1. The two-level random process is defined by

\[ X(t) = \sum_{i=-\infty}^{\infty} b_i p_T(t - iT) \]

where

\[ p_T(t) = \begin{cases} 
1 & 0 \leq t \leq T \\
0 & \text{otherwise}
\end{cases} \]

and \( \{b_i\}_{i=-\infty}^{\infty} \) is a sequence of independent identically distributed random variables with \( P(b_i = +1) = P(b_i = -1) = 1/2 \).

(a) Sketch a typical sample function of \( \{X(t)\} \).

(b) Find the mean value \( m_X(t) = E[X(t)] \).

(c) Find \( R_{XX}(0.5T, 0.7T) \) and \( R_{XX}(0.9T, 1.1T) \).

(d) Is the process wide sense stationary?

2. Let \( X \) and \( Y \) be two independent Gaussian random variables each with mean zero and variance one. For \( t \in (-\infty, +\infty) \) define

\[ Z(t) = X \cos(2\pi t) + Y \sin(2\pi t) \]

(a) Determine the mean and autocorrelation function of the random process \( \{Z(t)\} \). Is this process wide sense stationary?

(b) Prove that the process \( \{Z(t)\} \) is a Gaussian random process.

(c) Consider the three random variables obtained by sampling \( \{Z(t)\} \) at times \( t = 0, 0.25, 0.5 \). Find the covariance matrix for these random variables. Do these random variables have a joint probability density function? Explain.

3. Let \( X \) be a random variable uniformly distributed on the interval \([0, 1]\). Define the discrete time random process \( \{Y_k\}_{k=1}^{\infty} \) by

\[ Y_k = X^k - \frac{1}{k+1}, k = 1, 2, 3, ... \]

(a) Find the mean and autocorrelation function of \( \{Y_k\}_{k=1}^{\infty} \). Is this process WSS?

(b) Find the probability distribution function \( F_{Y_k}(y) \).

4. \( \{X_t\} \) and \( \{Y_t\} \) are two uncorrelated wide-sense stationary (WSS) Gaussian random processes each with mean zero. The power spectral density of \( \{X_t\} \) is given by

\[ S_X(f) = \begin{cases} 
1, & |f| \leq 5 \\
0, & \text{otherwise}
\end{cases} \]

and the autocorrelation function of \( \{Y_t\} \) is given by \( R_Y(\tau) = e^{-2|\tau|} \).
(a) Find the joint density function of $X_t$ and $Y_t$.

(b) ${X_t}$ and ${Y_t}$ are the inputs to two linear time invariant filters with the frequency responses $H_1(f)$ and $H_2(f)$, respectively, given by

$$
H_1(f) = \begin{cases} 
    a_1 & |f| \leq 1 \\
    0 & \text{otherwise}
\end{cases}
$$

and

$$
H_2(f) = \begin{cases} 
    a_2 & 2 \leq |f| \leq 4 \\
    0 & \text{otherwise}
\end{cases}
$$

Let $\{V_t\}$ and $\{W_t\}$ denote the output processes of the first and second filters, respectively.

Find the joint density function of $V_{s_1}$ and $W_{s_2}$.

5. Let $\{X_t\}$ be a wide sense stationary (WSS) random process with mean zero and power spectral density

$$
S_X(f) = \begin{cases} 
    f^2 & |f| \leq 3 \\
    0 & |f| > 3
\end{cases}
$$

$\{X_t\}$ is the input to two linear time-invariant filters with transfer functions

$$
H_1(f) = \begin{cases} 
    2f & |f| \leq 1 \\
    0 & \text{otherwise}
\end{cases}
$$

and

$$
H_2(f) = \begin{cases} 
    1 & 1 < |f| \leq 2 \\
    0 & \text{otherwise}
\end{cases}
$$

Let $\{Y_t\}$ and $\{Z_t\}$ denote the output processes of the first and second filter, respectively.

Find

(a) $\text{var}(Y_t)$,
(b) $\text{var}(Z_t)$,
(c) $\text{cov}(Y_{t_1}, Z_{t_2})$.

6. Find the power spectral density of the random process $\{Y(t)\}$ where

$$
Y(t) = X(t) \cos(2\pi f_1 t + \Theta).
$$

$\{X(t)\}$ is a wide sense stationary random process with autocorrelation function

$$
R_X(\tau) = \sigma^2 \frac{\sin(2\pi f_0 \tau)}{2\pi f_0 \tau}
$$

where $\sigma$ is a positive constant. $\Theta$ is a random variable uniformly distributed over $(0, 2\pi)$ and is independent of $\{X(t)\}$. 

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7. \( \{X(t)\} \) is a WSS random process. Let

\[
Y(t) = \frac{1}{2T} \int_{t-T}^{t+T} X(\tau) \, d\tau.
\]

Show that

\[
S_Y(f) = S_X(f) \left\{ \frac{\sin(2\pi f T)}{2\pi f T} \right\}^2
\]

Hint: Find the impulse response of the appropriate filter.

8. A random variable \( Y \) has the following probability mass function. \( P(Y = -k) = (1 - p)p^k \) for \( k = 0, 1, 2, \ldots \) where \( 0 < p < 1 \). The random process \( \{X_t : t \geq 0\} \) is defined as follows.

\[ X_t = e^{Y_t}. \]

(a) Draw three different sample path of this random process.

(b) Specify the state space of the random variable \( X_t \). If it is discrete, find the probability mass function of \( X_t \). If it is continuous, find its probability density function.

(c) Find \( E[X_t] \).

9. A random process \( \{X(t)\} \) is defined as follows:

\[ X(t) = a \cos(2\pi Zt + \Theta) \]

where \( Z \) is a random variable with probability density function \( f_Z(\alpha) \), \( \Theta \) is uniformly distributed over \( [0, 2\pi] \) and \( Z \) and \( \Theta \) are independent. Assume that \( f_Z(\alpha) \) is an even function.

Find the power spectral density of \( \{X(t)\} \). You must simplify your answer as much as possible to receive full credit.