1. (a) Let \( W \) = the event of winning at least one lottery. In the case that he buys 50 tickets in one lottery (in this case he can only have one winning ticket),

\[
P(W) = \frac{50}{100} = .5
\]

In the case that he buys one ticket in 50 lotteries, he can win 1, 2, 3, ..., 50 times.

\[
P(W) = 1 - P(W^c)
\]

Let \( A_i \) = event that he does not win in the \( i \)th lottery. Then

\[
W^C = \cap_{i=1}^{50} A_i \implies P(W^C) = P(\cap_{i=1}^{50} A_i)
\]

Now \( P(A_i) = \frac{99}{100} \) and \( A_1, A_2, \ldots, A_{50} \) are independent. Thus

\[
P(\cap_{i=1}^{50} A_i) = \prod_{i=1}^{50} P(A_i) = (.99)^{50}
\]

and

\[
P(W) = 1 - (.99)^{50} = .3949
\]

(b) Let \( G_1 \) be the winnings in one lottery.

i. When he buys 50 tickets in one lottery, he has the chance of winning one lottery with probability 0.5. Thus

\[
\bar{G}_1 = G_1 \times .5 = G_1/2.
\]

ii. When he buys 1 ticket in 50 lotteries he has the chance of winning 1, 2, 3, ..., or 50 lotteries and

\[
P(i) = \text{probability of winning exactly } i \text{ lotteries} = \binom{50}{i} (.01)^i (.99)^{50-i}
\]

Also if he wins \( i \) lotteries, his total winning would be \( iG_1 \). Thus

\[
\bar{G}_{50} = \sum_{i=1}^{50} \binom{50}{i} (.01)^i (.99)^{50-i} \times (iG_1) = G_1 \sum_{i=1}^{50} i \binom{50}{i} (.01)^i (.99)^{50-i} = .5G_1
\]

Therefore, his expected winning (in both cases) is the same.
2. (a) The sample space can be specified as

\[ S = \{(u_1, b), (u_1, r), (u_2, b), (u_2, r)\} \]

where

- \((u_1, b)\) = Outcome that urn 1 is selected and blue ball is drawn
- \((u_1, r)\) = Outcome that urn 1 is selected and red ball is drawn
- \((u_2, b)\) = Outcome that urn 2 is selected and blue ball is drawn
- \((u_2, r)\) = Outcome that urn 2 is selected and red ball is drawn

Now we need to find the probabilities of each outcome which are not directly available. For \(i = 1, 2\) let \(U_i = \) the event that urn \(i\) is selected. Also let \(B = \) the event that the ball that is drawn is blue and let \(R = \) the event that the ball that is drawn is red. Note that for example

\[ U_1 = \{(u_1, b), (u_1, r)\} \]

and

\[ U_2 = \{(u_2, b), (u_2, r)\} \]

Also

\[ B = \{(u_1, b), (u_2, b)\} \]

and

\[ R = \{(u_1, r), (u_2, r)\} \]

We know that

\[ P(U_1) = P(U_2) = 1/2 \]

Also

\[ P(B|U_1) = 1 - P(R|U_1) = 1/2 \]

and

\[ P(B|U_2) = 1 - P(R|U_2) = 1/4 \]

Thus

\[ P(\{(u_1, b)\}) = P(U_1 \cap B) = P(U_1)P(B|U_1) = 1/4 \]

\[ P(\{(u_1, r)\}) = P(U_1 \cap R) = P(U_1)P(R|U_1) = 1/4 \]

\[ P(\{(u_2, b)\}) = P(U_2 \cap B) = P(U_2)P(B|U_2) = 1/8 \]

\[ P(\{(u_2, r)\}) = P(U_2 \cap R) = P(U_2)P(R|U_2) = 3/8 \]

This completes the description of the probability space.

(b) We want \(P(U_2 | B)\).

Using the Bayes’ rule

\[ P(U_2 | B) = \frac{P(B|U_2)P(U_2)}{P(B)} \]
Now
\[ B = \{(u_1, b), (u_2, b)\} \implies P(B) = 1/4 + 1/8 = 3/8 \]
Thus
\[ P(U_2|B) = \frac{(1/4) \times (1/2)}{3/8} = 1/3 \]

**Note:** We can also calculate \( P(B) \) as
\[ P(B) = P(B|U_1)P(U_1) + P(B|U_2)P(U_2) = 3/8 \]

3. (a)
\[ S = \{(f, h), (b, h), (f, t), (b, t)\} \]

We need to find the probability measure \( P \). For this we find the probability of elementary events. Let \( H = \)event that head shows up, \( T = \)event that tail shows up, and let \( F = \)event that the wheel stops on \( f \). Then as subsets of \( S \) these are given by
\[ H = \{(f, h), (b, h)\}, T = \{(f, t), (b, t)\} \text{ and } F = \{(f, h), (f, t)\} \]
Now we have \( P(F) = .7 \), \( P(H|F) = .5 \) and \( P(H|F^c) = .2 \). Therefore,
\[
P(\{(f, h)\}) = P(H \cap F) = P(H|F)P(F) = .35
\]
\[
P(\{(b, h)\}) = P(H \cap F^c) = P(H|F^c)P(F^c) = .06
\]
\[
P(\{(f, t)\}) = P(T \cap F) = P(T|F)P(F) = .35
\]
\[
P(\{(b, t)\}) = P(T \cap F^c) = P(T|F^c)P(F^c) = .24
\]

(b)
\[ \text{Pr(Head is obtained)} = P(H) = .41 \]

(c) \[ \text{Pr}(f \text{ is obtained given that the final outcome is head}) \]
\[ = P(F|H) = \frac{P(H|F)P(F)}{P(H)} = \frac{35}{41} \]

4. Let \( R_n \) be the event that it rains on the \( n \)th day. Then
\[ P(R_n|R_{n-1}) = P(R_n^c|R_{n-1}^c) = p. \]

Let \( P_n = P(R_n) \). Then
\[
P_n = P(R_n|R_{n-1})P(R_{n-1}) + P(R_n|R_{n-1}^c)P(R_{n-1})
\]
\[
= pP_{n-1} + (1 - p)(1 - P_{n-1}) = (2p - 1)P_{n-1} + 1 - p
\]
Since the same relation holds for $P_{n-1}$ in terms of $P_{n-2}$, we get
\[
P_n = 1 - p + (2p - 1)\{ (2p - 1)P_{n-2} + 1 - p \} \\
= 1 - p + (1 - p)(2p - 1) + (2p - 1)^2P_{n-2}
\]
Continuing in this fashion (formally by an induction argument) we get
\[
P_n = (1 - p)\{ 1 + (2p - 1) + (2p - 1)^2 + (2p - 1)^3 + \ldots + (2p - 1)^{n-2} \} + (2p - 1)^{n-1}P_1.
\]
Or
\[
P_n = .5[1 - (2p - 1)^{n-1}] + (2p - 1)^{n-1}P_1.
\]
5. (a) Let $(\alpha_1, \alpha_2) = \text{event that both systems are operational}.$
   $(\alpha_1^c, \alpha_2) = \text{event that system 1 is not operational and system 2 is operational}.$
   $(\alpha_1, \alpha_2^c) = \text{event that system 1 is operational and system 2 is not operational}.$
   $(\alpha_1^c, \alpha_2^c) = \text{event that neither system is operational}.$
   Then $\mathcal{S} = \{ (\alpha_1, \alpha_2), (\alpha_1^c, \alpha_2), (\alpha_1, \alpha_2^c), (\alpha_1^c, \alpha_2^c) \}.$
   Now we need to find the probability measure on $\mathcal{S}$. We have
   \[ P((\alpha_1, \alpha_2)) = P(\alpha_1|\alpha_2)P(\alpha_2) = (3)(.5) = .15 \]
   Let $A = \text{event that system 1 is operational} = \{ (\alpha_1, \alpha_2), (\alpha_1, \alpha_2^c) \}.$ We have
   \[ P(A) = P(\alpha_1, \alpha_2) + P(\alpha_1, \alpha_2^c) = .2. \]
   Thus $P(\alpha_1, \alpha_2^c) = .2 - .15 = .05.$
   Similarly let $B = \text{event that system 2 is operational} = \{ (\alpha_1, \alpha_2), (\alpha_1^c, \alpha_2) \}.$ We have
   \[ P(B) = P(\alpha_1, \alpha_2) + P(\alpha_1^c, \alpha_2) = .5. \]
   Thus $P(\alpha_1^c, \alpha_2) = .5 - .15 = .35.$
   Finally, $P(\alpha_1^c, \alpha_2^c) = 1 - .15 - .05 - .35 = .45.$

(b) Let $C = \text{event that the system is not operational, i.e.,}$
   \[ C = \{ (\alpha_1^c, \alpha_2), (\alpha_1, \alpha_2^c), (\alpha_1^c, \alpha_2^c) \}. \]
   Thus $P(C) = .85.$ Now we need to find $P(B^c|C).$ Since $B^c \subset C,$ we have
   \[ P(B^c|C) = \frac{P(B^c)}{P(C)} = \frac{.5}{.85} = \frac{10}{17}. \]
6. (a) The event that the first man wins is given by
   \[ A_1 = \{ H, TTH, TTTTTH, TTTTTTTH, \ldots \} \]
   Therefore,
   \[ P(A_1) = P(\text{first man winning}) = P(H) + P(TTH) + P(TTTTTH) + \ldots \]
   \[ = p + (1 - p)^3p + (1 - p)^6p + \ldots \]
   \[ = p \sum_{k=0}^{\infty} (1 - p)^{3k} = \frac{p}{1 - (1 - p)^3}. \]
(b) The event that the second man wins is given by

\[ A_2 = \{ TH, TTTTH, TTTTTTTH, \ldots \} \]

and

\[ P(A_2) = P(\text{second man winning}) = \frac{p(1-p)}{1 - (1-p)^3} \]

Similarly

\[ P(A_3) = P(\text{third man winning}) = \frac{p(1-p)^2}{1 - (1-p)^3} \]

(c) We need to show that \( 1 - (1-p)^3 < 3p \) or \( 3 - p(3-p) < 3 \) which is true since \( p(3-p) > 0 \).

7. Let

\( (r, c) = \text{event that the resistor is good and the capacitor is good.} \)
\( (r^t, c) = \text{event that the resistor is bad and the capacitor is good.} \)
\( (r, c^t) = \text{event that the resistor is good and the capacitor is bad.} \)
\( (r^t, c^t) = \text{event that the resistor is bad and the capacitor is bad.} \)

(a) Then we can let \( \mathcal{S} = \{(r, c), (r^t, c), (r, c^t), (r^t, c^t)\} \). We need to also find the probability of each outcome. We have

\[ P(\{(r, c)\}) = P(R \cap C) = P(C|R)P(R) = \frac{1}{42} = \frac{1}{8} \]

\[ P(\{(r^t, c)\}) = P(R^c \cap C) = P(R^c|C)P(C) = \frac{2}{3} P(C) \]

We need \( P(C) \). But

\[ P(C) = P(C|R)P(R) + P(C|R^c)P(R^c) \]

Also

\[ P(C|R^c)P(R^c) = P(R^c|C)P(C) \]

Therefore,

\[ P(C) = \frac{P(C|R)P(R)}{1 - P(R^c|C)} = \frac{3}{8} \]

Thus \( P(\{(r^t, c)\}) = 1/4 \).

\[ P(\{(r, c^t)\}) = P(R \cap C^c) = P(C^c|R)P(R) = [1 - P(C|R)]P(R) = 3/8 \]
\[ P(\{(r^t, c^t)\}) = P(R^c \cap C^c) = P(R^c) - P(R^c \cap C) \]
\[ = P(R^c) - P(R^c|C)P(C) = 1/2 - (2/3)(3/8) = 1/4 \]
(b) $P(C) = 3/8$

(c) Let $A$ be the event that the filter does not work. Then

$A = \{(r^t, c), (r, c^t), (r^t, c')\}$

and $P(A) = 7/8$. We now need to find $P(R^c|A)$.

$$P(R^c|A) = \frac{P(R^c \cap A)}{P(A)}$$

Now since $R^c \subseteq A$, we have $R^c \cap A = R^c$ and

$$P(R^c|A) = \frac{P(R^c)}{P(A)} = \frac{1/2}{7/8} = 4/7$$