Fully Polynomial Approximation Algorithm for Collaborative Relaying in Sensor Networks under Finite Rate Constraints*

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Abstract. We take an algorithmic approach to a well-known communication channel problem and develop several algorithms for solving it. Specifically, we develop power control algorithms for sensor networks with collaborative relaying under bandwidth constraints, via quantization of finite rate (bandwidth limited) feedback channels. We first consider the power allocation problem under collaborative relaying where the tradeoff between minimizing ones own energy expenditure and the energy for relaying is considered under the constraints of packet outage probability and bandwidth constrained (finite rate) feedback. Then we develop bandwidth constrained quantization algorithms (due to the finite rate feedback) that seek the optimal way of quantizing channel quality and power values in order to minimize the total average transmission power and satisfy the given probability of outage. We develop two kinds of quantization protocols and associated quantization algorithms. For separate source-relay quantization, we reduce the problem to the well-known k-median problem [1] on line graphs and show a a simple $O((K_J)^2N)$ polynomial time algorithm, where $\log_2 K_J$ is the quantization bandwidth and N is the size of the discretized parameter space. For joint quantization, we first develop a simple 2-factor approximation of complexity $O(K_JN + N\log N)$. Then, for $\epsilon > 0$, we develop a fully polynomial approximation scheme (FPAS) that approximates the optimal quantization cost to within an $1+\epsilon$-factor. The running time of the FPAS is polynomial in $1/\epsilon$, size of the input N and also $\ln F$, where F is the maximum available transmit power.

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1 Introduction

Energy efficiency is an important consideration in wireless sensor networks. One technique for minimizing transmission energy in a cluster is collaborative relaying. Nodes can select partners to act as relays for forwarding their data to the clusterhead or sink. Relaying exploits cooperative diversity, the fact that sometimes the relay-clusterhead channel quality is significantly better than the direct source-clusterhead. Thus if the relay is able to receive and decode the source message, even if there are errors (packet outage) between the source and clusterhead, the relay can correctly transmit the packet to the destination.

Cooperative diversities under relaying can be exploited to further improve reliability and energy efficiency by using Channel State Information (CSI) [2, 3]. Communication channel quality is estimated and fed-back to the nodes in order to decide the metrics of relaying. The preceding cited works share a common feature in that they assume a set of relay nodes is already selected and the issue is to determine power allocations across all transmitting nodes without considering data originating from relay nodes themselves. No consideration is given toward the relay's own needs other than its function as a relay.

In our model, we assume that source sensor and relay sensor both have their own data to transmit to the clusterhead along with an individual quality of service requirement, e.g. outage probability P_{out} as a good approximation for frame error rate (FER). We do not consider partner selection protocols but assume a relay has been apriori selected. Each node divides its entire energy budget into two parts. One is for transmitting its own data, the other is devoted to relaying information. As a partner relationship is established between two nodes such that each of them helps the other forward/relay information, we are interested in a fundamental energy tradeoff question: What power allocation policy should be adopted by each node in order to minimize its own total energy consumption while meeting the outage probability constraints and complying with its obligation as a relay.

In [2, 3], perfect CSI at each node is assumed available to the source and relay nodes. However perfect CSI can only be available under the assumption of unlimited feedback channel capacity in order for the receiver to transmit back the measurements to its transmitter without any error. Adaptive signaling under the finite rate feedback constraint has attracted considerable attentions lately because of its more practical implications compared with the perfect CSI assumption. When the feedback channel is assumed error-free with limited capacity, there are in general three approaches in exploiting partial CSI at the transmitter side, namely, channel vector quantization, scalar quantization and quantized signal adaptation schemes [4] (and references therein).

Not much work has been done yet for adaptive signaling schemes in sensor networks with relay channels under the finite rate feedback constraint. In [5], the power control problem is tackled for relay channels with finite rate feedback. However, only amplify-and-forward relaying is considered, in which the issue of availability of CSI for the source-relay link is relatively easier to address than the decode-and-forward case. In addition, the majority of work in the literature on finite rate feedback problems approach the resulting quantization problems directly by finding out the optimal quantization regions of fading vectors, as well as associated power allocation functions [4].

In this paper, we take an algorithmic approach to collaborative relaying under finite rate feedback by using the technique of discretization of variables (in our case channel fading coefficients). We first briefly present results obtained in [**?**] where we optimize the total average power expenditure of both relay and source nodes under the assumption that nodes have *perfect* CSI in a network of two transmitting sensors and one clusterhead. Based on the power control strategies developed in [**?**] for decode-and-forward relaying, we develop bandwidth constrained quantization algorithms (due to the finite rate feedback) that seek the optimal way of quantizing channel quality and power values in order to minimize the total average transmission power and satisfy the given probability of outage.

We develop two kinds of quantization protocols and associated quantization algorithms. First we consider separate source and receiver quantization, where the clusterhead splits its available quantization bandwidth for feedback, independently between the source and relay node. We reduce this quantization problem to the well-known k-median problem [1] on line graphs and show a a simple $O(NK_J(K_J + \log N))$ polynomial time algorithm, where $\log K_J$ is the quantization bandwidth and N is the size of the discretized parameter space. Then we consider joint quantization. Here the base station can exploit the joint probability distributions of source and relay channels and power values and use the entire quantization bandwidth to jointly feedback both the source and relay. Unfortunately, the joint quantization problem is NP-hard by reduction from the k-median problem, which has itself been a subject of study for several decades (problem ND51 in [1]). Therefore, we develop a simple 2-factor approximation of complexity $O(N(K_J + \log N))$. Then, for $\epsilon > 0$, we develop a fully polynomial approximation scheme (FPAS) that approximates the optimal quantization cost to within an $1 + \epsilon$-factor. The running time of the FPAS is polynomial in $1/\epsilon$, size of the input N and also $\ln F$, where F is the maximum available transmit power.

The paper is organized as follows. We first present the system model in Section 2. Power control strategies with perfect CSI are then provided in Section 3. When finite rate feedback constraint is imposed, the independent and joint quantization algorithms for source and relay nodes are given in the next two Sections.

2 System Model

To illustrate the major idea of power control across relay nodes, we first consider a simple model in which there are two nodes N_1 and N_2 transmitting to a common receiver N_D with help from each other. Narrow-band quasi-static fading channel is assumed, where channel fading coefficients remain fixed during the transmission of a whole packet, but are independent from node to node. The complex channel coefficient $h_{i,j}$ captures the effects of both pathloss and the quasi-static fading on transmissions from node N_i to node N_j, where $i \in \{1, 2\}$, and $j \in \{2, 1, D\}$. Statistically, $h_{i,j}$ are modeled as zero mean, mutually independent proper complex Gaussian random variables with variances: $E|h_{i,j}|^2 = 2\sigma_{i,j}^2$. We first assume a non-causal system model in which amplitudes $|h_{i,j}|$ are available to all transmitters and receivers at the beginning of transmissions. In a quasi-static fading channel, CSI can be obtained by exploiting training sequences sent by transmitters [6].

Consider a time-division (TD) multiple access scheme in which an entire time period is divided into 4 slots [7, Fig. 2]. A repetition coding-based decode-and-forward strategy (R-DF)is assumed at N_j, $j = 1, 2$, where relay node transmits the same codeword as what source sends if its decoding is successful. The cooperative communication protocol can be described as follows: Based on the available CSI, N_1 can determine whether relaying from N_2 is needed or not, as explained in the power control algorithms below. If such collaboration is sought, N_1 transmits as a source to N_D in the first slot and then in the second slot N_2 forwards its decoded messages to the destination. If N_2 is not asked for relaying, N_1 transmits in the first 2 slots of on its own. Over the last two slots, N_1 and N_2 exchange their roles as a source and relay.

The mathematical characterization of the whole process is:

$$Y_{1,D}[k] = h_{1,D}S_1[k] + W_{1,D}[k], Y_{1,2}[k] = h_{1,2}S_1[k] + W_{1,2}[k]$$

for $k \in [0, N/4]$; and

$$Y_{2,R}[k] = h_{2,D}\tilde{S}_1[k] + W_{2,R}[k]$$

for $k \in (N/4, N/2]$, if relay N_2 is needed and decoding is successful. The figure N is the total number of degrees of freedom available over the entire transmission period, and $W_{i,j}$ are independent complex white Gaussian noise with two-sided power spectral density $\mathcal{N}_0 = 1$. For R-DF schemes, $\tilde{S}_j[k]$ are scaled versions of the transmitted Gaussian codewords $S_j[k]$. Over the last two slots, similar models can be set up for node 2 based on symmetry over $k \in (N/2, N]$.

Given CSI on $|h_{i,j}|$, transmission powers over various periods are denoted as: $E|S_1[k]|^2 = P_{1,D}$, $k \in [0, N/4]$ and $E|\tilde{S}_1[k]|^2 = P_{2,R}$, $k \in (N/4, N/2]$ if N_2 is needed and decoding is successful; $E|S_1[k]|^2 = P_{1,D}$, $k \in [0, N/2]$ and $E|\tilde{S}_1[k]|^2 = 0$, $k \in [0, N/2]$, if N_2 is not needed. Similarly, we define $E|S_2[k]|^2 = P_{2,D}$, $k \in (N/2, \frac{3}{4}N]$ and $E|\tilde{S}_2[k]|^2 = P_{1,R}$, $k \in (\frac{3}{4}N, N]$ if N_1 is needed and decoding is successful; $E|S_2[k]|^2 = P_{2,D}$, $k \in (N/2, N]$ and $E|\tilde{S}_2[k]|^2 = 0$, $k \in (N/2, N]$ if N_1 is not needed.

3 Total Energy Minimization for Collaborative Relaying with Perfect CSI

Under the constraint that each sensor node has an outage probability no greater than $P_{j,out}$, i.e. Pr $[I_j < R_j] \leq P_{j,out}$, where I_j is the mutual information of the overall link for transmitting node $j \in \{1, 2\}$'s information, our objective is to investigate power control policies under which the total energy of these two nodes is minimized in a complete collaborative manner. This **Collaborative Relaying** problem can be formulated as below:

$$\min \sum_{j=1}^{2} E\left[P_{j,D} + P_{j,R}\right], \text{ subject to } \Pr\left[I_k < R_k\right] \leq P_{k,out}, \tag{1}$$

for $k = 1, 2$.

Under the collaborative relaying approach, the optimal power allocation policy $[P_{i,D}, P_{j,R}]$ to solving problem (1) can be characterized by the following Theorem.

Theorem 1. *The optimal power allocation vector $[P_{i,D}, P_{j,R}]$ depends on channel strength ratios captured by $|h_{i,D}|/|h_{j,D}|$ and $|h_{i,D}|/|h_{i,j}|$ for $i \neq j$ and $i, j \in \{1, 2\}$. The resulting solutions are:*

$P_{i,D} = \hat{P}_{i,D}$, $P_{j,R} = \hat{P}_{j,R}$ if $h_{i,j}$ are in the set

$$A_i = \left\{|h_{i,j}| : \frac{|h_{i,D}|^2}{|h_{i,j}|^2} < \frac{2}{2^{R_i}+1} \text{ and } \frac{|h_{i,D}|^2}{|h_{i,j}|^2} + \frac{|h_{i,D}|^2}{|h_{j,D}|^2}\left(1 - \frac{|h_{i,D}|^2}{|h_{i,j}|^2}\right) \leq \frac{2}{2^{R_i}+1}\right\} \tag{2}$$

and $\hat{P}_{i,D} + \hat{P}_{j,R} \leq s_i^*$. Otherwise if $h_{i,j} \in A_i^c$, the complimentary set of A_i, i.e.

$$A_i^c = \left\{|h_{i,j}| : \frac{|h_{i,D}|^2}{|h_{i,j}|^2} \geq \frac{2}{2^{R_i}+1} \text{ or } \frac{|h_{i,D}|^2}{|h_{i,j}|^2} + \frac{|h_{i,D}|^2}{|h_{j,D}|^2}\left(1 - \frac{|h_{i,D}|^2}{|h_{i,j}|^2}\right) > \frac{2}{2^{R_i}+1}\right\} \tag{3}$$

and $2\tilde{P}_{i,D} \leq s_i^*$, the solution is $P_{i,D} = \tilde{P}_{i,D}$, $P_{j,R} = 0$. For all other cases, transmission powers are all set to zero $P_{i,D} = P_{j,R} = 0$.

Transmission power functions are defined as follows:

$$\tilde{P}_{i,D} \triangleq (2^{R_i} - 1)(|h_{i,D}|^2),\ \hat{P}_{i,D} \triangleq (2^{2R_i} - 1)(|h_{i,j}|^2),\ \hat{P}_{i,R} \triangleq \frac{2^{2R_j} - 1}{|h_{i,D}|^2}\left(1 - \frac{|h_{j,D}|^2}{|h_{j,i}|^2}\right). \tag{4}$$

The thresholds $s_i^*, i = 1, 2$ are determined by solving the following equations to meet outage probability constraints:

$$\begin{aligned} 1 - P_{i,out} = Pr&\left\{2\tilde{P}_{i,D} < s_i^*, \text{ for } \left(\frac{|h_{i,D}|^2}{|h_{i,j}|^2}, \frac{|h_{i,D}|^2}{|h_{j,D}|^2}\right) \in A_i^c\right\} \\ +Pr&\left\{\hat{P}_{i,D} + \hat{P}_{j,R} < s_i^*, \text{ for } \left(\frac{|h_{i,D}|^2}{|h_{i,j}|^2}, \frac{|h_{i,D}|^2}{|h_{j,D}|^2}\right) \in A_i\right\}. \end{aligned}$$

Proof. See [8].

4 Optimal Quantization for Optimal Collaborative Relaying

In the previous section, we assumed the availability of perfect CSI at each sensor node in order to develop an optimal power control algorithm for collaborative relaying. However, in reality, perfect CSI is not possible since bandwidth limitations prevent the full exchange of precise channel information between the source, relay and base-station[1]. This motivates the idea of developing power control algorithms for sensor networks with relaying under bandwidth constraints, specifically via quantization of finite rate (bandwidth limited) feedback channels.

In this paper, we develop optimal quantization algorithms for optimal sensor relaying by selecting appropriate quantization parameters and quantized values. Quantized information received at the source and relay nodes is then mapped to corresponding transmit powers. The overall objective of the quantization algorithm is to minimize the expected sum of source and relay transmit powers, as in the previous section. For the quantization algorithms, we need to consider the power consumed by source and relay to satisfy the outage probability of the source only during the first two mini-slots (the first half of the collaborative relaying process). The algorithm can then be separately applied to develop quantization for the source-relay pairs during the second half of the collaborative process (when source and relay switch roles).

4.1 Quantization Protocol

The proposed quantization algorithms are associated with a specific protocol for exchanging quantized information between the participants. We describe our protocol below. Quantized information is exchanged between the participants in four sequential steps as follows: In the first step, prior to data transmission, the source node broadcasts a training sequence to the base station as well as the relay. The clusterhead/basestation uses the training sequence to determine $h_{1,D}$ while the relay node simultaneously determines $h_{1,2}$. In the second step, the relay node broadcasts another training sequence along with the quantized value of measured $h_{1,2}$ using the quantization algorithm QRB (described subsequently) to the clusterhead and the source. This is used by the clusterhead to determine $h_{2,D}$. At this point, the clusterhead has perfect $h_{1,D}$ and $h_{2,D}$ measurements and quantized $h_{1,2}$, while the relay and source have measured and quantized $h_{1,2}$ values, respectively. Next, in the third step of the quantization protocol, using either joint or separate quantization algorithms (described subsequently) the station broadcasts quantized values to the source and relay. This value is sufficient for the source and relay to determine their respective transmit power levels for data transmission and relaying.

All algorithms can be implemented at all three nodes, so each node is aware of the mapping from quantization to power levels without separate information. Also each node is aware of the mapping for the other nodes. We also note as a characteristic of the algorithms that power values are not quantized through rounding, rather a set of feasible transmit power values is derived and there is a mapping from channel space to this power space.

[1] Imperfect CSI can also arise due to measurement errors and the time lag between channel state measurements and actual transmission. In this paper, we do not consider measurement errors and also assume slowly time-varying channel parameters.

4.2 Preliminaries

We develop the proposed quantization algorithms by discretizing the parameter space. For notational simplicity, let h denote any of the channel fading parameters $h_{1,2}$, $h_{1,D}$ and $h_{2,D}$. Let $\gamma > 0$ be an (arbitrary) discretization unit such that the range of each h is divided into M discrete and contiguous intervals $I_j = [h_j, h_{j+1})$, where $h_j = j\gamma$ and $j = 0, 1, \ldots M-1$. Each interval is of length γ, except the last interval $[h_{M-1}, \infty)$, which extends to infinity. The channel fading variables h are exponentially distributed and hence we can choose as a design parameter a maximum value, after which h is very small.

First, assume that the range of the source-relay fading coefficient is restricted, i.e., it is known that $h_{1,2} \in [h_a, h_b)$, where $h_a = k\gamma$, $h_b = l\gamma$, $l > k$. Now consider the discretized $\{h_{1,D}, h_{2,D}\}$ space as divided into $N = M^2$ blocks each of dimension $\gamma \times \gamma$. Let $b_{u,t}$ denote the $(u,t)^{th}$ block in this space and let $H_{u,t}$ be the apriori probability that the $h_{1,D}, h_{2,D}$ channel fading coefficients fall into $b_{u,t}$ where $H_{u,t} = Pr.\{u\gamma \leq h_{1,D} < (u+1)\gamma\} \cdot Pr.\{t\gamma \leq h_{2,D} < (t+1)\gamma\}$.

Also let $P_{u,t} = (P^s_{u,t}, P^r_{u,t})$ denote the minimum (source,relay) transmit power vector such that data can be collaboratively transmitted without outage if channel quality falls anywhere within block b_j. We define,

$$\left(P^s_{u,t} = \max\{P_{1,D}\},\, P^r_{u,t} = \max\{P_{2,D}\}\right) \forall (h_{1,D}, h_{2,D}) \in b_{u,t},\ \forall h_{1,2} \in [h_a, h_b) \tag{5}$$

where $P_{1,D}$ and $P_{2,D}$ are obtained using Theorem 1. Note that $P^s_{u,t} = 0$ ($P^r_{u,t} = 0$, resp.) if the block is one of those for which we require outage (non-cooperation from the relay, resp.) i.e the channel configuration corresponding to the given block falls under the threshold s_1^* (s_2^*, resp.). $H_{u,t}$ and $P_{u,t}$ can be obtained in $O(1)$ time for each block $b_{u,t}$.

Consider the N blocks in the discretized $h_{1,D}, h_{2,D}$ space. We state that a block $b_{i,j}$ s-covers (r-covers, resp.) block $b_{k,l}$ if $P^s_{i,j} \geq P^s_{k,l}$ ($P^r_{i,j} \geq P^r_{k,l}$, resp.). Consider a block that is s-covered as well as r-covered. If the source transmits at the source power of the s-covering block and the relay transmits at the relay power of the r-covering block, then we are guaranteed there will be no outage if the realized (actual) channel fading coefficients happen to fall within the covered block. Note that if the source and relay powers of a block are both zero, then we want the block to be in outage and there is no need to cover the block.

5 QBS and QBR: Independent Basestation-Source and Basestation-Relay Quantization Algorithms

We assume the total downlink quantization bandwidth (from clusterhead to source and relay) is k_J, i.e. the clusterhead has k_J bits available to transmit the results of quantizations QBS and QBR to the source and relay. Under independent quantization, the clusterhead, after measuring the exact $h_{1,D}, h_{2,D}$ values, transmits independent quantization information to the source and relay, such that the realized block (under measured h values) will be s-covered by the corresponding source power and and r-covered by the corresponding relay power.

Let k_s and k_r denote the choices for separate quantization bandwidths to source and relay respectively, where $k_s + k_r = k_J$. Let $K_J = 2^{k_J}$, $K_s = 2^{k_s}$ and $K_r = 2^{k_r}$.

The cost of the optimal independent quantization scheme OptIQ is given by

$$\text{Cost}_{\text{OptIQ}} = \min_{k_s+k_r=k_J} (QBS(k_s) + QBR(k_r)) \tag{6}$$

We show that optimal independent quantization algorithms can be obtained through simple reductions from the k-median problem, whose running time is polynomial in the discretization parameter N. As we show below, the running time of $QBS(k_s)$ and $QBR(k_r)$ are $O(NK_s + N\log N)$ and $O(NK_r + N\log N)$ respectively. Thus from Eq.6, the cost of the optimal independent quantization algorithm is $O(NK_J(K_J + \log N))$. We describe $QBS(k_s)$ and $QBR(k_r)$ below.

5.1 Algorithm $QBS(k_s)$

First, QRB quantizes $h_{1,2}$ and this encoded value is sent to the base station, which must implement either joint or separate quantization. Thus the quantization of the $\{h_{1,D}, h_{2,D}\}$ space is conditioned on the received quantized value of $h_{1,2}$, i.e. for every code of $h_{1,2}$, there is a particular quantization in the $\{h_{1,D}, h_{2,D}\}$ space. This quantization must be designed to minimize the source and relay power consumption. Since the optimal result also depends on the quantization of $h_{1,2}$, we must find the optimal quantization of $h_{1,2}$ for which the optimal quantization of the $\{h_{1,D}, h_{2,D}\}$ gives the minimal power consumption. QRB achieves this optimal recursive quantization as described in the last section.

Let $K_s = 2^{k_s}$, i.e. the $\{h_{1,D}, h_{2,D}\}$ space must be encoded by the clusterhead into K_s levels, given the restricted $h_{1,2}$ space. The objective of algorithm QBS is to find a set F_s of K_s blocks (equivalently K_s power levels) such that all blocks are s-covered and the expected transmission power of the source required to satisfy the outage probability over the entire $\{h_{1,D}, h_{2,D}\}$ space and restricted $h_{1,2}$ channel space is minimized. QBS can be expressed as the following minimization problem:

$$QBS : \underset{F_s}{\text{argmin}}\{\sum_{u,t} H_{u,t} \min_{b_{i,j}\in F_s | b_{i,j} s\text{−covers} b_{u,t}} P^s_{i,j}\} \tag{7}$$

We can now relate QBS to the k-median problem. The general k-median problem on a graph G can be formulated as finding the optimal set F of vertices (medians) that satisfies

$$\text{kcost}_G = \underset{F}{\text{argmin}}\{\sum_{u\in G} w_u \min_{v\in F} d_{uv}\} \tag{8}$$

where $|F| \leq k$, w_u is the weight of vertex u and d_{uv} is the minimum distance between u, v in G. While the k-median problem is known to be NP-hard in the general case, (ref. problem ND51 in [1]), it is solvable in polynomial time for trees [9–12] and lines (paths) [13–16]. In this case, QBS can be easily reduced to an instance of k-median on paths by using the fact that the s-cover relationship is transitive.

The reduction is as follows: Sort the N blocks in non-decreasing order of source power $P^s_{u,t}$. Construct the directed path G^* whose vertices are the elements of the sorted list in order. The directed edge cost between adjacent vertices $v_i = b_{u,t}$ and $v_{i+1} = b_{k,l}$ is set to $c_{i,i+1} = P^s_{k,l} - P^s_{u,t}$ while vertex v_i is assigned a weight $w_{v_i} = H_{u,t}$. After running the k-median algorithm on G^* (with $k = K_s$), the source power values of the k selected median nodes are mapped to the K_s quantization levels under QBS i.e the q^{th} quantization level corresponds to the power value of the q^{th} vertex in the k-median solution. Since QBS is implemented at both the source and clusterhead, the power-level to quantization mapping is apriori available to the source and it can transmit at the appropriate level when the the quantized level is fed-back by the clusterhead.

The cost of quantization algorithm QBS is obtained as:

$$\text{cost}_{QBS} = \text{kcost}_{G^*} + \sum_{u,t} H_{u,t} P^s_{u,t} \tag{9}$$

Since s-cover is transitive, the block represented by each vertex in G^*, s-covers all the blocks represented by vertices to its left. If v_i is selected to be one of the k-medians, then its contribution towards being a median for v_j is $(P^s_i - P^s_j)H_j$ while its contribution to being an s-cover for v_j is $P^s_i H_j$. For any set of k-medians from G^*, the difference in cost from QBS is the constant $\sum_j P^s_j H_j$ and thus there is a one-to-one correspondence between the optimal solution to k-median on G^* and the optimal quantization QBS. Thus we have,

Theorem 2. *QBS is an optimal quantization of the source channel.*

We note that k-median on a path can be implemented in $O(kn)$ time [9] Hence the time complexity of QBS is $O(NK_s + N \log N)$.

5.2 Algorithm $QBR(k_r)$

Algorithm $QBR(k_r)$ is identical to $QBS(k_s)$ with s-cover replaced by r-cover and all P^s values replaced by P^r.

6 Joint Source/Relay Quantization

We now consider the case when the clusterhead devotes the entire downlink quantization bandwidth to jointly quantize the source and relay transmit powers. Intuitively, this approach should prove more efficient in terms of total power minimization as the clusterhead can consider quantization over the joint probability distribution of transmit powers and channel fadings, as opposed to treating them independently. Unfortunately the related optimization problem is no longer polynomial. It can be shown that joint quantization is NP-hard by reduction from the general k-median problem. Therefore we consider bounded approximations.

The k-median problem has been the subject of study for several decades. There has been much work on developing efficient heuristics and approximation algorithms [17, 18, 9, 10], particularly on trees and line graphs, as cited earlier. For some more general cases, a constant factor approximation was presented in [18] for graphs with a Euclidean distance metric (a 6-factor approximation). Here, we first present a simple 2-factor approximation that exploits the much simpler structure of joint quantization (as opposed to general k-median) and is easy to implement. Then we develop a $(1+\epsilon)$-FPAS for joint quantization that can approximate the quantized total power to within an arbitrarily close ϵ factor.

For both algorithms, we assume that the total downlink bandwidth for joint quantization is k_J, with $K_J = 2^{k_J}$. As before, the relay first transmits a quantized value corresponding to a range of $h_{1,2}$. Thus both algorithms quantize the $\{h_{1,D}, h_{2,D}\}$ space into K_J values, given the restricted $h_{1,2}$ space. Each quantized value corresponds to a (source,relay) power level pair. After measuring channel quality, the clusterhead broadcasts the corresponding quantized value to the source and relay nodes. Subsequent data transmission is accomplished using the corresponding source and relay power levels. Note that the nodes are each aware of the others power requirements since the algorithm is implemented at both nodes. This is necessary since if the relay power is 0 (no relaying), the source can transmit at the required level during both slots.

6.1 2-factor Approximation for Joint Quantization

Consider an arbitrary block $b_{u,t}$. For notational simplicity, we drop the dual subscripts u, t and use b_j to denote the block. The minimum total power required to transmit this block without outage is given by $P_j = P_j^s + P_j^r$. H_j denotes the source-clusterhead and relay-clusterhead channel fading coefficient probability of b_j. We use P and H to denote this set of minimum total powers (per block) and channel fading probabilities over all blocks.

$QJ1$, the 2-approximation algorithm for joint quantization of source and relay powers is defined as follows: For each block b_j, replace (P_j^s, P_j^r) with $(P_j^{s'}, P_j^{r'}) = (\max(P_j^s, P_j^r), \max(P_j^s, P_j^r))$. Let $P_j' = P_j^{s'} + P_j^{r'}$ and sort the blocks in non-decreasing order of P_j'. Construct the line graph $G^*(P')$ on the vertices corresponding to this sorted list, similar to algorithm QBS, and run the k-median algorithm (with $k = K_J$) on G^*. Let F_J (with $|F_J| = K_J$) denote the subset of blocks corresponding to vertices returned by the k-median algorithm. Each block in F_J corresponds to a quantization level q, $0 \leq q \leq K_J - 1$. The corresponding source and relay transmit powers are $\max(P_i^s, P_i^r)$, where b_i is the block corresponding to quantization level q. In this case, the source and relay transmit powers are identical, thus they will transmit at the same power when a given quantization level is fedback from the clusterhead.

Theorem 3. *QJ1 is a 2-approximation to the optimal joint quantization algorithm.*

Proof. Let cost_{QJ^*} denote the cost of the optimal joint quantization algorithm for the given set of blocks. QJ^* finds a subset of K_J blocks such that all N blocks in the set are s- as well as r-covered by the source and relay power values represented by these K_J blocks and the average total power of the blocks minimally meets the outage probability requirements. Let $\text{cost}_{G^*(P)}$ denote the cost of the optimal k-median algorithm (with $k = K_J$) on directed line graph G^* using power values P and constructed as in the previous section.

We first note that $\text{cost}_{QJ^*} \geq \text{kcost}_{G^*(P)} + \sum_j H_j P_j$. Clearly, equality is met when $P_j^r = 0$ for all blocks. Further, every solution to QJ^* is a solution to k-median on $G^*(P)$. If b_j was a selected block in QJ^*, then $P_j > P_i$ for all blocks b_i that are simultaneously s-covered and r- covered by b_j. Thus in $G^*(P)$, vertex v_j would be to the right of all such vertices v_i. v_j can therefore be a median for these vertices. However the converse is not true and $G^*(P)$ need not correspond to a feasible quantization. The block corresponding to a median in G^* need not be a solution to QJ^*, since $P_j > P_i$ does not imply that b_j can simultaneously s-cover and r-cover b_i. Hence cost_{QJ^*} is larger than the right hand side in these cases.

Let cost_P denote the sum $\text{kcost}_{G^*(P)} + \sum_j H_j P_j$ for any set of powers P. Now consider the system of blocks with (P_j^s, P_j^r) replaced with $(P_j^{s''}, P_j^{r''}) = \left(P_j^s + P_j^r, P_j^s + P_j^r\right)$. Let $P_j'' = P_j^{s''} + P_j^{r''}$ i.e $P_j'' = 2P_j$. Clearly, $\text{cost}_{P"} = 2\text{cost}_P \leq 2\text{cost}_{QJ^*}$, from the discussion above. Note that every solution to k-median on $G^*(P'')$ can be converted to a lower cost feasible solution for k-median on $G^*(P')$ since $P_j'' \geq P_j'$ for all blocks b_j. Thus $\text{cost}_{P'} \leq \text{cost}_{P''}$. Putting the two observations together, we get $\text{cost}_{P'} \leq 2\text{cost}_{QJ^*}$ and hence $QJ1$ is a 2-approximation.

6.2 Fully Polynomial Approximation Scheme

For the (1+ϵ)-FPAS (labeled QJ2), we transform the problem from the $(h_{1,D}, h_{2,D})$ channel space to a covering problem in the 2-dimensional power space as follows: Each block b_t in the $(h_{1,D}, h_{2,D})$ channel space is characterized by the vector (P_t^s, P_t^r) in the power space, $1 \leq t \leq N$. Let $P^s = \{P_t^s\}_t$ and $P^r = \{P_t^r\}_t$ represent the set of source and relay powers. Without loss of generality, we assume that $|P^s| = |P^r| = N$. We construct an $N \times N$ grid of cells $C = (P^s \times P^r)$, where cell c_{ij} represents source power $P_i^s \in P^s$ and relay power $P_j^r \in P^r$, $1 \leq i, j \leq N$. As before, the total power of c_{ij} is represented by $P_{ij} = P_i^s + P_j^r$ while H_{ij} denotes the channel probability of c_{ij}, where $H_{ij} = H_k$ if c_{ij} corresponds to some block b_k, $1 \leq k \leq N$. Note that c_{ij} need not correspond to an actual block and in this case $H_{ij} = 0$.

We define s- and r-covering as before. For this problem we are interested only in joint s- and r-covering. The cells jointly covered by c_{ij} are defined by the rectangle with left bottom endpoint at the origin and top right corner at (P_i^s, P_j^r). However for the algorithm, we prefer to express the joint covering relationship as a directed graph G with $2N - 1$ levels numbered from 2 to $2N$. Level $2N$ consists of only one node c_{NN} with incoming edges from parents $c_{N-1,N}$ and

$c_{N,N-1}$ in level $2N-2$. In general, node c_{ij} is located in level $i+j$ and has two outgoing edges to its two children in level $i+j+1$ ($c_{i+1,j}$ and $c_{i,j+1}$) and two incoming edges from its two children in level $i+j-1$ ($c_{i-1,j}$ and $c_{i,j-1}$). The nodes in level $i+j$ are listed left to right in the order $c_{1,i+j-1}, \ldots, c_{i+j-1,1}$.

We use $\overline{H}_{ij}$ to represent the cumulative channel probability of all cells that are jointly covered by c_{ij}, where $\overline{H}_{ij} = \sum_{k=1}^{i} \sum_{l=1}^{j} H_{kl}$. Henceforth, we drop the dual subscript and use v_t to refer a generic node c_{ij} in G. We will also slightly abuse the notation and let $\overline{H}_a$ denote the set of nodes covered by a as well as the cumulative channel probability of these nodes. Thus for example, $\overline{H}_a \backslash \overline{H}_b$ denotes the nodes covered by a and not b as well as the cumulative value of their channel probabilities.

It can be seen that solving the k-median problem on directed graph G will also lead to a solution to the joint quantization problem. Recent results show that k-median can be solved in polynomial time on a directed tree [12]. However G is not a directed tree (removing the directions on edges leads to cycles) and this result cannot be applied. Instead we are able to develop an FPAS for this problem.

Let r-set $L_r = (v_1, v_2, \ldots, v_r)$ denote an ordered list of r nodes from G. The nodes in an r-set are ordered by increasing levels. For nodes in the same level, we impose a left to right ordering. Note that the ordering ensures $P_1 \leq P_2 \leq \ldots \leq P_r$. Let $QC(L_r)$ denote the quantization cost if all nodes from L_r (and only L_r) were chosen to represent quantization power levels. The total power required for transmitting each cell in the $(h_{1,D}, h_{2,D})$ space is the power level of its nearest ancestor in G belonging to L_r. Thus we have

$$QC(L_r) = \sum_{i=1}^{r} P_i \left(\overline{H}_i \backslash \left(\bigcup_{k=1}^{i-1} \overline{H}_k \right) \right) \tag{10}$$

Define $\overline{H}(L_r) = \bigcup_{k=1}^{r} \overline{H}_k$. $\overline{H}(L_r)$ represents the cumulative channel probability of nodes covered by L_r. Let $S_r^T = \{L_r^1, L_r^2, \ldots, L_r^T\}$ denote an ordered list of T distinct r-sets arranged in non-decreasing order of cost $QC(L_r^i)$, $1 \leq i \leq T$. Each r-set represents a potential sub-solution (with r levels) to the overall K_J level quantization problem. However, we would like to reduce the number of potential sub-solutions without losing essential information. Thus we prune the list by retaining only those potential solutions with a specific channel quality property.

Let $\delta > 0$ be an arbitrary parameter. The operation $\text{Prune}_{r,\delta}(S_r^T)$ returns the reduced list $S_r^{n(r)}$ of size $n(r)$ and is defined as follows:

Algorithm $Prune_{r,\delta}(S_r^k)$

1. Initialize $i \longleftarrow 1$, $j \longleftarrow 1$, $S_r^{n(r)} \longleftarrow \phi$.
2. $tempH \longleftarrow \overline{H}(L_r^i)$.
3. **While** $(QC(L_r^j) \leq (1+\delta) QC(L_r^i))$ **and** $(j \leq k)$
   if $(\overline{H}(L_r^j) \geq tempH)$ $\{tempH \longleftarrow \overline{H}(L_r^j); x \longleftarrow j; j++\}$
   Endif **Endwhile**

4. $S \longleftarrow S \bigcup L_r^x$.
5. **While** ($QC(L_r^j) \leq (1+\delta)QC(L_r^x)$) **and** $(\overline{H}(L_r^j) \leq \overline{H}(L_r^x))$ $\{j++\}$
6. $i \longleftarrow j$. **If** $j \leq k$ Go to Step 2.

Lemma 1. *For every $L_r^j \in S_r^T$, there exists an $L_r^x \in S_r^{n(r)}$, such that either ($QC(L_r^j)/(1+\delta) \leq QC(L_r^x)$ or $(1+\delta)QC(L_r^j) \geq QC(L_r^x)$) and $\overline{H}(L_r^x) > \overline{H}(L_r^j)$.*

Proof. Step 3 ensures that a representative L_r^x is found that has the highest $\overline{H}$ among all L_r^j's with $j \leq x$ and $QC(L_r^x) \leq (1+\delta)QC(L_r^j)$. Once L_r^x is found, step 5 ensures that we keep eliminating all L_r^j's within a δ-neighborhood of L_r^x that have smaller $\overline{H}$ values, i.e $QC(L_r^j) \leq (1+\delta)QC(L_r^x)$ and $\overline{H}(L_r^j) \leq \overline{H}(L_r^x)$.

Lemma 1 indicates a key requirement for the overall algorithm. By selecting the particular r-set with maximum $\overline{H}$ within each δ-neighborhood, we are minimizing the future cost of covering similar costing r-sets while potentially paying a factor of $(1+\delta)$ extra current cost.

We now define the key iterative step to be used in algorithm QJ2. Consider an arbitrary r-set $L_r = (v_1, v_2, \ldots, v_r)$. Let v_r correspond to actual node $u_j \in G$. We define the operation Create_{r+1} that creates new $r+1$ sets from L_r by considering $L_r \bigcup u_k, \forall k, k = j+1, j+2 \ldots$. New nodes are considered in increasing order as per our ordering convention. Thus the last node to be considered corresponds to cell c_{NN}. let $R = N^2$, the size of graph G. Then for each L_r, we create $R - j$ new $r+1$ sets.

$QC(L_{r+1})$ can be calculated in $O(R)$ time as follows: Assume all nodes covered by L_r are marked. Then $\overline{H}_{u_k} \backslash \overline{H}_{L_r}$ can be calculated and marked by breadth-first traversal of G starting from u_k in the reverse direction of arrows.

Finally, $QC(L_{K_J})$ is created by adding to each list in $QC(L_{K_J-1})$ the lowest possible node in G such that all nodes are covered.

Algorithm QJ2 is now defined below. We assume some arbitrary node u_i as the first member of the K_J quantization and proceed as follows:

Algorithm QJ2(u_i, ϵ)

1. $L_1^1 \longleftarrow (u_i)$, $S_1^1 \longleftarrow (L_1^1)$, $n(1) \longleftarrow 1$, $\delta \longleftarrow \frac{\epsilon}{2K_J}$.
2. **For** $r = 1$ to $K_J - 1$
   $S_{r+1}^T \longleftarrow \text{Create}_{r+1}(S_r^{n(r)})$; Sort S_{r+1}^T by Quantization Costs $QC(L_{r+1}^i)$'s ;
   $S_{r+1}^{n(r+1)} \longleftarrow \text{Prune}_{r+1,\delta}(S_{r+1}^T)$;
3. Cost(QJ2)$\longleftarrow QC(L_{K_J}^1)$. Return $L_{K_J}^1$.

The minimum cost algorithm is given by $QJ2 = \min_i QJ2(u_i)$. We now analyze the complexity and correctness of QJ2.

Theorem 4. *For $0 < \epsilon < 1$, $QJ2(\epsilon)$ is a FPAS for the joint quantization problem.*

Proof. We need to show that (a) the solution returned is within a factor of $1+\epsilon$ of the optimal solution and (b) the running time is polynomial in $1/\epsilon$.

For the first part, we have to show that our policy of selecting the r-set with the largest $\overline{H}$ within a δ-neighborhood is not suboptimal, i.e it does not create solutions with a cost that exceeds a $1+\epsilon$ factor of the optimal solution. Assume some L^q_{r-1} is optimal for the inductive hypothesis. Some L^1_1 is certainly optimal as we run R instances of the algorithm starting at each node. Let L^x_r be the r-set chosen during the r^{th} stage of pruning and let L^y_r be the optimal choice in the same δ-neighborhood, which was not chosen because of L^x_r. Let v_x and v_y be the two nodes that created the respective r-sets from L^q_{r-1}.

Let $A = \overline{H}_{v_x} \backslash \overline{H}_{L^q_{r-1}}$ and $B = \overline{H}_{v_y} \backslash \overline{H}_{L^q_{r-1}}$ be the marginal $\overline{H}$ contributions of the nodes. AP_x and BP_y are the marginal costs of adding v_x and v_y respectively to L^q_{r-1}. Also let $A_1 = A \bigcup \overline{H}_{L^q_{r-1}}$ and $B_1 = B \bigcup \overline{H}_{L^q_{r-1}}$

Since L^x_r was chosen over L^y_r, we know from lemma 1 that $A_1 \geq B_1$ and also $AP_x \leq (1+\delta)BP_y$. Now let u_t be a node that is added during step $r+1$. We consider two cases:

First, let u_t be the terminal node, i.e after u_t all nodes in G are covered. The cost of $L^x_r \bigcup u_t$ and $L^y_r \bigcup u_t$ are given by

$$C_1 = QC(L^q_{r-1}) + AP_x + P_t(\overline{H}_t \backslash A_1) \tag{11}$$

$$C_2 = QC(L^q_{r-1}) + BP_y + P_t(\overline{H}_t \backslash B_1), \tag{12}$$

respectively. From the above observations on A, B, A_1, B_1, we have $\overline{H}_t \backslash A_1 \leq \overline{H}_t \backslash B_1$ and thus $C_1 \leq (1+\delta)C_2 < (1+\epsilon)C_2$ since $\delta = \epsilon/2K_J$.

Suppose u_t is a non-terminal node. We argue that there always exists another vertex u_w to be added in future whose cost will be within a $1+\epsilon$ factor by going with L^x_r instead of L^y_r. Suppose now $\overline{H}_t \backslash A > \overline{H}_t \backslash B$ even though $A > B$. Thus u_t has a larger overlap with B. Clearly the Quantization Cost of $L^y_r \bigcup u_t$ can be unboundedly smaller than the cost of $L^x_r \bigcup u_t$. Now consider another additional node u_w that is added later than u_t such that A and B are both covered. u_w exists since u_t is non-terminal and A and B have to be covered before the algorithm terminates. By definition, $P_w \geq P_t$. Consider the costs of $L^x_r \bigcup u_t \bigcup u_w$ and $L^y_r \bigcup u_t \bigcup u_w$ given by

$$C_1 = QC(L^q_{r-1}) + AP_x + P_t(\overline{H}_t \backslash A_1) + P_w(\overline{H}_w \backslash (\overline{H}_t \bigcup A_1)) \tag{13}$$

$$C_2 = QC(L^q_{r-1}) + BP_y + P_t(\overline{H}_t \backslash B_1) + P_w(\overline{H}_w \backslash (\overline{H}_t \bigcup B_1)), \tag{14}$$

respectively. Now using the fact that $P_w \geq P_t$ and $A_1 \geq B_1$, we can see that again $C_1 \leq (1+\delta)C_2 < (1+\epsilon)C_2$ as desired. Hence we have shown that choosing the L^x_r representative as defined in the algorithm is not suboptimal by larger than a $1+\delta$ factor at each stage.

Since at each stage we are no more than a $1+\delta$ factor from the optimal, after K_J stages, we will be within a factor

$$(1+\delta)^{K_J} = (1 + \frac{\epsilon}{2K_J})^{K_J} \leq (1+\epsilon) \tag{15}$$

For the second part, we need to show that the running time is polynomial in $1/\epsilon$. Algorithm Prune takes clearly takes time $O(T)$ assuming all costs and $\overline{H}$ values are known. Let $F = QC(L_r^T)$, i.e F is the maximum quantization cost in S_r^T. Note that $F \leq P_{max}$, the maximum allowed transmission power for source and relay. This is usually imposed as a practical limitation. Now the size $n(r)$ of the pruned list can be determined as follows: Let $S_r^{n(r)} = \{L_r^1, \ldots, L_r^{n(r)}\}$. By lemma 1, we have $QC(L_r^i) > (1+\delta)QC(L_r^{i-1}$ and $\overline{H}(L_r^i) > \overline{H}(L_r^{i-1}$. Since successive elements in $S_r^{n(r)}$ differ by at least a $(1+\delta)$ factor, we get

$$n(r) \quad \leq \quad 2 + \log_{1+\delta} F \quad = \quad 2 + \frac{\ln F}{\ln 1+\delta} \quad \leq \quad 2 + \frac{2\ln F}{\delta} \quad = \quad O(\frac{K_J \ln F}{\epsilon}) \quad (16)$$

The time complexity of $QJ2(\epsilon)$ can now be determined as follows. The Create$_{r+1}$ operation of step 2 takes $O(|n(r)|R^2) = O(K_J \ln FR^2/\epsilon)$ time since $O(R)$ nodes are separately added to each existing list and each addition takes $O(R)$. The size T of the new $r+1$-list S_{r+1}^T is $O(|n(r)|R)$ and so sorting takes $O(|n(r)|R\log(|n(r)|R))$ time. Finally, the pruning operation takes $O(|n(r)|R)$ time. Hence the overall complexity is dominated by the first step which is $O(K_J \ln FR^2/\epsilon)$. Since the algorithm is called $O(R)$ times (one for each u_i), the total complexity is $O(K_J \ln FR^3/\epsilon)$ which is polynomial in $1/\epsilon$, K_J and F.

7 Relay to Clusterhead Quantization Algorithm QRB

Finally, we describe the Quantization between relay and base station. Let k_b be the relay to clusterhead quantization bandwidth and $K_b = 2^{k_b}$. If the clusterhead is a more powerful node than an ordinary sensory, then $K_b << \{K_s, K_r\}$. Let $QRB(k, t\gamma)$ represent the optimal cost of quantizing the range of $h_{1,2}$ represented by $0 < h_{1,2} < t\gamma$ into k levels, $1 \leq t \leq M$. In the case of independent clusterhead to source/relay quantization, QRB can be specified by the following dynamic program.

$$QRB_{k,t\gamma} = \min_{1 \leq r < t} \{QRB_{k-1,r\gamma} + \text{OptIQ}_{K_J}^{r\gamma,t\gamma}\} \tag{17}$$

where the second term is a call to the independent quantization algorithm with restricted $h_{1,2}$ and total bandwidth parameters as described. The boundary conditions are evaluated at $QRB_{1,t\gamma_1}$ for $1 \leq t \leq N_1$, using the recursive calls and the fact that $QRB_{0,t\gamma_1} = 0$. QRB is calculated in a bottom-up manner with increasing t and k.

8 Conclusions

In this paper, we address the problem of developing power control algorithms for sensor networks with collaborative relaying under bandwidth constraints, via quantization of finite rate feedback channels. We develop a system model using channel fading parameters as the metric and are able to develop power control policies that minimize aggregate source and relay power. Perfect Channel State Information is not available due to bandwidth constraints and thus we focused on developing quantization algorithms. We develop two quantization protocols: independent quantization and joint quantization of source and relay channels by the clusterhead. Our proposed quantization problem is related to the k-median problem. Independent quantization can be reduced to k-median on line graphs and hence easily solved in polynomial time. However joint quantization is NP-hard and therefore we are forced to develop approximations. We show an easy to implement 2-factor approximation and then develop a Fully Polynomial Approximation Scheme that can approach the optimal to within a $(1+\epsilon)$ factor. In future work, we will work on further simplification of the FPAS.

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