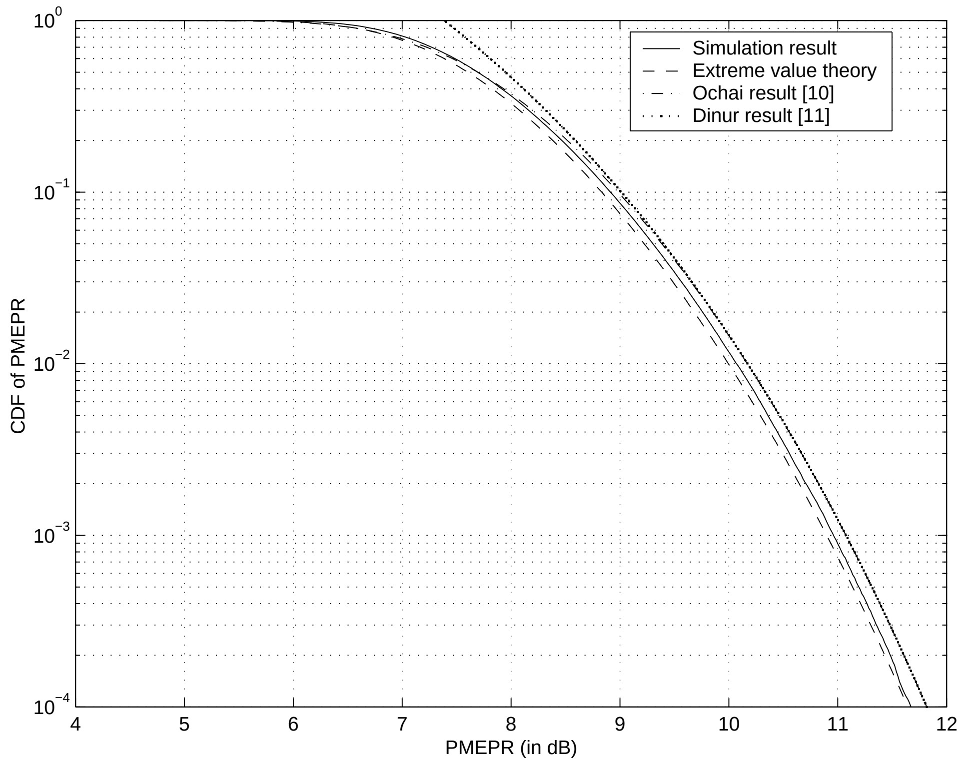

Figure 5: Complementary cumulative distribution function (CDF) of the peak-to-mean envelope power ratio (PMEPR) of a non-interleaved (2,1,6) convolutionally-coded OFDM system with 100 subcarriers employing QPSK: simulation, the proposed expression (33), and that of (36) and (37), with equal power distributions across subcarriers. Note the close agreement of the proposed expression with the simulated PMEPR; thus, as expected, the closed form of the CDF of the PMEPR of an uncoded OFDM signal still holds for this coded system. The simulation curves are obtained by running 10^6 independent OFDM symbols.

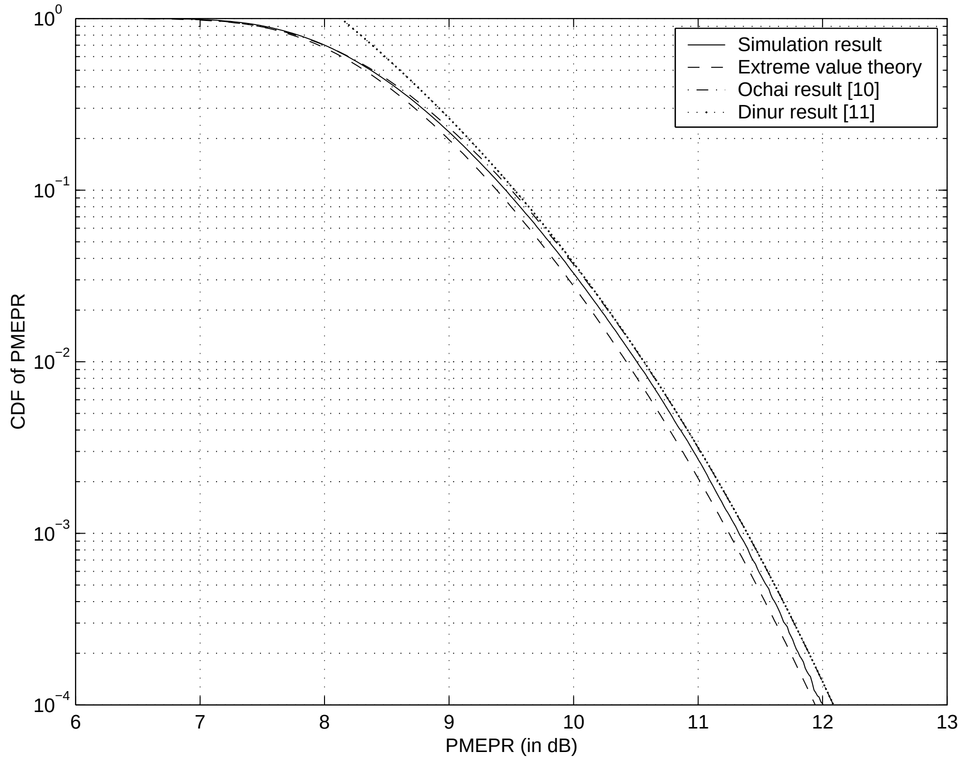

Figure 6: Complementary cumulative distribution function (CDF) of the peak-to-mean envelope power ratio (PMEPR) of a non-interleaved (2,1,6) convolutionally-coded OFDM system with 256 subcarriers employing QPSK: simulation, the proposed expression (33), and that of (36) and (37), with equal power distributions across subcarriers. Note the close agreement of the proposed expression with the simulated PMEPR; thus, as expected, the closed form of the CDF of the PMEPR of an uncoded OFDM signal still holds for this coded system. The simulation curves are obtained by running 10^6 independent OFDM symbols.

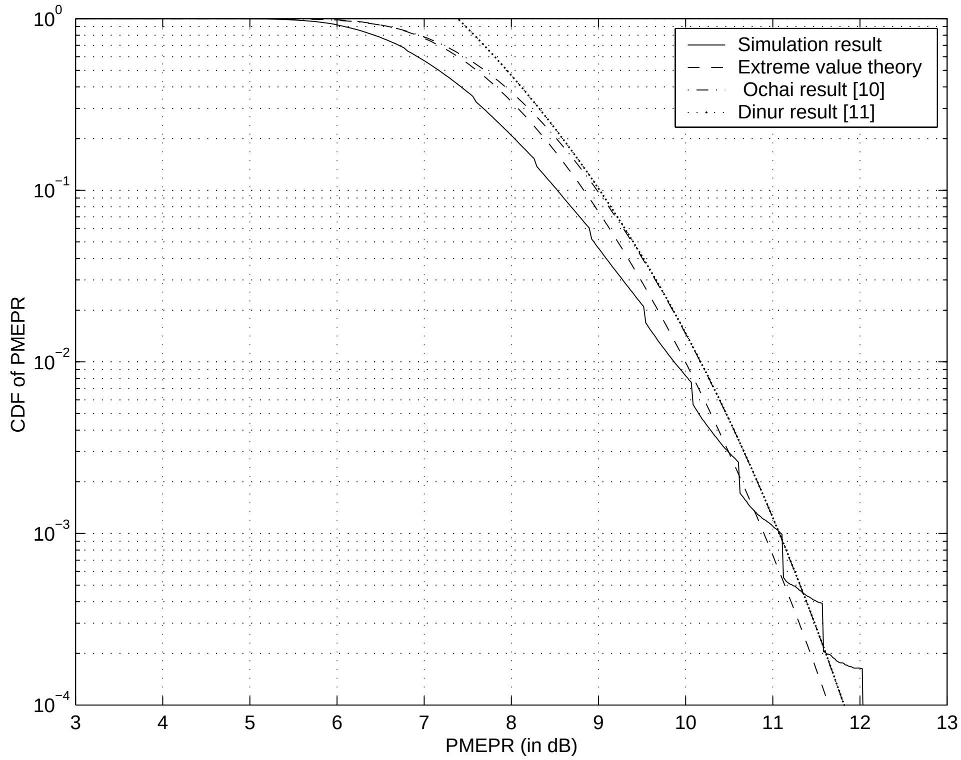

Figure 7: Complementary cumulative distribution function (CDF) of the peak-to-mean envelope power ratio (PMEPR) of a coded (8×8 block symbol-wise interleaver, $(2, 1, 6)$ convolutional code), OFDM system with 100 subcarriers employing QPSK: simulation, the proposed expression (33), and that of (36) and (37), with equal power distributions across subcarriers. The simulation curves are obtained by running 10^6 independent OFDM symbols.

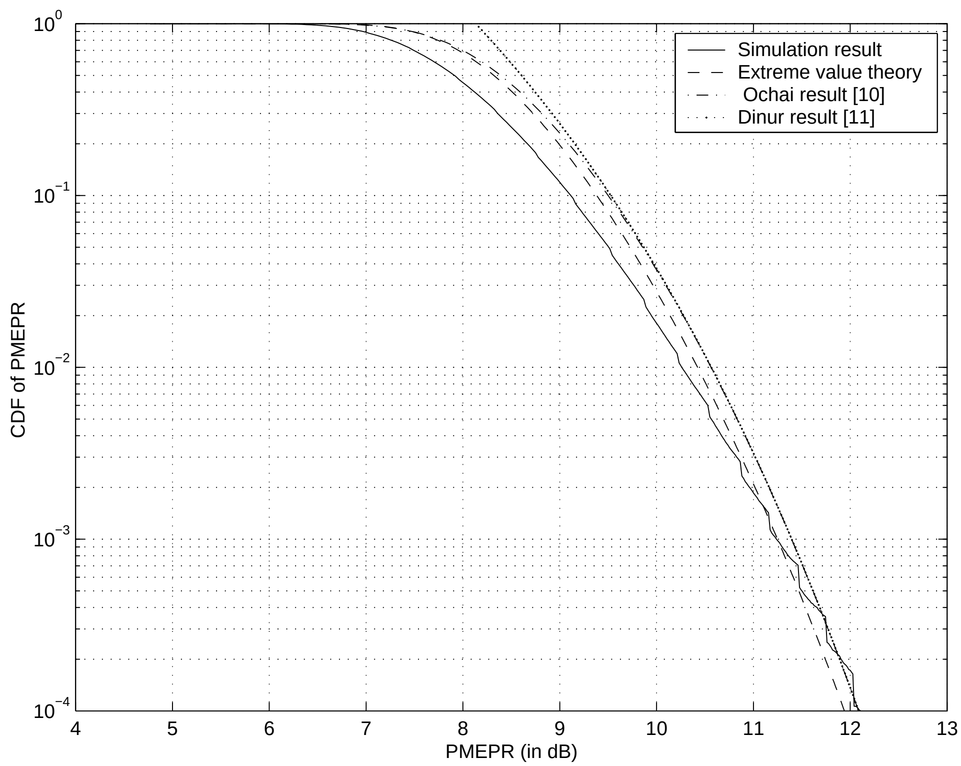

Figure 8: Complementary cumulative distribution function (CDF) of the peak-to-mean envelope power ratio (PMEPR) of a coded (8×8 block symbol-wise interleaver, $(2, 1, 6)$ convolutional code) OFDM system with 256 subcarriers employing QPSK: simulation, the proposed expression (33) and that of (36) and (37), with equal power distributions across subcarriers. The simulation curves are obtained by running 10^6 independent OFDM symbols.

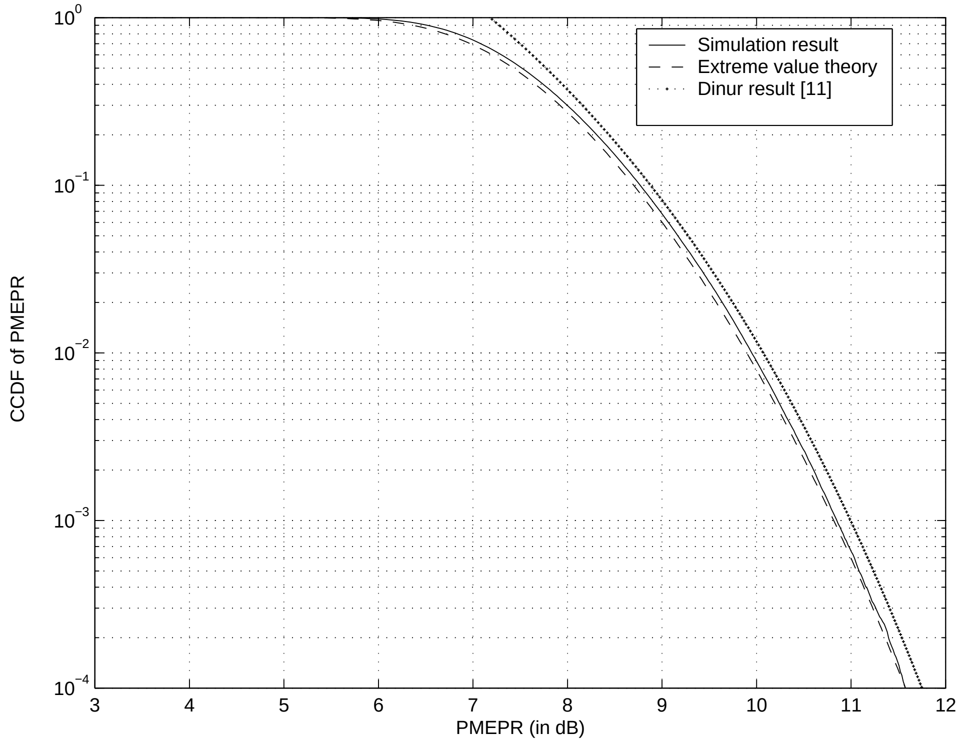

Figure 9: Complementary cumulative distribution function (CDF) of the peak-to-mean envelope power ratio (PMEPR) of a uncoded OFDM signal with unequal power distribution across subcarriers determined by (54), with 100 subcarriers employing QPSK: simulation, the proposed expression (52), and the upper bound (53), where $\tilde{\lambda}$ is given by (55). There is close agreement of the proposed expression with the simulated PMEPR for a number of subcarriers as small as 100. The simulation curves are obtained by running 10^6 independent OFDM symbols.

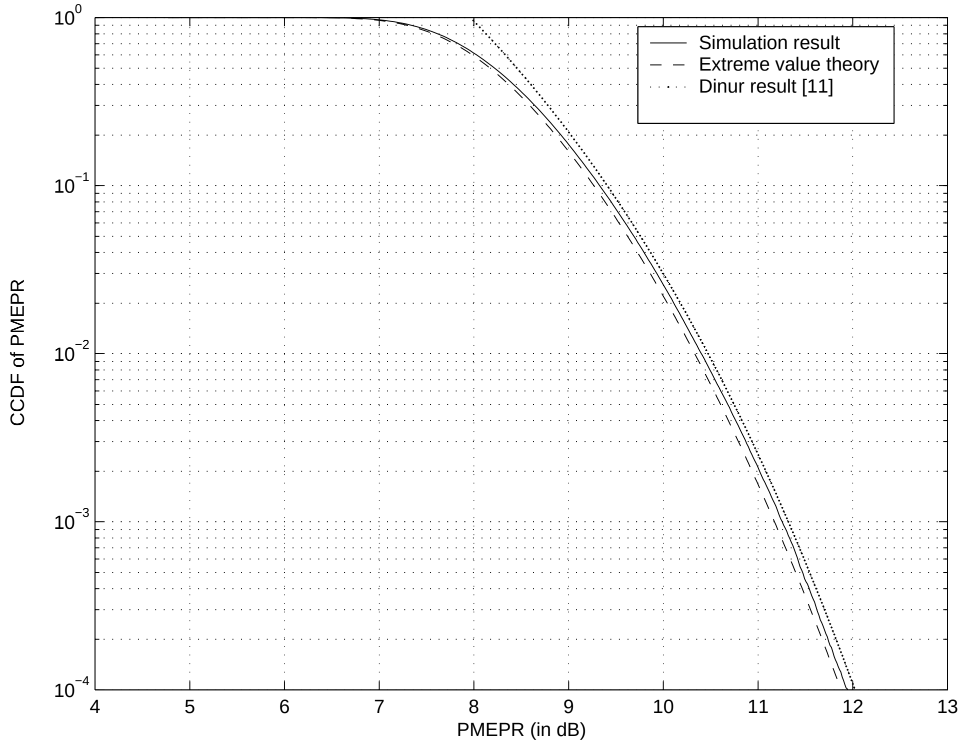

Figure 10: Complementary cumulative distribution function (CDF) of the peak-to-mean envelope power ratio (PMEPR) of a uncoded, OFDM signal with unequal power distribution across subcarriers determined by (54), with 256 subcarriers employing QPSK: simulation, the proposed expression (52), and and the upper bound (53), where $\tilde{\lambda}$ is given by (55). Note the close agreement of the proposed expression with the simulated PMEPR. The simulation curves are obtained by running 10^6 independent OFDM symbols.