1 Optimum Receiver Principles

1.1 Transmitter

The communication model we are concerned with is shown in the following figure.

Source: The source is an iid process \( \{V_n\} \) with alphabet

\[ \Omega_V = \{v_1, v_2, \cdots, v_M\} \]

and probability mass function \( p_V(v_i), \ i = 1, 2, \cdots, M \). We assume that the source emits symbols at the rate of one symbol every \( T \) seconds. The source rate is \( R = 1/T \) source symbols per second.

Modulator: The modulator is specified by the signal set

\[ S = \{s_1(t), s_2(t), \cdots, s_M(t)\} \]

where each \( s_i(t) = 0 \) for \( t \notin [0, T] \). For the interval \( [nT, (n+1)T] \), if the source puts out the symbol \( V_n = v_k \), the modulator produces the signal \( s_k(t - nT) \) in response to the source’s output. This is called a symbol by symbol modulation.

Channel: The channel is the AWGN channel with noise process \( \{N_w(t)\} \) having zero mean and the power spectral density \( S_{N_W}(f) = \frac{N_0}{2} \).

Therefore for \( t \in [nT, (n+1)T] \), the received signal is

\[ r(t) = s(t) + N_W(t) \]

where \( s(t) \in S \).

1.2 Optimum Demodulator

The goal of the demodulator is to make a decision about \( \{V_k\} \) based on \( R(t) \), \( t \in [kT, (k+1)T] \), where

\[ R(t) = s(t) + N_W(t) \]

We wish our demodulator to have small average error probability, i.e.,

\[ P(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} P(V_k \neq \hat{V}_k) \]
is to be small. It can be shown that since the source is independent and
the noise is white, it is sufficient to consider a modulator that minimizes the
probability of error for each source symbol, i.e.,

\[ P(E) = P(V_k \neq \hat{V}_k) \quad \text{for all } k \]

### 1.3 Best decision rule

We observe \( R(t) \) for \( t \in (-\infty, \infty) \) and want to make a decision about each \( V_k \).

**Symbol be symbol detection**

In a memoryless modulation system with an iid message sequence and
AWGN channel, it is sufficient to use only the received signal \( R(t) \) for \( t \in [nT, (n+1)T] \) to make a decision about \( V_n \).

Hereafter we use only the received signal \( R(t) \) for \( t \in [0, T] \) to make a decision about \( V_0 \), which we will denote as \( V \).

According to the MAP rule we let

\[ g[R(t)] = v_i \]

if and only if

\[ p_{V|R(t)}(v_i|r(t), t \in [0, T]) \]

is largest among all messages \( v_k \neq v_i \).

We can’t evaluate this probability because \( R(t) \) is a waveform, i.e., random
process. Therefore we look for a sufficient statistic for decisions about \( V \) based
on \( R(t) \) (a simplifying procedure).

Suppose we are given a set of functions

\[ \{ \phi_1(t), \phi_1(t), ..., \phi_L(t) \} \]

such that

1. (a) \( \phi_i(t) = 0 \) if \( t \notin [0, T] \)
   (b) \( \int_0^T |\phi_i(t)|^2 dt < \infty \).
2. Each $s_k(t)$ can be expressed as a linear combination of the set

$$\{\phi_1(t), \phi_1(t), ..., \phi_L(t)\}$$

i.e.,

$$s_k(t) = \sum_{i=1}^{L} a_{ki} \phi_i(t)$$

Now given $R(t)$ compute

$$\mathbf{R} = (R_1, R_2, ..., R_L)$$

where

$$R_k = \int_0^T R(t) \phi_k(t) dt$$

**Claim:** $\mathbf{R}$ is a sufficient statistic for decisions about $V$ based on $R(t), \ 0 \leq t \leq T$

This claim will be proved later.

Now we use $\mathbf{R}$ to estimate $V$. The best decision rule for $V$ based on $\mathbf{R}$ is:

Given

$$\mathbf{R} = \mathbf{r}$$

let

$$g(\mathbf{r}) = m_i$$

if and only if

$$p_{X|\mathbf{R}}(m_i|\mathbf{r}) \geq p_{X|\mathbf{R}}(v_k|\mathbf{r}) \text{ for all } k \neq i$$

with ties broken arbitrarily.

Equivalently,

Given

$$\mathbf{R} = \mathbf{r}$$

let

$$g(\mathbf{r}) = v_i$$

if and only if
$$p_X(v_i)p_{R|X}(r|v_i) \geq p_X(v_k)p_{R|X}(r|v_k) \text{ for all } k \neq i$$

For this we need to calculate $p_{R|X}(r|v_i)$ for all $i$.

Now given that $X = v_i$, $s_i(t)$ is transmitted and

$$R(t) = s_i(t) + N_W(t)$$

Thus

$$R_k = \int_0^T R(t)\phi_i(t)dt$$

$$= \int_0^T s_i(t)\phi_k(t)dt + \int_0^T N_W(t)\phi_k(t)dt$$

Then

$$R = s_i + N$$

where

$$s_i = (s_{i1}, s_{i1}, ..., s_{iL})$$

and

$$N = (N_1, N_2, ..., N_L)$$

where

$$s_{ik} = \int_0^T s_i(t)\phi_k(t)dt$$

and

$$N_k = \int_0^T N_W(t)\phi_k(t)dt$$

Therefore,

$$p_{R|X}(r|v_i) = p_N(r - s_i)$$

Thus we need to find $p_N(\alpha)$.

Now $N$ is a Gaussian random vector. We compute

1. $E[N_i] = 0$.

2. $E[N_iN_j] = \frac{N_0}{2} \int_0^T \phi_i(t)\phi_j(t)dt$
Everything becomes simpler if we choose $\phi_i(t)$’s so that the following conditions are satisfied.

$$\int_0^T \phi_i(t)\phi_j(t)dt = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

We then get

$$\text{cov}(N_i, N_j) = E[N_iN_j] = \begin{cases} \frac{N_0}{2} & i = j \\ 0 & i \neq j \end{cases}$$

Thus

$$p_N(x) = \prod_{j=1}^L p_{N_j}(x_j) = \frac{1}{(\pi N_0/2)^{L/2}} e^{-\sum_{j=1}^L x_j^2}$$

and,

$$p_{R|X}(r|v_i) = \frac{1}{(\pi N_0/2)^{L/2}} e^{-\sum_{j=1}^L (r_j - s_{ij})^2}$$

The best decision rule is now given by:

Given

$$R = r$$

let

$$g(r) = v_i$$

if and only if

$$\sum_{j=1}^L (r_j - s_{ij})^2 - \frac{N_0}{2} \ln p_X(v_i) \leq \sum_{j=1}^L (r_j - s_{kj})^2 - \frac{N_0}{2} \ln p_X(v_k) \text{ for all } k \neq i$$

Note that

$$\sum_{j=1}^L (r_j - s_{ij})^2 = ||r - s_i||^2$$

is the Euclidean distance between the two vectors $r$ and $s_i$.

The receiver structure is now as follows:

**Figure for optimal receiver**

The following questions remain to be resolved.
1. How to find the set of functions

\{\phi_1(t), \phi_2(t), ..., \phi_L(t)\}

so that they satisfy the following properties.

(a) \[
\int_0^T |\phi_i(t)|^2 \, dt = 1, \quad \forall i = 1, 2, \cdots L
\]

(b) Each \(s_k(t)\) can be expressed as a linear combination of the set

\{\phi_1(t), \phi_2(t), ..., \phi_L(t)\}

i.e.,

\[ s_k(t) = \sum_{i=1}^{L} a_{ki} \phi_i(t) \]

Note:

- A function that satisfies the property in (3) is said to be normalized.
- A set of functions that satisfy the property (4) are said to be orthogonal.
- A set of functions satisfying (3) and (4) are said to be orthonormal (O.N.).

2. How to find the \(a_{ki}\)’s.

3. Show that \(\mathbf{R}\) is a sufficient statistic.

4. Simplify receiver, equivalent implementation.

5. Compute \(P(E)\).

6. Compare \(P(E)\) for different signal sets, each with its own optimum receiver.
We first give the answer to question 2 since it is the easiest to answer. By our assumption on \( \{ \phi_i(t) \} \),

\[
s_{kj} = a_{kj}
\]

**Proof:***

\[
s_{kj} = \int_0^T s_k(t)\phi_j(t) \, dt = \int_0^T \left[ \sum_{i=1}^{L} a_{ki}\phi_i(t) \right] \phi_j(t) \, dt
\]

\[
= \sum_{i=1}^{L} a_{ki} \int_0^T \phi_i(t)\phi_j(t) \, dt = a_{kj}
\]