1. Prove that if the number of states in a Markov chain is $M$, and if state $j$ can be reached from state $i$, then it can be reached in $M$ steps or less.

2. A transition probability matrix $P$ is called doubly stochastic if the sum over each column of $P$ equals one, i.e., $\sum_i P_{ij} = 1$, for all $j$. If such a chain is irreducible and aperiodic and has $M + 1$ states $0, 1, \ldots, M$, show that its limiting probabilities are given by

$$\pi_j = \frac{1}{M + 1}, \quad j = 0, 1, \ldots, M$$

3. A particle moves on a circle through points which have been marked (in a clockwise order) $0, 1, 2, 3, 4$. At each step it has a probability $p$ of moving to the right (clockwise) and $1 - p$ of moving to the left (counterclockwise). Let $X_n$ denote the position of the particle after the $n$th step.

(a) Argue that $\{X_n, n \geq 0\}$ is a Markov chain. Find the transition probability and the limiting probabilities of this chain.

4. Classify the states of the markov chains whose transition probability matrix are given below.

$$P_1 = \begin{bmatrix}
.5 & 0 & .5 & 0 & 0 \\
.25 & .5 & .25 & 0 & 0 \\
.5 & 0 & .5 & 0 & 0 \\
0 & 0 & 0 & .5 & .5 \\
0 & 0 & 0 & .5 & .5 \\
\end{bmatrix}$$

$$P_2 = \begin{bmatrix}
1/4 & 3/4 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1/3 & 2/3 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

5. Let $X_n$ denote the capital of a gambler at the end of the $n$th play. His strategy is as follows. If his capital is 4 dollars or more, then he bets 2 dollars which earns him 4, 3, or 0 dollars with probabilities .2, .3 and .5, respectively. If his capital is 1,2 or 3 dollars then he bets 1 dollar which earns him 2 or 0 dollars with probabilities .55 and .45, respectively. When his capital is 0, he stops.

(a) Let $Y_n$ denote his earning at the $n$th play. Calculate $P(Y_{n+1} = k | X_n = i)$.
(b) Show that $\{X_n\}$ is a Markov chain.
(c) Compute the transition probability matrix of \( \{X_n\} \).

(d) Classify the states.

6. Consider an equipment which is currently in use. If it fails it is replaced with an identical one. If that one fails it is again replaced with an identical one, and so on. Let \( p_k \) denote the probability that a new equipment lasts for \( k \) units of time. Let \( Y_n \) denote the age of the equipment at the end of the \( n \)th period. For any equipment \( Y_{n+1} = 0 \) if the equipment failed during the \( n + 1 \)st period and \( Y_{n+1} = Y_n + 1 \) if it did not.

(a) Show that \( \{Y_n, n \geq 0\} \) is a Markov chain.

(b) Find its transition probability matrix.

(c) Classify the states.

7. Consider a Markov chain with states 0, 1, 2, 3 and 4. Suppose \( P(0, 4) = 1 \) and when the chain is in any state \( i, i > 0 \), the next state is equally likely to be any one of the states 0, 1, \ldots, \( i - 1 \). Find the limiting probabilities of this chain.