Convergence of the Complex Envelope of Bandlimited OFDM Signals

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***Abstract*—Orthogonal frequency division multiplexing (OFDM) systems have been used extensively in wireless communications in recent years; thus, there is significant interest in analyzing the properties of the transmitted signal in such systems. In particular, a large amount of work has focused on analyzing the variation of the complex envelope of the transmitted signal and on designing methods to minimize this variation. In this paper, it is established that the complex envelope of a bandlimited uncoded OFDM signal converges weakly to a Gaussian random process as the number of subcarriers goes to infinity. This shows that the properties of the OFDM signal will asymptotically approach those of a Gaussian random process over any finite time interval. The convergence proof is then extended to two important cases, namely, coded OFDM systems and systems with an unequal power allocation across subcarriers.**

***Index Terms*—Convergence, extreme value theory, Gaussian random process, orthogonal frequency division multiplexing (OFDM), peak-to-mean envelope power ratio.**

I. Introduction

A MAJOR goal of modern communication systems is to allow high-speed communication, regardless of the location or mobility of the system users. However, this goal is difficult to achieve due to the multipath fading that affects wireless communication signals. One alternative for achieving high-speed wireless communication in the presence of multipath fading is to employ a multicarrier system, generally implemented as an orthogonal frequency division multiplexing (OFDM) system [1], in conjunction with error control coding. Such coded OFDM systems have been employed or are being considered for a number of applications, including digital audio broadcast and digital video broadcast in Europe [2], wireless local area networks [3], broadband fixed wireless access [4], and cellular data [5].

One of the challenges to be overcome when employing an OFDM system in low-power peer-to-peer wireless communication systems is that the complex envelope of the transmitted OFDM signal can demonstrate significant variation; in other words, its peak-to-mean envelope power ratio (PMEPR) can be much larger than that of an analogous single-carrier system [1], [6]. This large PMEPR can require significant backoff of the average operating power of the power amplifier in the transmitter if it is to be operated in the linear region, which results in significant power inefficiency [8], [9]. Thus, there has been a large body of work in the analysis of the variation of the complex envelope of the OFDM signal and in methods to reduce this variation [6], [7] (and references therein). Here, the focus is on the analysis problem.

In the literature, there are in general two approaches to analyzing the PMEPR distribution for OFDM signals. The first approach is to seek bounds on the PMEPR distribution without requiring the statistical characterization of the baseband OFDM signals [11], [13], [15], [18]. These results have motivated PMEPR reduction techniques that modify the constellation on each subcarrier [14], [16]–[18].

The second approach taken by many recent papers that have analyzed the PMPER of the transmitted OFDM signal [19]–[22], [24] or its effects [23] often assume that the complex envelope of the transmitted OFDM signal converges in some sense to a Gaussian random process as the number of subcarriers becomes large. For example, in the work of [21] and [22], the assumption of such convergence is used when studying the PMEPR distribution to justify the use of Rice's level-crossing results for the envelope of a complex Gaussian random process [26]. However, there exists no rigorous investigation into the limiting form of the complex envelope of the transmitted OFDM signal, despite the theoretical and practical importance of such an endeavor. Thus, in this paper, a formal proof that an uncoded bandlimited OFDM signal converges weakly to a Gaussian random process is rigorously established for the first time in literature.

Using the assumption that the envelope of the transmitted OFDM signal is asymptotically Gaussian, previous work [21], [22] has relied largely on the work of Rice [26] to develop results for the PMEPR distribution of the OFDM signal. The work of [21] employs [26] in conjunction with a number of approximations and a parameter obtained through simulation to arrive at a final expression for the PMEPR. The work of [22] finds lower and upper bounds for the PMEPR distribution through the use of extensive manipulation on top of the results found in [26]. At first glance, it might appear that the work here can be used to make these previous results rigorous. However, since

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the proof of weak convergence consists of demonstrating convergence over any finite interval and the symbol period for a bandlimited OFDM signal approaches infinity as the number of subcarriers goes to infinity, our convergence proof cannot be applied to complete the rigorous justification for the work in [21], [22]. And, unfortunately, the extension of our proof to an infinite interval has proved elusive.

After showing the complex envelope of an uncoded OFDM system converges in distribution to a complex Gaussian random process, we then turn our attention to coded systems. Because an OFDM system effectively forms a large number of frequency-nonselective subchannels, it is well-known that uncoded OFDM systems will perform poorly on wireless communication channels due to a lack of diversity. Thus, wireless OFDM systems almost always employ some form of error control coding. This introduces statistical dependence among the symbols placed on the subcarriers, and thus the technique in the proof for the uncoded case cannot be applied directly. However, by invoking results from modern central limit theory for sums of dependent random variables, it is possible to prove that the complex envelope of coded baseband OFDM signals also converges to a Gaussian random process over any finite time interval, which generalizes the results for the uncoded case to many block coded and convolutionally coded systems.

Another extension made is to the convergence study of the complex envelope of an OFDM system employing an unequal power distribution over multiple carriers. In [24], an analytical expression of the PMEPR distribution was obtained by applying the extreme value theory [35], [36] for χ^2 random process to OFDM systems with an unequal power distribution across subcarriers. To justify the application of extreme value theory, [24] directly cited the proof of convergence in [12], which only applies to the case when power is divided evenly across subcarriers. Thus, a proof is provided here which shows the conditions under which OFDM systems with an unequal power allocation will exhibit a similar convergence result as those with an equal power allocation. In addition, it should be noted that the similar results to those in [24] on the PMEPR distribution in the unequal power distribution case were obtained independently in [25] by also exploiting extreme value theory, which, however, bears a more general formulation than that of [24].

This paper is organized as follows. Section II provides the proofs of the main results of the paper. First, the appropriate convergence of the real part of the baseband OFDM signal is established. This proof then provides a foundation to the proof of convergence of the complex envelope of baseband OFDM signal to a complex Gaussian random process as the number of subcarriers grows to infinity, as well as the extension to coded OFDM systems in Section III. In Section IV, the extension is made to uncoded OFDM systems with an unequal power distribution across subcarriers. Finally, conclusions are drawn in Section V.

II. Convergence of the Envelope of Uncoded Baseband OFDM Signals

The convergence of the complex envelope of an uncoded OFDM systems serves as the basis for the remainder of the results in the paper, including the important coded OFDM case. The proof for the convergence of the real part of the baseband OFDM signal given in (1) is considered first, and then its straightforward extension to the complex envelope is performed.

Theorem 1: Consider the real part of the complex envelope of the transmitted signal of an OFDM system with N subcarriers:

$$x_N(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(A_k^R \cos\left(2\pi \frac{k}{NT_c} t\right) - A_k^I \sin\left(2\pi \frac{k}{NT_c} t\right) \right) \quad (1)$$

for $t \in T$, where $T \subseteq R$ is any closed and finite interval, the complex sequence $\{A_k = A_k^R + jA_k^I, k = 0, \cdots, N-1\}$ is an independent and identically distributed (i.i.d.) sequence of complex random variables, and the real part (A_k^R) and imaginary part (A_k^I) are bounded ($|A_k^R| \le \bar{A}$ and $|A_k^I| \le \bar{A}$), with $E[A_k^R] = E[A_k^I] = 0$, $E[A_k^R A_k^I] = 0$, and $E[(A_k^R)^2] = E[(A_k^I)^2] = \sigma^2$. Then

$$\{x_N(t), t \in T\} \xrightarrow{\mathcal{D}} \{x(t), t \in T\}$$

where $x(t)$ is a zero-mean stationary random process defined over T, with autocorrelation function

$$E[x(t_i)x(t_j)] = \sigma^2 \text{sinc}\left(\frac{2(t_j - t_i)}{T_c}\right), \quad \forall t_i, \quad t_j \in T.$$

Before providing the proof of Theorem 1, we first define a number of key terms to set the context. The implied weak convergence of the underlying measures in Theorem 1 is in the metric space (C, ρ), where C is the space of continuous functions on the interval T, and $\rho(x,y) = \sup_{t \in T} |x(t) - y(t)|$. In this paper, all probabilities are defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is the outcome space, $\mathcal{F}$ is the σ-field on Ω, and $\mathcal{P}$ is the probability measure defined on $\mathcal{F}$. Measurability of the appropriate quantities is then easily established [25].

The notion of weak convergence on C is now defined formally. A sequence $\{x_N\}$ of random functions of C *converges in distribution* to the random function x, denoted by

$$x_N \xrightarrow{\mathcal{D}} x$$

or

$$\{x_N(t), t \in T\} \xrightarrow{\mathcal{D}} \{x(t), t \in T\} \quad (2)$$

if the following is true [27]:

$$\lim_{N \to \infty} \mathcal{P}\{x_N \in A\} = \mathcal{P}\{x \in A\} \text{ for all } x\text{−continuity sets } A \quad (3)$$

where a set A in $\mathcal{C}$ is an x-continuity set if $\mathcal{P}\{x \in \partial A\} = 0$, where ∂A is the boundary of A [27], consisting of those points that are limits of sequences of points in A and are also limits of sequences of points outside A.

The finite-dimensional distributions corresponding to $x_N(t)$ in (1) are crucial to the proof and are defined next. For points $t_1, \ldots, t_k$ in $[0,1]$, let $\pi_{t_1,\ldots,t_k}$ be the continuous mapping that carries the point x of C to the point

$(x(t_1),\ldots,x(t_k))$ in the k-dimensional Euclidean space R^k. The finite-dimensional sets are now defined as sets of the form $\pi^{-1}_{t_1,\ldots,t_k}H$ with $H \in \mathcal{R}^k$, where $\mathcal{R}^k$ is the class of Borel sets in R^k. Then each random function x_N of C induces on $(R^k,\mathcal{R}^k)$ a unique probability measure $P_N\pi^{-1}_{t_1,\ldots,t_k}$, defined by $P_N\pi^{-1}_{t_1,\ldots,t_k}(A)=\mathcal{P}\{\omega : x_N(\omega)\in \pi^{-1}_{t_1,\ldots,t_k}(A)\}$, for $A\in\mathcal{R}^k$. Then, $P_N\pi^{-1}_{t_1,\ldots,t_k}$ is called the *finite-dimensional distribution* corresponding to x_N [27, p. 19, 30].

Now, we turn to the proof of Theorem 1. For lemmas in Section II, proofs that are omitted can be found in [25]. To prove convergence in distribution of a sequence of random functions $\{x_N\}$ to some $\{x(t), t\in T\}$ in C, it is sufficient to show that the sequence $\{x_N\}$ is tight and that each of the finite-dimensional distributions $P_N\pi^{-1}_{t_1,\ldots,t_k}$ of x_N converges weakly to the measure $\mu_{t_1,\ldots,t_k}$ induced by x on $(R^k,\mathcal{R}^k)$, for each $(t_1,\ldots,t_k)$ [27, p. 47]. The sequence $\{x_N\}$ of random functions of C is tight if and only if it satisfies the following two conditions [27, p. 55].

Condition 1: For each positive η, there exists an a such that

$$\mathcal{P}\{|x_N(0)|>a\}\le\eta,\quad N\ge 1. \tag{4}$$

Condition 2: For each positive ε and η, there exists a δ, with $0<\delta<1$, and an integer N_0 such that

$$\mathcal{P}\left\{\sup_{\substack{|s-t|<\delta\\ s,t\in[0,1]}}|x_N(s)-x_N(t)|\ge\varepsilon\right\}\le\eta,\quad N\ge N_0. \tag{5}$$

Lemma 1: Let $\{x_N\}$ be defined as in (1). Then, for each positive η, there exists an a such that

$$\mathcal{P}\{|x_N(0)|>a\}\le\eta,\quad N\ge 1. \tag{6}$$

Establishing Condition 2 is the crux of the entire proof. First, a preliminary lemma is presented and then Condition 2 is established. Note that only Lemma 2 restricts the class of signals to which the convergence results apply, and the OFDM signals of interest are shown to be part of this class.

Lemma 2:

$$E|x_N(t+h)-x_N(t)|^2\le\beta h^2,\quad \beta=\frac{4}{3}\left(\frac{\pi\sigma}{T_c}\right)^2$$

$\forall N\ge 1, h\in\mathcal{R}, h\ne 0.$

Proof: Note that the derivatives of the functions $x_N(t)$ have variances that are uniformly bounded (in N and t). This can be used to show that there is a finite constant β such that for all N and t, the variance of $x_N(t+h)-x_N(t)$ is bounded by βh^2. (A full proof by alternate means can be found in [25].) ∎

Lemma 3: Let $\{x_N\}$ be defined as in (1). Then, for each positive ε and η, there exists a δ, with $0<\delta<1$, and an integer N_0 such that

$$\mathcal{P}\left\{\sup_{\substack{|s-t|<\delta\\ s,t\in[0,1]}}|x_N(s)-x_N(t)|\ge\varepsilon\right\}\le\eta,\quad N\ge N_0. \tag{7}$$

The proof of Lemma 3, which is the crux of the entire result, can be found in Appendix A. Hence, for the sequence $\{x_N\}$ in (1) of random functions of C, both Condition 1 and Condition 2 are satisfied, and thus $\{x_N\}$ is tight [27, p. 55]. Given Lemma 3, establishing Theorem 1 only requires a demonstration that the finite-dimensional distribution $P_N\pi^{-1}_{t_1,\ldots,t_k}$ of x_N, which is determined by the random vector $(x_N(t_1),\ldots,x_N(t_k))$, converges weakly to the measure $\mu_{t_1,\ldots,t_k}$ induced by x on $(R^k,\mathcal{R}^k)$, for each $(t_1,\ldots,t_k)$ [27, p. 54]. First, a technical lemma is presented, and then the Cramér-Wold Theorem [27, p. 49] is employed in a straightforward manner to establish the result.

Lemma 4:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\cos\left(\frac{2\pi k}{NT_c}\tau\right)=\mathrm{sinc}\left(\frac{2\tau}{T_c}\right),$$

where $\mathrm{sinc}(x)=\frac{\sin(\pi x)}{\pi x}$.

Lemma 5: Let $x_N(t)$ be defined as in (1), and pick any integer $L\ge 1$ and collection of sample times $\{t_1,t_2,\ldots,t_L\}$. Then

$$\underline{\Gamma}_N=(x_N(t_1),x_N(t_2),\ldots,x_N(t_L))^T\xrightarrow{\mathcal{D}}\underline{\Gamma}$$

where $\underline{\Gamma}=(\Gamma_1,\Gamma_2,\ldots,\Gamma_L)^T$ is an L-dimensional vector with jointly Gaussian components, mean vector $\underline{0}$, and covariance matrix Σ, where the $(i,j)^{\mathrm{th}}$ element of Σ is given by

$$\Sigma_{i,j}=E[\Gamma_i\Gamma_j]=\sigma^2\mathrm{sinc}\left(\frac{2(t_i-t_j)}{T_c}\right). \tag{8}$$

The proof of Lemma 5 can be found in Appendix A. Thus, Theorem 1 is established, which then provides a foundation for the straightforward proof of convergence of the complex envelope of baseband OFDM signal to a complex Gaussian random process as the number of carriers grows to infinity, as given in Theorem 2 and Theorem 3, respectively. The reader interested in the detailed proofs of Theorem 2 and Theorem 3 is referred to [25].

Theorem 2: Consider the complex envelope of the transmitted signal in an OFDM system with N subcarriers

$$s_N(t)=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}A_ke^{j\omega_kt} \tag{9}$$

where $\omega_k=\frac{2\pi k}{NT_c}$, $T_c\in(0,\infty)$, and $\{A_k, k=0,\ldots,N-1\}$ is an independent and identically distributed (i.i.d.) sequence of complex random variables, where the real part (A_k^R) and imaginary part (A_k^I) are bounded ($|A_k^R|\le\bar{A}$ and $|A_k^I|\le\bar{A}$), with $E[A_k^R]=E[A_k^I]=0$, $E[A_k^RA_k^I]=0$, and $E[(A_k^R)^2]=E[(A_k^I)^2]=\sigma^2$. Then, as $N\to\infty$, for any closed and finite interval $T\subseteq R$,

$$\{s_N(t), t\in T\}\xrightarrow{\mathcal{D}}\{s(t), t\in T\}$$

where $\xrightarrow{\mathcal{D}}$ implies *convergence in distribution* and $s(t)$ is a zero-mean stationary Gaussian random process defined over the interval T, with real part $x(t)$ and imaginary part $y(t)$ such that

$$E[x(t_i)x(t_j)] = E[y(t_i)y(t_j)] = \sigma^2 \mathrm{sinc}\left(\frac{2(t_j - t_i)}{T_c}\right)$$

and

$$E[x(t_i)y(t_j)] = \sigma^2 \frac{\sin^2\left(\frac{(t_j - t_i)\pi}{T_c}\right)}{\frac{\pi(t_j - t_i)}{T_c}}$$

for all t_i and t_j in T.

The implied weak convergence of the underlying measures is in the metric space $(C \times C, \bar{\rho})$, where C is the space of continuous functions on the interval T, and

$$\bar{\rho}((x_1, x_2), (y_1, y_2)) = \max\{\rho(x_1, y_1), \rho(x_2, y_2)\} \tag{10}$$

where x_1, x_2, y_1, y_2 are in C_T and $\rho(x, y) = \sup_{t \in T} |x(t) - y(t)|$. Theorem 2 can then be used to prove the following analogous result for the complex baseband representation of the transmitted signal in multicarrier systems that are symmetric about the carrier [21].

Theorem 3: Consider the complex signal

$$V_N(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k e^{j\omega_k t}$$

where $\omega_k = \frac{2\pi}{NT_c}(k - \frac{N-1}{2})$ and $\{A_k,\, k = 0, \ldots, N-1\}$ is as defined above. Then, as $N \to \infty$, for any closed and finite interval $T \subseteq R$,

$$\{V_N(t),\, t \in T\} \xrightarrow{\mathcal{D}} \{V(t),\, t \in T\}$$

where $V(t)$ is a zero-mean stationary complex Gaussian random process defined over the interval T with independent real and imaginary parts, each with autocorrelation function

$$\sigma^2 \mathrm{sinc}\left(\frac{(t_j - t_i)}{T_c}\right), \quad \forall t_i, \quad t_j \in T.$$

The implied weak convergence of the underlying measures is on the metric space $(C \times C, \bar{\rho})$, as defined above in Theorem 2.

Remarks: Note that we only require that A_K^R and A_k^I are uncorrelated, and not the stronger condition of independence between the real and imaginary parts of each symbol [13]. This assumption holds not only for quadrature amplitude modulation (QAM) constellations, but also for phase-shift keying (PSK) constellations.

III. The Convergence of the Complex Envelope of Coded OFDM Systems

One of the guiding tenets of wireless OFDM systems is that the bandwidth of each subcarrier should be less than the coherence bandwidth of the wireless channel, which results in no intersymbol interference (ISI) on a given subcarrier and thus obviates the need for complex equalization at the receiver. However, by definition, this makes the effective channel on each subcarrier a frequency non-selective fading channel, which implies that uncoded OFDM systems will perform very poorly. Thus, it has been widely recognized that some form of error control coding is necessary in wireless OFDM systems. However, when error control coding is applied, the assumption of independence between symbols required for the results of Theorem 2 and Theorem 3 are violated. Thus, in this section, the results of the previous sections are extended to systems employing error control coding.

It is clear from the work of other researchers that error control coding can have a significant impact on the distribution of the PMEPR of OFDM systems; in fact, a recent line of research has exploited such a fact to develop error control codes for OFDM systems that greatly reduce the PMEPR (see [9] and references therein). In this section, it is shown that, despite the dependence of the symbols at the output of the error control coder on one another, analogous results to those of Theorem 2 and Theorem 3 hold under very broad conditions. In particular, the results hold well for any system with enough "mixing" of codewords.

To establish an analog to Theorem 2, first consider the type of symbol sequence that is employed in a coded system in place of the i.i.d. symbol sequence of the uncoded OFDM system. Clearly, the sequence output from the coded modulation in a system employing some form of error control coding contains dependent symbols, for the introduction of such dependence *is* the role of the error control coder. However, most good codes for random errors do not introduce correlation into the symbol stream [30, p. 527], [31], and thus, although it certainly contains dependence, the coded symbol stream can be modeled as uncorrelated. Also, note that such a symbol stream is only locally dependent for traditional codes (i.e., codes that do not introduce the long-term dependence exemplified by, for example, turbo codes [32]). For block codes, symbols separated in index by more than a block length are independent; for convolutional codes, symbols separated in index by more than the constraint length are independent. Thus, the random process at the output of the coded modulation is a form of random process known as "m-dependent" [27], which will be important to establish the mixing results required in the proof of Theorem 4. Finally, note that most coded OFDM systems employ some form of interleaving between the coded modulator and the IFFT in order to obtain some form of diversity; thus, it is important to allow for the possibility of such, although it should be noted that it is *not* required for the results. These assumptions lead to the statement of Theorem 4, which is a generalization of Theorem 2. The proof of Theorem 4 follows from the work of Section II and [33]; for details, see Appendix A.

Theorem 4: Consider the complex envelope of the transmitted signal in a coded OFDM systems

$$s_N(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} B_{k,N} e^{j\omega_k t} \tag{11}$$

where $\omega_k = \frac{2\pi k}{NT_c}$, $T_c \in (0, \infty)$, and $\{B_{k,N},\, k = 0, \ldots, N-1\}$ is defined by $\underline{B}_N = (B_{0,N}, B_{1,N}, \ldots, B_{N-1,N})^T = P_N \underline{A}_N$, where P_N is an arbitrary $N \times N$ permutation matrix, which permutes the entries of $\underline{A}_N = (A_0, A_1, \ldots, A_{N-1})^T$, and let

$\{A_k, k = 0, 1, \ldots, N-1\}$ be drawn from a stationary sequence of identically distributed (but not necessarily independent) random variables where, for all k and l

1) $E[A_k^R] = E[A_k^I] = 0$.
2) $E[(A_k^R)^2] = E[(A_k^I)^2] = \sigma^2 < \infty$.
3) $|A_k^R| < \bar{A}$ and $|A_k^I| < \bar{A}$.
4) A_l and A_k are uncorrelated, $k \neq l$.
5) A_k^R and A_k^I are uncorrelated.
6) There exists an integer n_0 such that A_k and A_l are independent if $|k-l| \geq n_0$.

Then, as $N \to \infty$, for any closed and finite interval $T \subseteq R$

$$\{s_N(t), t \in T\} \xrightarrow{\mathcal{D}} \{s(t), t \in T\}$$

where $\xrightarrow{\mathcal{D}}$ implies *convergence in distribution* and $s(t)$ is a zero-mean stationary Gaussian random process defined over the interval T, with real part $x(t)$ and imaginary part $y(t)$ such that

$$E[x(t_i)x(t_j)] = E[y(t_i)y(t_j)] = \sigma^2 \mathrm{sinc}\left(\frac{2(t_j - t_i)}{T_c}\right)$$

and

$$E[x(t_i)y(t_j)] = \sigma^2 \frac{\sin^2\left(\frac{(t_j - t_i)\pi}{T_c}\right)}{\frac{\pi(t_j - t_i)}{T_c}}$$

for all t_i and t_j in T.

The implied weak convergence of the underlying measures is on the metric space $(C \times C, \bar{\rho})$ as defined above in Theorem 2. Thus, a convergence result analogous to that demonstrated in Section II for uncoded systems has been established for many coded systems.

IV. Extension to Systems With Unequal Power Distributions

In previous sections, it has been assumed that the power allocated on each subcarrier of the OFDM system is identical, i.e., $E|\frac{1}{\sqrt{N}}A_k^2| = \frac{2\sigma^2}{N}, k = 0, \ldots, N-1$. However, since OFDM systems are usually used in channels with nonflat frequency response, it is often desirable to allocate different amounts of power to different subcarriers [22], particularly if some sort of channel state information is available at the transmitter. Let $s_N(t)$ be a complex OFDM symbol, which is redefined as

$$s_N(t) = \sum_{k=0}^{N-1} s_{N,k} e^{j\omega_k t} \tag{12}$$

where $s_{N,k}, \quad k = 0, \ldots, N-1$, are independent complex random variables, and $\omega_k = \omega_0 + \frac{2\pi}{NT_c}k$. Let $s_{N,k}^R$ and $s_{N,k}^I$ be the real and imaginary parts of $s_{N,k}$, which have the following statistical characteristics: $E[s_{N,k}^R] = E[s_{N,k}^I] = 0$, $E[(s_{N,k}^R)^2] = E[(s_{N,k}^I)^2] = g_N(k)$, and $E[s_{N,k}^R s_{N,k}^I] = 0$. Assume there exists a finite constant D_0, such that

$$P\left[\frac{s_{N,k}^R}{\sqrt{g_N(k)}} \leq D_0\right] = P\left[\frac{s_{N,k}^I}{\sqrt{g_N(k)}} \leq D_0\right] = 1. \tag{13}$$

The function $g_N(k)$ gives the amount of power allocated to the kth subcarrier. Here, it is assumed that the OFDM system is designed to approximate some given power spectral density $G(\omega)$ [22]. The function $G(\omega)$ is assumed to be Riemann-integrable in the interval $[\omega_0, \omega_0 + 2\pi/T_c]$, and bounded by some constant M_G, with

$$\int_{\omega_0}^{\omega_0 + 2\pi/T_c} G(\omega)\, d\omega = \sigma^2. \tag{14}$$

This power distribution may be approximated if the power allocated to the kth subcarrier is

$$g_N(k) = \sigma^2 \frac{G(\omega_k)}{\sum_{m=0}^{N-1} G(\omega_m)}. \tag{15}$$

in which case the average power of the baseband OFDM signal is $P_{\text{av}} = 2\sum_{k=0}^{N-1} g_N(k) = 2\sigma^2$. Let λ_1 and λ_2 be the first and second normalized moment of $\frac{1}{\sigma^2}G(\omega)$, respectively, as defined in [22]

$$\begin{aligned}
\lambda_1 &= \lim_{N\to\infty} \frac{1}{\sigma^2} \sum_{k=0}^{N-1} g_N(k)\omega_k = \frac{1}{\sigma^2}\int_{\omega_0}^{\omega_0 + BW} \omega G(\omega) d\omega \\
\lambda_2 &= \lim_{N\to\infty} \frac{1}{\sigma^2} \sum_{k=0}^{N-1} g_N(k)\omega_k^2 = \frac{1}{\sigma^2}\int_{\omega_0}^{\omega_0 + BW} \omega^2 G(\omega) d\omega
\end{aligned} \tag{16}$$

where $BW = \frac{2\pi}{T_c}$.

As before, let $s_N(t) = x_N(t) + jy_N(t)$ and $R_{s_N}(\tau) = E[s_N^*(t)s_N(t+\tau)]$; then

$$\begin{aligned}
E[s_N^*(t)s_N(t+\tau)] &= \sum_{k=0}^{N-1} E|s_{N,k}|^2 e^{j\omega_k \tau} \\
&= 2\sigma^2 \frac{\sum_{k=0}^{N-1} \frac{BW}{N} G(\omega_k) e^{j\omega_k \tau}}{\sum_{k=0}^{N-1} \frac{BW}{N} G(\omega_k)} \\
&\to 2\int_{\omega_0}^{\omega_0 + BW} G(\omega) e^{j\omega\tau} d\omega \\
&= R_s(\tau)
\end{aligned} \tag{17}$$

as $N \to \infty$, where $R_{x_N}(\tau) = E[x_N(t)x_N(t+\tau)]$ and $R_{(y_N, x_N)}(\tau) = E[x_N(t)y_N(t+\tau)]$. It can be shown that

$$R_{x_N}(\tau) = R_{y_N}(\tau) = \sum_{k=0}^{N-1} g_N(k)\cos(\omega_k \tau)$$

$$R_{(y_N, x_N)}(\tau) = \sum_{k=0}^{N-1} g_N(k)\sin(\omega_k \tau).$$

Therefore, the autocorrelation functions of the random processes $x_N(t)$ and $y_N(t)$ and their cross-correlation function have the following relationships:

$$\begin{aligned}
R_{s_N}(\tau) &= 2(R_{x_N}(\tau) + jR_{(y_N, x_N)}(\tau)) \\
R_{(y_N, x_N)}(\tau) &= -R_{(x_N, y_N)}(\tau) = -R_{(y_N, x_N)}(-\tau).
\end{aligned} \tag{18}$$

Let $s(t), x(t)$ and $y(t)$ be the random processes to which $s_N(t), x_N(t)$ and $y_N(t)$ are converging in distribution, respectively. The convergence of these random processes will be proved in the coming paragraphs. Hence, as $N \to \infty, R_{x_N}(\tau) \to R_x(\tau) = \mathrm{Re}\{\int_{\omega_0}^{\omega_0+BW} G(\omega)e^{j\omega\tau}d\omega\}$, and $R_{(y_N,x_N)}(\tau) \to R_{(y,x)}(\tau) = \mathrm{Im}\{\int_{\omega_0}^{\omega_0+BW} G(\omega)e^{j\omega\tau}d\omega\}$. Since $R_{(y_N,x_N)}(\tau) = -R_{(y_N,x_N)}(-\tau), R_{(y_N,x_N)}(0) = 0$; in other words, $x_N(t)$ and $y_N(t)$ are uncorrelated for each t, as are $x(t)$ and $y(t)$.

To prove that $x_N(t)$ is converging to a Gaussian random process $x(t)$ with autocorrelation function $R_x(\tau) = \mathrm{Re}\{\int_{\omega_0}^{\omega_0+BW} G(\omega)e^{j\omega\tau}d\omega\}$, it is sufficient to show the tightness of $\{x_N(t)\}$ and convergence of the finite dimensional distributions of arbitrary finite samplings of $x_N(t)$ as has been done in the previous sections for the equal power case. Using (13) and the fact that $G(\omega)$ is upper bounded by M_G, it is trivial to prove the convergence of the finite dimensional distributions, as well as the conditions for tightness. Thus, all that is required is to show a counterpart to Lemma 2.

Lemma 6: $\forall\epsilon > 0$, there exists $N_0(\epsilon)$, such that

$$E|x_N(t+h) - x_N(t)|^2 \le \beta h^2, \quad \forall N \ge N_0.$$

Proof:

$$\begin{aligned} E|x_N(t+h) - x_N(t)|^2 &= 4\sum_{k=0}^{N-1} g_N(k)\sin^2\left(\frac{\omega_k h}{2}\right) \\ &\le \sum_{k=0}^{N-1} g_N(k)\omega_k^2 h^2 \\ &= h^2\sigma^2 \frac{\sum_{k=0}^{N-1} G(\omega_k)\omega_k^2 \frac{BW}{N}}{\sum_{k=0}^{N-1} G(\omega_k)\frac{BW}{N}} \\ &\to h^2 \int_{\omega_0}^{\omega_0+BW} \omega^2 G(\omega)\,d\omega \\ &= h^2\lambda_2 \end{aligned}$$

as $N \to \infty$. ∎

Then, $\forall\epsilon > 0$, there exists N_0 such that $E|x_N(t+h) - x_N(t)|^2 \le \beta h^2$, if $N \ge N_0$, where $\beta = \lambda_2 + \epsilon$. As a result, in (32) of the proof of Lemma 3, the corresponding lower bound of M will be

$$M \ge \log_2\left[\frac{2(\lambda_2+\epsilon)}{\varepsilon^2 \cdot \eta} G(q)\right] \tag{19}$$

and the rest follows in an identical fashion to the proof of Lemma 3.

For $\{s_N(t)\}$, it can be shown in a straightforward manner that tightness and the convergence of the finite-dimensional distributions is assured. Therefore, we have Theorem 5 proven, namely, the following.

Theorem 5: As $N \to \infty$, the sequence of complex random processes $\{s_N(t)\}$ converges in distribution to a complex Gaussian random process $s(t) = x(t) + jy(t)$, with zero mean and autocorrelation function $R_s(\tau) = \int_{\omega_0}^{\omega_0+BW} G(\omega)e^{j\omega\tau}d\tau$, where $G(\omega)$ determines the power allocation function as given in (15), satisfying (14).

It can seen that since $R_{(y,x)}(0) = 0, x(t)$ and $y(t)$ are independent of each other at each t, implying that $|s(t)|^2 = x^2(t) + y^2(t)$ is a $\chi^2(2)$-process.

V. Conclusion

Many approximations to the PMEPR of OFDM systems are based on assumptions of Gaussianity that invoke the Central Limit Theorem for a large number of subcarriers in the system. Here, we have considered this justification in more detail than previous efforts. In particular, we have demonstrated that the complex envelope of the transmitted OFDM signal converges weakly to a limiting Gaussian random process under broad conditions. In particular, the convergence holds for uncoded systems with a uniform power allocation across subcarriers, a broad class uncoded systems with an unequal power allocation across subcarriers, and for many coded systems.

Unfortunately, as the number of subcarriers goes to infinity, the symbol period of an OFDM system grows without bound. Hence, the weak convergence demonstrated here, which applies to any finite interval, cannot be used to complete the rigorous justification of many results that have relied on asymptotic Gaussianity for PMEPR characterization. Hence, it is of considerable interest to extend the results contained here to an infinite interval, although such an extension appears nontrivial.

Appendix

II. Establishing Tightness for the Real Part of the Complex Envelope of an OFDM Baseband Signal: Proof of Lemma 3

Lemma 3: Let $\{x_N\}$ be defined as in (1). Then, for each positive ε and η, there exists a δ, with $0 < \delta < 1$, and an integer N_0 such that

$$\mathcal{P}\left\{\sup_{\substack{|s-t|<\delta \\ s,t\in[0,1]}} |x_N(s) - x_N(t)| \ge \varepsilon\right\} \le \eta, \quad N \ge N_0. \tag{20}$$

Proof: Based on the proposition in [28, pp. 55,56], since $\{x_N(t), t \in T\} \in C$, then every countable set S dense in T is a separating set, which means, with probability 1

$$\sup_{\substack{t,s\in S \\ |t-s|<\delta}} |x_N(t) - x_N(s)| = \sup_{\substack{t,s\in T \\ |t-s|<\delta}} |x_N(t) - x_N(s)| \tag{21}$$

for $0 < \delta < 1$.

Define the set S to be the set of dyadic rationals

$$S = \left\{\frac{k}{2^n}, k = 0, 1, \ldots, 2^n - 1; \quad n = 0, 1, 2, \cdots\right\}. \tag{22}$$

Define the random variables

$$Z_v^{(N)}(\omega) = \sup_{0\le k\le 2^v-1} \left|x_N\left(\omega, \frac{k+1}{2^v}\right) - x_N\left(\omega, \frac{k}{2^v}\right)\right| \tag{23}$$

for $\omega \in \Omega$, where $v \in \mathcal{Z}^+$ is a positive integer, we then have [28, p. 56],

$$\sup_{\substack{t,s\in S\\|t-s|<2^{-M}}} |x_N(\omega,t) - x_N(\omega,s)| \le 2\sum_{v=M+1}^{\infty} Z_v^{(N)}(\omega) \tag{24}$$

for $\omega \in \Omega$, where M is a positive integer. By employing (21) and (24)

$$\begin{aligned}\mathcal{P}&\left\{\sup_{\substack{s,t\in T\\|t-s|<2^{-M}}} |x_N(s) - x_N(t)| \ge \varepsilon\right\}\\ &= \mathcal{P}\left\{\sup_{\substack{s,t\in S\\|t-s|<2^{-M}}} |x_N(s) - x_N(t)| \ge \varepsilon\right\}\\ &\le \mathcal{P}\left\{\sum_{v=M+1}^{\infty} Z_v^{(N)} \ge \frac{\varepsilon}{2}\right\}\\ &\le \mathcal{P}\left\{\bigcup_{v=M+1}^{\infty}\left\{Z_v^{(N)} \ge \frac{D(\epsilon)}{q^v}\right\}\right\}\\ &\le \sum_{v=M+1}^{\infty}\mathcal{P}\left\{Z_v^{(N)} \ge \frac{D(\epsilon)}{q^v}\right\}\end{aligned} \tag{25}$$

where $D(\epsilon)$ and q are constants. The constant q will be specified later, and the constant $D(\epsilon)$ can be determined by the following equation for $q > 1$:

$$\sum_{v=M+1}^{\infty}\frac{D(\epsilon)}{q^v} = D(\epsilon)\frac{1/q^{M+1}}{1-1/q} = \frac{1}{2}\varepsilon. \tag{26}$$

From (23)

$$\begin{aligned}\mathcal{P}&\left\{Z_v^{(N)} \ge \frac{D(\epsilon)}{q^v}\right\}\\ &= \mathcal{P}\left\{\sup_{0\le k\le 2^v-1}\left|x_N\left(\frac{k+1}{2^v}\right) - x_N\left(\frac{k}{2^v}\right)\right| \ge \frac{D(\epsilon)}{q^v}\right\}\\ &\le \sum_{k=0}^{2^v-1}\mathcal{P}\left\{\left|x_N\left(\frac{k+1}{2^v}\right) - x_N\left(\frac{k}{2^v}\right)\right| \ge \frac{D(\epsilon)}{q^v}\right\}.\end{aligned} \tag{27}$$

By Lemma 2 and Chebyshev's inequality [29]

$$\begin{aligned}\mathcal{P}&\left\{\left|x_N\left(\frac{k+1}{2^v}\right) - x_N\left(\frac{k}{2^v}\right)\right| \ge \frac{D(\epsilon)}{q^v}\right\}\\ &\le \frac{E\left|x_N\left(\frac{k+1}{2^v}\right) - x_N\left(\frac{k}{2^v}\right)\right|^2}{(D(\epsilon)/q^v)^2}\\ &\le \frac{\beta\left(\frac{1}{2^v}\right)^2}{D(\epsilon)^2\frac{1}{q^{2v}}} = \frac{\beta}{D(\epsilon)^2}\left(\frac{q^2}{4}\right)^v.\end{aligned} \tag{28}$$

Then, from (27)

$$\mathcal{P}\left\{Z_v^{(N)} \ge \frac{D(\epsilon)}{q^v}\right\} \le \frac{\beta}{D(\epsilon)^2}\left(\frac{q^2}{2}\right)^v. \tag{29}$$

Given (A) and (29), if $1 < q < \sqrt{2}$

$$\begin{aligned}\mathcal{P}&\left\{\sup_{\substack{s,t\in T\\|t-s|<2^{-M}}} |x_N(s) - x_N(t)| \ge \varepsilon\right\}\\ &\le \sum_{v=M+1}^{\infty}\mathcal{P}\left\{Z_v^{(N)} \ge \frac{D(\epsilon)}{q^v}\right\}\\ &\le \frac{\beta}{D(\epsilon)^2}\frac{\left(\frac{q^2}{2}\right)^{M+1}}{1-\frac{q^2}{2}}\end{aligned} \tag{30}$$

$\forall N \ge 1$. By substituting in $D(\epsilon)$ from (26)

$$\frac{\beta}{D(\epsilon)^2}\frac{\left(\frac{q^2}{2}\right)^{M+1}}{1-\frac{q^2}{2}} = \frac{2\beta\, G(q)}{\varepsilon^2 2^M} \tag{31}$$

where $G(q) = [(1-1/q)^2 \cdot (1-q^2/2)]^{-1}$.

Thus, for any positive ε and η, select $1 < q < \sqrt{2}$ and positive integer M to satisfy

$$M \ge \log_2\left[\frac{2\beta}{\varepsilon^2\cdot\eta}G(q)\right] = \log_2\left[\frac{8\pi^2\sigma^2}{3T_c^2\varepsilon^2\eta}G(q)\right] \tag{32}$$

and let $\delta = 2^{-M}$. Then, the condition of (5) is satisfied

$$\mathcal{P}\left\{\sup_{\substack{|s-t|<\delta\\s,t\in[0,1]}} |x_N(s) - x_N(t)| \ge \varepsilon\right\} \le \eta, \quad N \ge N_0 = 1.$$

Since ε and η were arbitrary, this establishes Condition 2 of the tightness definition for the sequence of random signals in (1). ■

III. Convergence of the Finite-Dimensional Distributions: Proof of Lemma 5

Lemma 5: Let $x_N(t)$ be defined as in (1), and pick any integer $L \ge 1$ and collection of sample times $\{t_1, t_2, \ldots, t_L\}$. Then

$$\underline{\Gamma}_N = (x_N(t_1), x_N(t_2), \ldots, x_N(t_L))^T \xrightarrow{\mathcal{D}} \underline{\Gamma}$$

where $\underline{\Gamma} = (\Gamma_1, \Gamma_2, \ldots, \Gamma_L)^T$ is an L-dimensional vector with jointly Gaussian components, mean vector $\underline{0}$, and covariance matrix Σ, where the (i,j)th element of Σ is given by

$$\Sigma_{i,j} = E[\Gamma_i\Gamma_j] = \sigma^2\mathrm{sinc}\left(\frac{2(t_i - t_j)}{T_c}\right). \tag{33}$$

Proof: Pick any integer $L \ge 1$ and collection of sample times $\{t_1, t_2, \ldots, t_L\}$. The Cramér-Wold Theorem [27, p. 49] will be employed; thus, consider any linear combination

$$Z_N = \sum_{l=1}^{L} a_l x_N(t_l)$$

where $a_1, a_2, \ldots, a_L$ are real constants. Then

$$\begin{aligned} Z_N &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k^R \sum_{l=1}^{L} a_l \cos\left(2\pi \frac{k}{NT_c} t_l\right) \\ &\quad - \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k^I \sum_{l=1}^{L} a_l \sin\left(2\pi \frac{k}{NT_c} t_l\right) \\ &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k^R r_{k,N} - \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k^I i_{k,N} \\ &= \sum_{k=0}^{N-1} \gamma_{k,N} \end{aligned} \tag{34}$$

where

$$\begin{aligned} r_{k,N} &= \sum_{l=1}^{L} a_l \cos\left(2\pi \frac{k}{NT_c} t_l\right), \\ i_{k,N} &= \sum_{l=1}^{L} a_l \sin\left(2\pi \frac{k}{NT_c} t_l\right), \\ \gamma_{k,N} &= \frac{1}{\sqrt{N}} \left(A_k^R r_{k,N} - A_k^I i_{k,N}\right). \end{aligned}$$

Noting $|r_{k,N}| < \sum_{l=1}^{L} |a_l|$ and $|i_{k,N}| < \sum_{i=1}^{L} |a_l|$, and $|A_k^R| \le \bar{A}, |A_k^I| \le \bar{A}$, Lindeberg's condition for triangular arrays [29, p. 116] is satisfied as follows. Since

$$\begin{aligned} |A_k^R r_{k,N} - A_k^I i_{k,N}| &\le \left|A_k^R\right| |r_{k,N}| + \left|A_k^I\right| |i_{k,N}| \\ &\le 2\bar{A} \sum_{l=0}^{N-1} |a_l| = C_0 \end{aligned}$$

and, for any $\epsilon > 0$, there exists N_0, such that when $N \ge N_0, \sqrt{N}\epsilon > C_0$. Therefore, if $N \ge N_0$

$$\begin{aligned} &\sum_{k=0}^{N-1} E\left\{|\gamma_{k,N}|^2; |\gamma_{k,N}| > \epsilon\right\} \\ &= \sum_{k=0}^{N-1} E\left\{|\gamma_{k,N}|^2; \left|A_k^R r_{k,N} - A_k^I i_{k,N}\right| > \sqrt{N}\epsilon\right\} = 0. \end{aligned} \tag{35}$$

The limiting value of the variance of Z_N will determine two separate cases. Thus, noting $E[Z_N] = 0$, the variance of Z_N is computed as follows. First, note

$$E[Z_N^2] = \sum_{l=1}^{L} \sum_{m=1}^{L} a_l a_m E[x_N(t_l) x_N(t_m)].$$

Next, note

$$\begin{aligned} &E[x_N(t_l) x_N(t_m)] \\ &\quad = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{v=0}^{N-1} E \\ &\qquad \times \left[\left(A_k^R \cos\left(\frac{2\pi k}{NT_c} t_l\right) - A_k^I \sin\left(\frac{2\pi k}{NT_c} t_l\right)\right)\right. \\ &\qquad \times \left.\left(A_v^R \cos\left(\frac{2\pi v}{NT_c} t_m\right) - A_v^I \sin\left(\frac{2\pi v}{NT_c} t_m\right)\right)\right] \\ &\quad = \frac{1}{N} \sum_{k=0}^{N-1} \left(\sigma^2 \cos\left(\frac{2\pi k}{NT_c} t_l\right) \cos\left(\frac{2\pi k}{NT_c} t_m\right)\right. \\ &\qquad + \left.\sigma^2 \sin\left(\frac{2\pi k}{NT_c} t_l\right) \sin\left(\frac{2\pi k}{NT_c} t_m\right)\right) \\ &\quad = \frac{\sigma^2}{N} \sum_{k=0}^{N-1} \cos\left(\frac{2\pi k (t_m - t_l)}{NT_c}\right) \\ &\quad \longrightarrow \sigma^2 \mathrm{sinc}\left(\frac{2(t_m - t_l)}{T_c}\right) \end{aligned} \tag{36}$$

which implies

$$\begin{aligned} \psi^2 &\triangleq \lim_{N\to\infty} E[Z_N^2] \\ &= \sum_{l=1}^{L} \sum_{m=1}^{L} \sigma^2 a_l a_m \mathrm{sinc}\left(\frac{2(t_m - t_l)}{T_c}\right). \end{aligned}$$

If $\psi^2 > 0$, Lindeberg's conditions for triangular arrays [29, p. 116] are thus satisfied; therefore, $Z_N \xrightarrow{\mathcal{D}} Z$, where Z is normal, $E[Z] = 0$, and $E[Z^2] = \psi^2$.

If $\psi^2 = 0$: Chebyshev's inequality [29] yields

$$P(|Z_N| \ge \eta) \le \frac{E[Z_N^2]}{\eta^2} \longrightarrow 0$$

for any $\eta > 0$, which implies $Z_N \xrightarrow{\mathcal{D}} 0$. Thus, Z_N converges in distribution to a Gaussian random variable with mean 0 and variance 0.

The two cases together imply that $Z_N \xrightarrow{\mathcal{D}} Z \sim N(0, \psi^2)$ for any ψ^2. Now, for the same constants $a_1, a_2, \ldots, a_L$, define $U = \sum_{l=1}^{L} a_l \Gamma_l$, where Γ_i denotes the ith element of $\underline{\Gamma}$. U is normal with mean $E[U] = 0$ and variance $E[U^2] = \psi^2$. Thus, $U \stackrel{\mathcal{D}}{=} Z$ for any L and collection of $\{a_1, a_2, \ldots, a_L\}$. By the Cramér-Wold Theorem,

$$\underline{\Gamma}_N = (x_N(t_1), x_N(t_2), \ldots, x_N(t_L))^T \xrightarrow{\mathcal{D}} \underline{\Gamma}$$

■

IV. Convergence of the Complex Envelope of Coded OFDM Systems: Proof of Theorem 4

This Appendix contains the proof of Theorem 4, which extends our results from uncoded OFDM systems to many coded systems. It is straightforward to verify measurability of the appropriate quantities and tightness of the sequence of measures in a manner analogous to that in Section II for the uncoded case. In particular, establishing tightness for the uncoded case, which was the crux of the proof, depended only on the second order statistics of the random process. Since the second order statistics of the random process of Theorem 4 are identical to those of the random process of Theorem 2, tightness follows in an identical fashion. However, the proof of the analog of Lemma 5 is greatly complicated, since the dependence of the summands greatly complicates central limit theory. However, a result of Withers [33] is sufficient for Theorem 4; in fact, it is clear that Theorem 4 holds under much broader conditions for the random process input to the IFFT. The statement and proof of Lemma 7, which is the analog to Lemma 5 for complex signals, is given here. Since the remainder of the arguments follow those of Theorem 2 [25], Lemma 7 completes the proof of Theorem 4.

Lemma 7: Pick any L and collection of sample times $\{t_1, \ldots, t_L\}$, and define

$$x_N(t_i) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(B_{k,N}^R \cos\left(2\pi \frac{k}{NT_c} t_i\right) - B_{k,N}^I \sin\left(2\pi \frac{k}{NT_c} t_i\right)\right)$$

and

$$y_N(t_i) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(B_{k,N}^R \sin\left(2\pi \frac{k}{NT_c} t_i\right) + B_{k,N}^I \cos\left(2\pi \frac{k}{NT_c} t_i\right)\right).$$

Then

$$\underline{\tilde{\Gamma}_N} = (x_N(t_1), y_N(t_1), \ldots, x_N(t_L), y_N(t_L))^T \xrightarrow{\mathcal{D}} \underline{\tilde{\Gamma}}$$

where

$$\underline{\tilde{\Gamma}} = (x_1, y_1, x_2, y_2, \ldots, x_L, y_L)^T$$

is a $2L$-dimensional jointly Gaussian distributed random vector, with mean vector $\underline{0}$, and covariance matrix defined by

$$E[x_i x_j] = E[y_i y_j] = \sigma^2 \text{sinc}\left(\frac{2(t_j - t_i)}{T_c}\right) \tag{37}$$

and

$$E[x_i y_j] = \sigma^2 \frac{\sin^2\left(\frac{(t_j - t_i)\pi}{T_c}\right)}{\frac{\pi(t_j - t_i)}{T_c}}. \tag{38}$$

Proof: The Cramér-Wold Theorem will be employed. Thus, consider any linear combination

$$\tilde{Z}_N = \sum_{l=1}^{L} (\alpha_l x_N(t_l) + \beta_l y_N(t_l))$$

where $\alpha_1, \beta_1, \ldots, \alpha_L, \beta_L$ are real constants. Then

$$\begin{aligned}
\tilde{Z}_N
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} B_{k,N}^R \sum_{l=1}^{L} \left(\alpha_l \cos\left(\frac{2\pi k}{NT_c} t_l\right) + \beta_l \sin\left(\frac{2\pi k}{NT_c} t_l\right)\right) \\
&\quad - \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} B_{k,N}^I \sum_{l=1}^{L} \left(\alpha_l \sin\left(\frac{2\pi k}{NT_c} t_l\right) - \beta_l \cos\left(\frac{2\pi k}{NT_c} t_l\right)\right) \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k^R \sum_{l=1}^{L} \left(\alpha_l \cos\left(\frac{2\pi p_N(k)}{NT_c} t_l\right) + \beta_l \sin\left(\frac{2\pi p_N(k)}{NT_c} t_l\right)\right) \\
&\quad - \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k^I \sum_{l=1}^{L} \left(\alpha_l \sin\left(\frac{2\pi p_N(k)}{NT_c} t_l\right) - \beta_l \cos\left(\frac{2\pi p_N(k)}{NT_c} t_l\right)\right)
\end{aligned} \tag{39}$$

where $\{p_N(\cdot)\}$ is the sequence of functions that maps indexes from symbols in $\{A_k, k = 0, \ldots, N-1\}$ to indexes of symbols in $\{B_{k,N}, k = 0, \ldots, N-1\}$; in other words, $B_{p_N(i),N} = A_i$. Now define

$$\tilde{r}_{k,N} = \sum_{l=1}^{L} \left(\alpha_l \cos\left(\frac{2\pi p_N(k)}{NT_c} t_l\right) + \beta_l \sin\left(\frac{2\pi p_N(k)}{NT_c} t_l\right)\right)$$

$$\tilde{i}_{k,N} = \sum_{l=1}^{L} \left(\alpha_l \sin\left(\frac{2\pi p_N(k)}{NT_c} t_l\right) - \beta_l \cos\left(\frac{2\pi p_N(k)}{NT_c} t_l\right)\right)$$

such that

$$\tilde{Z}_N = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k^R \tilde{r}_{k,N} - \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k^I \tilde{i}_{k,N}$$

and let

$$X_{k,N} = A_k^R \tilde{r}_{k,N} - A_k^I \tilde{i}_{k,N}.$$

Then, $E[X_{k,N}] = 0, \forall k, \forall N$, and

$$\begin{aligned}
\psi_{k,N}^2 &\triangleq E[X_{k,N}^2] = \sigma^2 \left(\tilde{r}_{k,N}^2 + \tilde{i}_{k,N}^2\right) \\
&= \sigma^2 \sum_{l=1}^{L} \sum_{m=1}^{L} \Bigg((\alpha_l \alpha_m + \beta_l \beta_m) \times \cos\left(\frac{2\pi p_N(k)}{NT_c}(t_m - t_l)\right) \\
&\quad + (\alpha_l \beta_m - \alpha_m \beta_l) \sin\left(\frac{2\pi p_N(k)}{NT_c}(t_m - t_l)\right)\Bigg)
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
\psi^2 &\triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \psi_{k,N}^2 \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \sigma^2 \times \sum_{l=1}^{L} \sum_{m=1}^{L} \Bigg((\alpha_l \alpha_m + \beta_l \beta_m) \cos\left(\frac{2\pi p_N(k)}{NT_c}(t_m - t_l)\right) \\
&\quad + (\alpha_l \beta_m - \alpha_m \beta_l) \sin\left(\frac{2\pi p_N(k)}{NT_c}(t_m - t_l)\right)\Bigg) \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \sigma^2 \times \sum_{l=1}^{L} \sum_{m=1}^{L} \Bigg((\alpha_l \alpha_m + \beta_l \beta_m) \cos\left(\frac{2\pi k}{NT_c}(t_m - t_l)\right)
\end{aligned}$$

$$+(\alpha_l\beta_m - \alpha_m\beta_l)\sin\left(\frac{2\pi k}{NT_c}(t_m - t_l)\right)\Bigg)$$
$$= \sum_{l=1}^{L}\sum_{m=1}^{L}\sigma^2\left((\alpha_l\alpha_m + \beta_l\beta_m)\,\mathrm{sinc}\left(\frac{2(t_m-t_l)}{T_c}\right)\right.$$
$$\left. +(\alpha_l\beta_m - \alpha_m\beta_l)\frac{\sin^2\left(\frac{\pi(t_m-t_l)}{T_c}\right)}{\frac{\pi(t_m-t_l)}{T_c}}\right). \tag{41}$$

Also, let $S_N \triangleq \sum_{k=1}^{N} X_{k,N}$ and $S_N(a,b) \triangleq \sum_{k=a+1}^{a+b} X_{k,N}, a \geq 0, N - a \geq b \geq 1$ and $\psi_N^2 = \mathrm{Var}[S_N] = \sum_{j=1}^{N}\psi_{j,N}^2$ such that

$$\lim_{N\to\infty}\frac{\psi_N^2}{N} = \psi^2.$$

For $\psi^2 > 0$: $\frac{S_N}{\psi_N} \xrightarrow{\mathcal{D}} N(0,1)$ can be established by verifying the conditions of Theorem 2.1 of [33] hold; in particular, it is sufficient to show that $\exists\varepsilon > 0, \gamma \geq 0$, such that follows.

Condition 1: $\mathcal{X} = \{X_{k,N}, N \geq 1, N \geq k \geq 1\}$ is l-mixing with

$$l(j,u) = o\left(j^{\frac{-2\gamma}{\varepsilon}}\right)$$

as $j \to \infty$, where we have (42), shown at the bottom of the page.

Condition 2:

$$\sup_{a,N} E[|S_N(a,b)|^{2+\varepsilon}] = O\left(b^{1+\frac{\varepsilon}{2}+\gamma}\right)$$

as $b \to \infty$.

Condition 3: $\psi_N^2 \to \infty$ as $N \to \infty$ and $\sum_{k=0}^{\infty}\tilde{c}(k) < \infty$, where

$$\tilde{c}(k) = \max_{\{N:N\geq k\}}\tilde{c}_N(k)$$

and

$$\tilde{c}_N(k) = \sup_{\{u,v:|u-v|\geq k, 1\leq u\leq N, 1\leq v\leq N\}}|\mathrm{Cov}(X_{u,N}, X_{v,N})|.$$

To show Condition 1, note that A_k^R and A_k^I are m-dependent with $m = n_0$; thus, the $X_{k,N}$ are m-dependent, which implies $l_N(j,u) = 0$ for $j > n_0$ and $l(j,u) = 0$ for $j > n_0$ and any real u. Thus, Condition 1 holds for all $\varepsilon > 0, \gamma \geq 0$.

From [33, Prop. 2.1(a)], Condition 2 will hold for $\gamma = \varepsilon, \varepsilon \leq 2$, if

$$\max_{k,N} E[X_{k,N}^4] < \infty$$

and

$$\max_{a,b,N}\frac{E[S_N^2(a,b)]}{b} < \infty.$$

Thus, consider

$$\begin{aligned} E\left[X_{k,N}^4\right] &= E\left[\left(A_k^R\tilde{r}_{k,N} + A_k^I\tilde{i}_{k,N}\right)^4\right] \\ &= \tilde{r}_{k,N}^4 E\left[\left(A_k^R\right)^4\right] + \tilde{i}_{k,N}^4 E\left[\left(A_k^I\right)^4\right] \\ &\quad + 4\tilde{r}_{k,N}^3\tilde{i}_{k,N}E\left[\left(A_k^R\right)^3 A_k^I\right] \\ &\quad + 4\tilde{r}_{k,N}\tilde{i}_{k,N}^3 E\left[\left(A_k^I\right)^3 A_k^R\right] \\ &\quad + 6\tilde{r}_{k,N}^2\tilde{i}_{k,N}^2 E\left[\left(A_k^R\right)^2\left(A_k^I\right)^2\right]. \end{aligned} \tag{43}$$

Now, $|\tilde{r}_{k,N}|$ and $|\tilde{i}_{k,N}|$ are each upper bounded by $\zeta \triangleq \sum_{l=1}^{L}(|\alpha_l| + |\beta_l|)$. Thus

$$E[X_{k,N}^4] \leq 16\zeta^4\bar{A}^4 < \infty.$$

Next, exploiting the uncorrelated nature of the sequences

$$E[S_N^2(a,b)] = \sum_{k=a+1}^{a+b}\psi_{k,N}^2.$$

For each k and $N, \psi_{k,N}^2 \leq \zeta^2$. Thus

$$E[S_N^2(a,b)] \leq b\zeta^2, \quad \forall N, a, b$$

and

$$\max_{a,b,N}\frac{E[S_N^2(a,b)]}{b} \leq \zeta^2 < \infty$$

thus establishing Condition 2.

To establish Condition 3, note that

$$\tilde{c}_N(k) = \begin{cases} 0, & k \geq 1 \\ \max_{l\leq N}\psi_{l,N}^2, & k = 0 \end{cases}$$

and

$$\tilde{c}(k) = \begin{cases} 0, & k \geq 1 \\ \max_{\{N, l\leq N\}}\psi_{l,N}^2, & k = 0. \end{cases}$$

$$\begin{aligned} l(j,u) &= \sup_{\{N:N\geq j\}}\{l_N(j,u)\}, \\ l_N(j,u) &= \max_{\{1\leq v\leq N-j\}}\ \sup_{\text{any choice of } \delta_w\text{'s,each}\in\{0,1\}}|\mathrm{Cov}(e^{iuP}, e^{-iuF})| \\ P &= \frac{1}{\psi_N}\sum_{w=1}^{v}\delta_w X_{w,N} \end{aligned} \tag{42}$$

and

$$F = \frac{1}{\psi_N}\sum_{w=v+j}^{N}\delta_w X_{w,N}.$$

Recalling $\psi_{l,N}^2 \leq \zeta^2$ for all l and N yields

$$\sum_{j=0}^{\infty} \tilde{c}(j) \leq \zeta^2 < \infty.$$

Condition 3 is now established by noting that $\psi_N^2 \to \infty$ for any $\psi^2 > 0$.

Thus, for $\psi^2 > 0$

$$\frac{1}{\sqrt{N}} \sum_{k=1}^{N} X_{k,N} \xrightarrow{\mathcal{D}} Z \sim N(0, \psi^2).$$

If $\psi^2 = 0$, Chebyshev's inequality [29] yields

$$P(|S_N| \geq \eta\sqrt{N}) \leq \frac{\psi_N^2}{N\eta^2} \to 0$$

as $N \to \infty$ for any $\eta > 0$, which implies $Z_N \xrightarrow{P} 0$. Thus, Z_N converges in distribution to a Gaussian random variable with mean 0 and variance 0.

The two cases together imply that $Z_N \xrightarrow{\mathcal{D}} Z \sim N(0, \psi^2)$ for any ψ^2. Then for the same constants $\alpha_1, \beta_1, \cdots, \alpha_L, \beta_L$, define $\tilde{U} = \sum_{l=1}^{L}(\alpha_l x_l + \beta_l y_l)$. $\tilde{U}$ is a Gaussian random variable with mean 0 and $E[\tilde{U}^2] = \psi^2$.

Thus, $\tilde{U} \stackrel{\mathcal{D}}{=} Z$ for any $2L$ collection of $\{\alpha_1, \beta_1, \cdots, \alpha_L, \beta_L\}$. By the Cramér-Wold Theorem

$$\tilde{\underline{\Gamma}}_N = (x_N(t_1), y_N(t_1), \ldots, x_N(t_L), y_N(t_L))^T \xrightarrow{\mathcal{D}} \tilde{\underline{\Gamma}}.$$

■

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