On the Minimax Robustness of the Uniform Transmission Power Strategy in MIMO Systems

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***Abstract*— In this letter, it is shown that the uniform power allocation across transmit antennas is optimal in the sense that this strategy will maximize the minimum *average* mutual information of a multiple-input-multiple-output (MIMO) system across the class of any arbitrary correlated fading channels, with constraints on the the total fixed transmit power (P_Q), total power of the fades at the transmitter side (P_T), and total power of the fades at the receiver side (P_R), if the channel state information (CSI) is perfectly known at the receiver side only.**

I. Introduction

Employing multiple antennas at both the transmitter and receiver of a communication system operating over a narrowband wireless communications channel can significantly increase the Shannon capacity in the scenario with independent fading [1] [2], or correlated fading [3] across different antenna pairs. Consider a MIMO system with n_T transmit antennas and n_R receive antennas, and let $H_{i,j}$ be the fading coefficient between the jth transmit antenna and the ith receive antenna. Assume the receiver has the perfect channel state information (CSI), while the transmitter does not have CSI. If $H_{i,j}$ and $H_{k,l}$, for any disparate pairs i, j and k, l, are independent complex Gaussian random variables, it is shown in [2] that the optimal strategy to maximize the average mutual information of such a MIMO system is to transmit statistically independent identically distributed complex Gaussian codewords across n_T antennas with equal power P_Q/n_T, where P_Q is the total transmit power. If $H_{i,j}$ and $H_{k,l}$ are correlated, and in addition to lacking CSI, the transmitter is also ignorant of values of the correlations, it is assumed without rigorous justification in [3]-[4] that in order to maximize the average mutual information of a MIMO system under correlated fades, the uniform power distribution across transmit antennas will "naturally" be employed [3]. In [5], assuming that channel can play the role of a malicious nature by altering fading values $H_{i,j}$ to perform an inverse water-filling, it has been shown that the uniform power allocation strategy can maximize the minimum average mutual information of a MIMO system. However, under the constraints only on the variances of the fading values, when the channel can only set correlation properties of the fades, the optimality of the uniform power allocation strategy is still an open problem. In this paper, by following the line of the work [5], this strategy will be demonstrated to be minimax robust [6], which maximizes the minimum *average* mutual information of a MIMO system with arbitrary correlated fades under certain power constraints.

II. System Model

Throughout the paper, the following notations will be used: I_N for the $N \times N$ identity matrix, $A^\dagger$ for transpose conjugate of the matrix A, A^* for conjugate of the matrix A, $\det(A)$ for determinant of the square matrix A, $A^{'}$ for transpose of the matrix A, and $\underline{X}$ for column vector.

The discrete-time equivalent system model is given by: $\underline{Y} = H\underline{X} + \underline{Z}$, where $\underline{X}$ is an $n_T \times 1$ column vector whose jth component represents the signal transmitted by the jth antenna. Similarly, the received signal and received noise are represented by $n_R \times 1$ complex column vectors, $\underline{Y}$ and $\underline{Z}$, respectively. The noise vector $\underline{Z}$ is an additive white Gaussian random vector, whose entries $\{Z_i, i = 1, \ldots, n_R\}$ are i.i.d circularly symmetric complex Gaussian random variables with mean zero and unit variance, thus $Z_i \sim \tilde{N}(0,1)$.

It is assumed here that the total average power transmitted across the n_T transmit antennas is fixed, i.e. $E\left[\sum_{k=1}^{N} |X_k|^2\right] = P_Q$. Entries of the channel fading matrix H are assumed to be circularly symmetric complex Gaussian random variables with zero mean, and thus a Rayleigh fading channel is being assumed. Constraints on variances of $H_{i,j}$ will be described below. Per above, it is assumed that the *transmitter* has neither knowledge of the entries of H nor knowledge of the correlation statistics of the entries, but that the *receiver* has perfect knowledge of $H_{i,j}$. Hence, as in [2], if the input vector $\underline{X}$ is a proper complex Gaussian random vector, whose covariance matrix is $E[\underline{X} \cdot \underline{X}^\dagger] = Q$, the mutual information $\Phi(Q)$ of this MEA system (conditioned on H) is $\Phi(Q) = \log_2 \det\left(I_{n_R} + H \cdot Q \cdot H^\dagger\right)$ bps/Hz.

It is assumed that the covariance matrix of the random variables $H_{i,j}$ has the following general covariance structure, as described in [3]: $E[H_{i,k}H^*_{j,l}] = \Psi^T_{k,l}\Psi^R_{i,j}$, where Ψ^T and Ψ^R are $n_T \times n_T$ and $n_R \times n_R$ covariance matrices generated by the transmit and receive antennas, respectively. As in [3], the matrix H can be factored in the form $H \stackrel{\mathcal{D}}{=} \left(\Psi^R\right)^{\frac{1}{2}} W \left(\Psi^T\right)^{'\frac{1}{2}}$, where the entries of W are i.i.d with $\tilde{N}(0,1)$, and $x \stackrel{\mathcal{D}}{=} y$ means random variables x and y have the same distribution.

Our goal here is to find the minimax robust Q_0 [6], under the constraint of $\text{Tr}(Q) = P_Q$, $\text{Tr}(\Psi^T) = P_T$ and $\text{Tr}(\Psi^R) = P_R$, where $\text{Tr}(A)$ is the trace of matrix A [7], to maximize the infimum average mutual information $E\left[\Phi\left(Q\right)\right]$, i.e.,

$$Q_0 = \arg\max_{Q \in S_Q} \inf_{\Psi^R \in S_R, \Psi^T \in S_T} E\left[\Phi\left(Q\right)\right] \quad (1)$$

where $S_Q = \{Q : \text{Tr}(Q) = P_Q\}$, $S_T = \left\{\Psi^T : \text{Tr}(\Psi^T) = P_T\right\}$ and $S_R = \left\{\Psi^R : \text{Tr}(\Psi^R) = P_R\right\}$, are the sets of non-negative definite matrices with the constraint of fixed trace, which are all convex sets. The expectation $E[\cdot]$ is over the statistical distribution of the fading entries of H under the given correlation matrices Ψ^T, Ψ^R and Q. The trace constraints for channel correlation matrices Ψ^T and Ψ^R imply that the total power of the fades caused by scatterings around transmit and receive antennas are fixed as P_T and P_R, respectively.

III. Theorem and Proof

Theorem 1: The minimax robust solution to (1) is $Q_0 = P_Q I_{n_T}/n_T$, and

$$\max_{Q\in S_Q} \inf_{\Psi^R\in S_R, \Psi^T\in S_T} E\left[\Phi\left(Q\right)\right] = E\left[\log_2\left(1+P_T P_R \frac{P_Q}{n_T} y\right)\right], \tag{2}$$

where y is an exponentially distributed random variable with unit mean.

Proof:

By singular value decomposition (SVD) [7], it can be shown that $Q = U_Q D_Q U_Q^\dagger$, $\Psi^T = U_T D_T U_T^\dagger$, and $\Psi^R = U_R D_R U_R^\dagger$, where U_Q, U_T and U_Q are unitary matrices, and D_Q, D_T and D_R are diagonal matrices whose diagonal entries $\left\{\lambda_k^Q\right\}$, $\left\{\lambda_k^T\right\}$ and $\left\{\lambda_k^R\right\}$ are the eigenvalues of Q, Ψ^T and Ψ^R, respectively, in a decreasing order.

By substituting $H \overset{\mathcal{D}}{=} \left(\Psi^R\right)^{\frac{1}{2}} W \left(\Psi^T\right)^{'\frac{1}{2}}$ into $\Phi(Q)$, and recognizing that for any unitary matrices U and V, $UWV^\dagger$ has the same statistical distribution as W [2], where entries of W are independently distributed as $\tilde{N}(0,1)$, it can be shown that

$$\begin{aligned} E\left[\Phi(Q)\right] &= E\left[\log_2 \det\left(I_{n_R} + D_R^{\frac{1}{2}} W D_T^{\frac{1}{2}} V D_Q V^\dagger D_T^{\frac{1}{2}} W^\dagger D_R^{\frac{1}{2}}\right)\right] \quad (3)\\ &= E\left[\log_2 \det\left(I_{n_R} + D_R^{\frac{1}{2}} W B_2 W^\dagger D_R^{\frac{1}{2}}\right)\right]\\ &= E\left[\log_2 \det\left(I_{n_R} + D_R^{\frac{1}{2}} W D_B W^\dagger D_R^{\frac{1}{2}}\right)\right], \quad (4)\end{aligned}$$

where $V = U_T^{'} U_Q$ is a unitary matrix, $B_1 = D_T^{\frac{1}{2}} V D_Q^{\frac{1}{2}}$, and $B_2 = B_1 B_1^\dagger$. D_B is the diagonal matrix whose diagonal entries are eigenvalues of B_2, λ_k^B, in a decreasing order, with $\sum_{k=1}^{n_T} \lambda_k^B = \mathrm{Tr}(B_2) = P_B$.

Letting $\alpha_k = \lambda_k^B / P_B$,

$$\begin{aligned} E\left[\Phi(Q)\right] &\overset{(a)}{\geq} \sum_{k=1}^{n_T} \alpha_k E\left[\log_2\left(1 + P_B \sum_{j=1}^{n_R} \lambda_j^R \left|w_{j,k}\right|^2\right)\right]\\ &\overset{(b)}{=} E\left[\log_2\left(1 + P_B \sum_{j=1}^{n_R} \lambda_j^R \left|w_{j,1}\right|^2\right)\right]\\ &\overset{(c)}{\geq} E\left[\log_2\left(1 + P_B P_R \left|w_{1,1}\right|^2\right)\right]\\ &\overset{(d)}{\geq} E\left[\log_2\left(1 + P_T P_R \lambda_{min}^Q \left|w_{1,1}\right|^2\right)\right], \quad (5)\end{aligned}$$

where the inequality (a) is due to the concavity of the function $\log\det A$ on the convex set of non-negative Hermitian matrices [7, pp. 466], as well as the representation of $D_B = P_B \sum_{k=1}^{n_T} \alpha_k \underline{e}_k \underline{e}_k^\dagger$, where $\underline{e}_k$ is a column vector with 1 as its kth component and 0's elsewhere. Equality (b) is because the sum term is identically distributed. Inequality (c) is due to the concavity of the function $\log x$ over the region $x > 0$, and $\sum_{j=1}^{n_R} \lambda_j^R = P_R$. Inequality (d) is because $P_B = \sum_{i=1}^{n_T} \sum_{j=1}^{n_T} \lambda_i^Q \lambda_j^T \left|v_{i,j}\right|^2 \geq P_T \lambda_{min}^Q$, where the last step is due to $\lambda_i^Q \geq \lambda_{min}^Q, \forall i$, and $\sum_{i=1}^{n_T} \left|v_{i,j}\right|^2 = 1$, since V is a unitary matrix. Therefore, for any covariance matrices Q, Ψ^T and Ψ^R, $E\left[\Phi(Q)\right]$ can be lower bounded as that in (5).

For any given transmission strategy Q, the lower bound in (5) can be achieved by setting the eigenvalue $\lambda_1^T = P_T$, $\lambda_k^T = 0, k = 2, \ldots, n_T$, and $\lambda_1^R = P_R$, $\lambda_j^R = 0, j = 2, \ldots, n_R$, with $U_T = U_Q^*$ and U_R any arbitrary unitary matrix. Thus, the worst case of channel puts all of its energy at the transmission side in the direction of the weakest eigenvector of Q, while the energy of the channel at the receiver side is concentrated in any eigen-direction. Thus,

$$\inf_{\Psi^R\in S_R, \Psi^T\in S_T} E\left[\Phi(Q)\right] = E\left[\log_2\left(1 + P_T \lambda_{min}^Q P_R \left|w_{1,1}\right|^2\right)\right]. \tag{6}$$

Our goal is in (1) to find the minimax robust Q, and since $\sum_{k=1}^{n_T} \lambda_k^Q = P_Q$, $\lambda_{min}^Q \leq P_Q/n_T$ (otherwise, the condition of the fixed total transmission power will be violated), the upper bound of (6) can be achieved by setting $Q = P_Q I_{n_T}/n_T$, which is exactly the transmission strategy of the uniform power distribution across transmit antennas. Hence, (2) is proven to be true.

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It can be observed that if $Q_0 = P_Q I_{n_T}/n_T$, an arbitrary choice of the eigenvectors of Ψ_0^T and Ψ_0^R will minimize the mutual information as long as only one eigenvalue of each is non-zero. This set of $\left(Q_0, \Psi_0^T, \Psi_0^R\right)$ is not a saddle point [6], since given such $\left(\Psi_0^T, \Psi_0^R\right)$, the uniformly distributed power allocation does not achieve the maximum of $E\left[\Phi(Q)\right]$.

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