1. (a) The sample space for this experiment is given by \( \Omega = \{(a,b) : a, b = W, O, B\} \) where \( W, O \) and \( B \) represent the white, orange and black balls, respectively.

i. The set of values for \( X \) are given by \( \mathcal{X} = \{0, 1, 2\} \).

ii. The values for \( P(X = k) \) are calculated as follows. Let \( A = \) the event that the first ball is orange, and let \( B = \) the event that the second ball is orange.

Then \( p(A) = \frac{3}{8} = \frac{1}{3} \). Also \( P(B|A) = \frac{2}{8} = \frac{1}{4} \) and \( P(B|A^c) = \frac{3}{8} \). Also, \( P(B^c|A^c) = \frac{5}{8} \). and \( P(B^c|A) = \frac{6}{8} = \frac{3}{4} \).

Now
\[
P(X = 0) = P(A^c \cap B^c) = P(A^c)P(B^c|A^c) = \frac{2}{3} \times \frac{5}{8} = \frac{5}{12}
\]
\[
P(X = 1) = P(A) + P(A^c \cap B) = P(A) + P(A^c \cap B) + P(A^c \cap B)
\]
\[
= P(B^c|A^c)P(A) + P(B^c|A)P(A^c) = \frac{1}{2}
\]
\[
P(X = 2) = P(A \cap B) = P(B|A)P(A) = \frac{1}{12}
\]

2. (a) The sample space of the random variables are given below.

i. \( X \in \Omega_X = \{1, 2, 3, 4, 5, 6\} \).

ii. \( Y \in \Omega_Y = \{1, 2, 3, 4, 5, 6\} \).

iii. \( Z \in \Omega_Z = \{2, 3, 4, \ldots, 12\} \).

iv. \( W \in \Omega_W = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\} \).

(b) i.

\[
P(X = 1) = 1/36, \ P(X = 2) = 3/36 = 1/12, \ P(X = 3) = 5/36, \ P(X = 4) = 7/36, \ P(X = 5) = 9/36, \ P(X = 6) = 11/36
\]

ii.

\[
P(Y = 6) = 1/36, \ P(Y = 5) = 3/36 = 1/12, \ P(Y = 4) = 5/36, \ P(Y = 3) = 7/36, \ P(Y = 2) = 9/36, \ P(Y = 1) = 11/36
\]

iii.

\[
P(Z = 2) = 1/36, \ P(Z = 3) = 2/36, \ P(Z = 4) = 3/36, \ P(Z = 5) = 4/36, \ P(Z = 6) = 5/36, \ P(Z = 7) = 6/36, \ P(Z = 8) = 5/36, \ P(Z = 9) = 4/36, \ P(Z = 10) = 3/36, \ P(Z = 11) = 2/36, \ P(Z = 12) = 1/36.
\]
iv.

\[ P(W = -5) = 1/36, \ P(W = -4) = 2/36, \ P(W = -3) = 3/36, \]

\[ P(W = -2) = 4/36, \ P(W = -1) = 5/36, \ P(W = 0) = 6/36, \]

\[ P(W = 1) = 5/36, \ P(W = 2) = 4/36, \ P(W = 3) = 3/36, \]

\[ P(W = 4) = 2/36, \ P(W = 5) = 1/36 \]

3. 

\[ \text{Prob(At most one six appears)} \]

\[ = \left( \frac{5}{6} \right)^3 + 3 \times \frac{1}{6} \times \left( \frac{5}{6} \right)^2 \]

4. The sample space of \( X \) is \( \Omega_X = \{ n - 2h : \text{for } h = 0, 1, 2, ..., n \} \).

Let \( Y \) the number of tails that show up. Then \( X = n - 2Y \).

\[ P(X = n-2h) = P(n-2Y = n-2h) = P(Y = h) = \binom{n}{h} (1-p)^h p^{n-h} \text{ for } h = 0, 1, ..., n \]

5. (a) The sample space of \( X \) is given by \( \Omega_X = \{ r, r+1, r+2, r+3, ... \} \).

(b) Clearly for \( n < r, P(X = n) = 0 \). For \( n \geq r \),

\[ P(X = n) = \text{Prob}(r - 1 \text{ heads appear in the first } n - 1 \text{ tosses and head appears on the } n\text{th toss}) \]

\[ = \binom{n-1}{r-1} p^{r-1}(1-p)^{n-1-(r-1)} \times p = \binom{n-1}{r-1} p^r(1-p)^{n-r} \text{ for } n \geq r \]

(c) Two properties need to be verified.

i. \( P(X = n) \geq 0 \) for all \( n \), which is of course true.

ii. \( \sum_{n=r}^{\infty} P(X = n) = 1 \). We need to show that

\[ \sum_{n=r}^{\infty} \binom{n-1}{r-1} p^r(1-p)^{n-r} = 1 \]

or

\[ \sum_{j=0}^{\infty} \binom{j + r - 1}{r-1} (1-p)^j = \frac{1}{p^r} \]

Let \( f(x) = \sum_{j=0}^{\infty} \binom{j + r - 1}{r-1} x^j \). We will show that \( f(x) = \frac{1}{(1-x)^r} \). To show this we use induction on \( r \). For \( r = 1 \) and \( r = 2 \) this is easy to verify. Now suppose it is true for an arbitrary \( r \). We will show that it is also true for \( r+1 \).
\[
\sum_{j=0}^{\infty} \left( \frac{j+r}{r} \right) x^j = \sum_{j=0}^{\infty} \frac{j+r}{r} \left( \frac{j+r-1}{r-1} \right) x^j
\]

\[
= \frac{1}{r} \sum_{j=0}^{\infty} (j+1) \left( \frac{j+r-1}{r-1} \right) x^j + \frac{r-1}{r} \sum_{j=0}^{\infty} \left( \frac{j+r-1}{r-1} \right) x^j
\]

Now the first summation above is equal to \( \frac{d}{dx} xf(x) \) and the second term is \( f(x) \). Thus

\[
\sum_{j=0}^{\infty} \left( \frac{j+r}{r} \right) x^j = \frac{1}{r} \frac{d}{dx} xf(x) + \frac{r-1}{r} f(x) = \frac{1}{(1-x)^{r+1}}
\]

6. The joint probability density function of \( X \) and \( Y \) is given by

\[
f_{XY}(x, y) = \frac{e^{-y}}{y}, \quad 0 < x < y, \quad 0 < y < \infty
\]

We have

\[
E[X^2|Y = y] = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y) \, dx
\]

Thus we need to find \( f_{X|Y}(x|y) \). This is computed as follows.

\[
f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}
\]

Now

\[
f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) \, dx = \int_{0}^{y} \frac{e^{-y}}{y} \, dx = e^{-y} \quad \text{for} \quad 0 < y < \infty
\]

Thus

\[
f_{X|Y}(x|y) = \frac{1}{y} \quad \text{for} \quad 0 < x < y, \quad 0 < y < \infty
\]

Therefore

\[
E[X^2|Y = y] = \int_{0}^{y} x^2 \frac{1}{y} \, dx = \frac{y^2}{3}
\]

7. Let \( X \) denote the time it takes for the miner to get to safety and let \( Y \) denote the tunnel he chooses the first time.

(a)

\[
E[X] = \frac{1}{3} \left( E[X|Y = 1] + E[X|Y = 2] + E[X|Y = 3] \right)
\]

\[
= \frac{1}{3} \left( 2 + E[X] + 5 + 10 + E[X] \right) \quad \implies \quad E[X] = 17 \text{ hours.}
\]
(b) 

\[ E[X] = \frac{1}{3} (E[X|Y = 1] + E[X|Y = 2] + E[X|Y = 3]) \]

\[ = \frac{1}{3} (2 + 1/2 \times 5 + 1/2 \times (10 + 5)) + \frac{1}{3} (5) + \frac{1}{3} (10 + 1/2 \times (2 + 5) + 1/2 \times 5) \]

\[ = 11 \text{ hours.} \]

(c) In the first case,

\[ E[X^2] = \frac{1}{3} (E[X^2|Y = 1] + E[X^2|Y = 2] + E[X^2|Y = 3]) \]

\[ = \frac{1}{3} \left( E(2 + X)^2 + 5^2 + E(10 + X)^2 \right) \implies E[X^2] = 537 \]

Thus \( \text{var}(X) = 248. \)

In the second case,

\[ E[X^2] = \frac{1}{3} (E[X^2|Y = 1] + E[X^2|Y = 2] + E[X^2|Y = 3]) \]

\[ = \frac{1}{3} \times \frac{1}{2} \times ((2+5)^2+(2+10+5)^2)+\frac{1}{3} (5^2)+\frac{1}{3} \times \frac{1}{2} \times ((10+2+5)^2+(10+5)^2) = 150.33 \]

Thus \( \text{var}(X) = 29.33. \)

8. (a) Let \( C_i \) for \( i = 1, 2, 3 \) denote the coin that what selected. Then

\[ P(N = n) = \sum_{i=1}^{3} P(N = n|C_i)P(C_i) = \frac{1}{3} \left( \sum_{i=1}^{3} P(N = n|C_i) \right) \]

where

\[ P(N = n|C_i) = \binom{10}{n} p^n(1-p)^{10-n} \]

where \( p = .2, .5, .7 \) for \( i = 1, 2, 3, \) respectively.

(b) The game is fair if the average number of heads equals the average number of tails. In other words if \( E[N] = 5. \) But in this case \( E[N] = \frac{1}{3}(2 + 5 + 7) = \frac{14}{3}. \)