Mixed $\mathcal{L}_1/\mathcal{H}_2/\mathcal{H}_\infty$ Control Design: Numerical Optimization Approaches *

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Abstract

This paper presents some new approaches to mixed performance control problems of linear systems. The design techniques proposed in this paper are based on numerical search of the norm bounded stable transfer matrix Q in the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ suboptimal controller parameterizations so that the additional performance specifications are satisfied. The design problems are then converted to some finite dimensional nonlinear unconstrained optimization problems by explicitly parameterizing the $\mathcal{H}_\infty$ or $\mathcal{H}_2$ norm bounded stable transfer matrix Q for any fixed order. Finally, some two-stage optimization algorithms are applied to find the optimal parameters. Numerical examples have shown significant performance improvements of the proposed algorithms over those in the existing literature.

1 Introduction

The fundamental objective of a feedback control system design is to achieve desired performance despite of model uncertainties and external disturbances. It is well-known in the control community that there are intrinsic conflicts between achievable performance and system robustness. A well thought control system design is to make some suitable tradeoffs between performance and system robustness. It is therefore desirable to develop design techniques that can optimally and systematically perform such performance and robustness tradeoffs. It is therefore not surprising that multiobjective (or mixed performance) optimal control has become a crucially important research area in the last decade or so.

Many approaches have been proposed in the literature to solve mixed performance problems. An overview of various approaches to multi-objective design is presented by Vroemen and Jager (1997). However, it is impossible to review all approaches and works related to mixed performance problems. Hence only some most related work will be described here. Bernstein and Haddad (1989) proposed for the first time the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ as a way to formulate a meaningful optimization problem to the standard $\mathcal{H}_\infty$ control problem by using the Lagrange multiplier method. Later on, Khargonekar and Rotea (1991), and Halder *et al* (1997) addressed the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem and converted the state feedback mixed design into a convex optimization problem. A time domain signal formulation for a dual mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem was presented and characterized by Doyle *et al* (1994) and Zhou *et al* (1990,1994). In a different direction, Limebeer *et al* (1994) and Chen and Zhou (2001) considered Nash game approaches to these mixed problems. Solutions based on linear matrix inequalities (LMI) to multi-objective problems have also been proposed for output feedback control by Oliveira *et al* (1999), and Scherer *et al* (1997). In this design, the objectives are formulated in terms of a common Lyapunov function. However, this formulation tends to be very conservative. Clement and Duc (2000) presented an extension to this method in order to use different Lyapunov functions for each design objective. Hindi *et al* (1998), and Scherer (1995, 2000) proposed to combine LMI's with the Youla parameterization in order to search for the optimal Youla parameter in a finite dimensional space. Similarly, Qi *et al* (2001) proposed finite dimensional approximations for the Youla parameter in order to solve mixed problems with time-domain contraints. Thus, an optimal solution is reached by approximating it from below and above. Due to its physical interpretation, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design has also been applied to the optimal filtering problem, e.g. (Khargonekar *et al* 1996, Rotstein *et al* 1996). Multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ problems have also been characterized in terms of their duality description (Djouadi *et al* 2001). Furthermore, Chen *et al* (2000) extended the $\mathcal{H}_2/\mathcal{H}_\infty$ design to nonlinear systems by using a fuzzy output feedback controller.

Another mixed problem, the $\mathcal{L}_1/\mathcal{H}_\infty$ control was synthesized via convex optimization of finite dimensional approximations by Sznaier and Bu (1998). Haddad *et al* (1998) revisited the $\mathcal{L}_1/\mathcal{H}_\infty$ problem, but now a fixed structure controller is suggested and optimality conditions are derived for its solution. Sznaier and Blanchini (1994) proposed the design of rational mixed $\mathcal{L}_1/\mathcal{H}_\infty$ suboptimal controllers by solving a sequence of finite dimensional auxiliary problems. On the other hand, solutions to the $\mathcal{H}_2/\mathcal{L}_1$ problems were presented by Amishima *et al* (1988), Voulgaris (1994), and Wu and Chu (1999) where quadratic programming problems were introduced to reach a solution. In addition, Salapaka and Khammash (1998) and Salapaka *et al* (1999) approached the $\mathcal{H}_2/\mathcal{L}_1$ problem by obtaining upper and lower bound convergence methods that involve the Q parameter in the Youla parameterization. Sznaier and Amishima (1998) studied the $\mathcal{H}_2$ problem with time constraints and suggested approximations to the optimal solutions by solving some quadratic programming problems. In general, most of approaches presented in the literature do not solve directly the true mixed performance problem by either optimizing an upper bound of the true performance or using a related performance criterion since the original mixed performance problem is a highly complex and nonlinear constrained optimization problem.

In recent years, evolutionary schemes have been extensively used to solve nonlinear constrained optimization problems where multi-local minima can restrict global convergence. Evolutionary schemes are inspired by the natural selection criteria where the stronger organisms are likely to survive after generations. Thus, a parallelism can be drawn with an optimization problem where the evolution period is considered as the optimization time and the most fitted organism in the population will represent the optimal solution. Two evolutionary schemes, evolution algorithms and genetic algorithms, are most commonly used. These algorithms present two main characteristics:

a multi-directional (random) search and an information exchange among best solutions. These properties can generate new search directions in order to avoid local minima. Applications of genetic algorithms to control and signal processing have been reported in literature: digital IIR filter design (Man *et al* 1999), adaptive recursive filtering (White and Flockton 1997), active noise control (Tang *et al* 1995), systems model reduction (Li *et al* 1997), weighting function design for $\mathcal{H}_\infty$ loop-shaping (Whidborne *et al* 1995), etc.

Motivated from those successful applications of the evolutionary algorithms, it seems nature to apply these algorithms such as genetic algorithms to the above mentioned multiobjective optimization problems. Nevertheless, it is unlikely to produce any reasonable results if these algorithms are applied blindly since the optimization parameter spaces are too large. In this paper, we shall propose design techniques that explicitly parameterize the free transfer matrix Q in the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ suboptimal controller parameterizations such that the optimization parameter spaces are highly restricted and then evolutionary algorithms such as genetic/evolution algorithms can be applied effectively to produce the desired results. In addition to the computational advantage, the proposed technique may produce mcuh lower order controllers than those using Youla parameterization and convex optimization.

The rest of the paper is organized as follows. First, notations and some definitions used in the paper are presented in Section 2. Next, the suboptimal $\mathcal{H}_\infty$ and $\mathcal{H}_2$ controller parameterizations are introduced in Section 3. Section 4 gives some explicit parameterizations of norm bounded $\mathcal{H}_\infty$ and $\mathcal{H}_2$ functions. In Section 5, various multi-objective design problems are presented and optimization schemes are proposed. Section 6 outlines a two-stage optimization scheme that will be applied in Section 7 to solve those multi-objective problems. Finally, some numerical examples are shown in Section 7 and the paper is concluded with some remarks in Section 8.

2 Notations and Definitions

Let $G(s)$ be an MIMO transfer matrix with the following state-space realization

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \begin{bmatrix} G_{11}(s) & \dots & G_{1m}(s) \\ \vdots & \ddots & \vdots \\ G_{p1}(s) & \dots & G_{pm}(s) \end{bmatrix} \tag{1}$$

Let $\mathcal{H}_2$ denote the space of all strictly proper and stable transfer matrices. The $\mathcal{H}_2$ norm is defined as

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} Trace[G^*(j\omega)G(j\omega)]d\omega \tag{2}$$

It can be computed by using the state space representation of G. For a stable and strictly proper G (i.e. A is stable and $D = 0$), we have

$$\|G\|_2^2 = Trace(B^T QB) = Trace(CPC^T) \tag{3}$$

where P and Q are the controllability and observability Gramians which can be obtained by solving the following Lyapunov equations

$$AP + PA^T + BB^T = 0 \qquad A^T Q + QA + C^T C = 0. \tag{4}$$

Let $\mathcal{RH}_\infty$ denote the space of all proper and real rational stable transfer functions. The $\mathcal{H}_\infty$ norm is defined as:

$$\|G\|_\infty = \sup_{Re(s)>0} \bar{\sigma}[G(s)] = \sup_{\omega\in\mathcal{R}} \bar{\sigma}[G(j\omega)]. \tag{5}$$

The $\mathcal{L}_1$ norm of a stable transfer matrix is defined as

$$\|G\|_1 = \max_{1\le i\le p} \sum_{j=1}^{m} \|g_{ij}\|_1 \tag{6}$$

where $g_{ij}(t) = \mathcal{L}^{-1}\{G_{ij}(s)\}$ and

$$\|g_{ij}\|_1 = \int_0^\infty |g_{ij}(t)|dt \tag{7}$$

Consider a feedback system described by the block diagram in Figure 1 where the generalized plant G and the controller K are assumed to be real-rational and proper with $y(t) \in \mathcal{R}^{p_2}$ and $u(t) \in \mathcal{R}^{m_2}$. Let G be partitioned accordingly as

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \tag{8}$$

and

$$K = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right]$$

Then the transfer function from w to z is given by

$$T_{zw} = \mathcal{F}_l(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} = \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right] \tag{9}$$

where

$$A_{cl} = \begin{bmatrix} A + B_2\tilde{R}^{-1}D_kC_2 & B_2\tilde{R}^{-1}C_k \\ B_kR^{-1}D_{21} & A_k + B_kR^{-1}D_{22}C_k \end{bmatrix}$$

$$
\begin{aligned}
B_{cl} &= \begin{bmatrix} B_1 + B_2 \tilde{R}^{-1} D_k D_{21} \\ B_k R^{-1} D_{21} \end{bmatrix} \\
C_{cl} &= \begin{bmatrix} C_1 + D_{12} D_k R^{-1} C_2 & D_{12} \tilde{R}^{-1} C_k \end{bmatrix} \\
D_{cl} &= D_{11} + D_{12} D_k R^{-1} D_{21} \\
R &= I - D_{22} D_k, \quad \tilde{R} = I - D_k D_{22}
\end{aligned}
$$

and $\mathcal{F}_l(\cdot,\cdot)$ is called a lower linear fractional transformation.

3 $\mathcal{H}_\infty$ and $\mathcal{H}_2$ Suboptimal Controller Parameterizations

Consider the generalized feedback system described in Figure 1 with the generalized plant G given by (8) with some suitable assumptions. Then it is well known that all stabilizing controllers $K(s)$ satisfying the suboptimal $\mathcal{H}_\infty$ condition, $\|T_{zw}\|_\infty < \gamma$ for a given $\gamma > 0$, can be parameterized as $K = \mathcal{F}_l(M_\infty, Q)$ with $Q \in \mathcal{RH}_\infty$, $\|Q\|_\infty < \gamma$ where

$$
M_\infty = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right] = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{10}
$$

and M_∞ is constructed from the solutions of two Riccati equations (Doyle *et al* 1999, Zhou *et al* 1996, Zhou and Doyle 1997). In the case of $Q = 0$, the solution, $K = M_{11}$, is called the central controller. Note that there is no guarantee that M_∞ is itself stable even though the closed-loop system is stable. It is noted that M_∞ and Q are $(p_2 + m_2) \times (p_2 + m_2)$ and $m_2 \times p_2$ transfer matrices respectively.

To consider the $\mathcal{H}_2$ optimal control problem, we shall assume for simplicity that $D_{11} = 0$ and $D_{22} = 0$. The case where $D_{22} \neq 0$ can be dealt with easily. It is also well known that all stabilizing controllers for the generalized plant G can be written as $K = \mathcal{F}_l(M_2, Q)$ with

$$
M_2 = \left[\begin{array}{c|cc} \hat{A}_2 & -L_2 & B_2 \\ \hline F_2 & 0 & I \\ -C_2 & I & 0 \end{array}\right] \tag{11}
$$

where $\hat{A}_2 = A + B_2 F_2 + L_2 C_2$ and F_2 and L_2 can be constructed from the solutions of two related Riccati equations, see (Zhou *et al* 1996, Zhou and Doyle 1997). It is then clear that the closed-loop $\mathcal{H}_2$ norm is given by

$$
\|T_{zw}\|_2^2 = \|G_c B_1\|_2^2 + \|F_2 G_f\|_2^2 + \|Q\|_2^2 \tag{12}
$$

where the definition of the transfer matrices G_c and G_f can be found in (Zhou *et al* 1996, Zhou and Doyle 1997). Consequently, $Q = 0$ represents the optimal solution to the $\mathcal{H}_2$ problem. As in the $\mathcal{H}_\infty$ parameterization, M_2 and Q are $(p_2 + m_2) \times (p_2 + m_2)$ and $m_2 \times p_2$ transfer matrices respectively. Let $\gamma_{opt}^2 = \|G_c B_1\|_2^2 + \|F_2 G_f\|_2^2$. Then for any $\gamma > \gamma_{opt}$, all suboptimal $\mathcal{H}_2$ controllers satisfying $\|T_{zw}\|_2 < \gamma$ can be parameterized as $K = \mathcal{F}_l(M_2, Q)$ with $Q \in \mathcal{RH}_2$ and $\|Q\|_2^2 < \gamma^2 - \gamma_{opt}^2$.

4 Parameterizations of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ Norm Bounded Functions

It is now clear that if a controller is required to satisfy both the $\mathcal{H}_\infty$ norm constraint, $\|T_{zw}\|_\infty < \gamma$, and some additional performance objectives, it has to come from the family of $\mathcal{H}_\infty$ controllers parameterized in the last section. In other words, a stable Q with $\|Q\|_\infty < \gamma$ must be found to satisfy the additional performance objectives. Similar observations can be made for problems involving $\mathcal{H}_2$ performance objectives. To find a suitable $Q \in \mathcal{RH}_\infty$ with $\|Q\|_\infty < \gamma$ or a $Q \in \mathcal{RH}_2$ with $\|Q\|_2 < \sqrt{\gamma^2 - \gamma_{opt}^2}$, it is desirable to have more explicit characterizations of these norm bounded analytic functions that are appropriate for numerical optimization.

Stein and Bosgra (1991) presented a parameterization for an $\mathcal{H}_\infty$ norm bounded strictly proper and stable transfer matrix. That result was extended to the proper case (Campos and Zhou 2001) by the following lemma.

Lemma 1 *Let $\gamma > 0$ and let Q be a stable transfer matrix of degree n_q and $\|Q\|_\infty < \gamma$. Then Q can be represented as $Q = \left[\begin{array}{c|c} A_q & B_q \\ \hline C_q & D_q \end{array}\right]$ with $A_q = A_{q_k} + A_{q_s}$ for some $A_{q_k} = -A_{q_k}^T \in \mathcal{R}^{n_q \times n_q}$, $B_q \in \mathcal{R}^{n_q \times p_2}$, $C_q \in \mathcal{R}^{m_2 \times n_q}$, $D_q \in \mathcal{R}^{m_2 \times p_2}$, and*

$$\begin{aligned} A_{q_s} &= \frac{1}{2}\left(-B_q R^{-1} D_q^T C_q - C_q^T D_q R^{-1} B_q^T - B_q R^{-1} B_q^T - C_q^T (I + D_q R^{-1} D_q^T) C_q\right) \quad (13) \\ \bar{\sigma}(D_q) &< \gamma \quad (14) \end{aligned}$$

where $R = \gamma^2 I - D_q^T D_q$.

Proof: Assume that $Q = \left[\begin{array}{c|c} \hat{A}_q & \hat{B}_q \\ \hline \hat{C}_q & D_q \end{array}\right] \in \mathcal{RH}_\infty$ is a n_q^{th} order observable realization and $\|Q\|_\infty < \gamma$, then according with the Bounded Real Lemma (Zhou *et al* 1996) $\bar{\sigma}(D_q) < \gamma$ and $\exists Y > 0$ such that

$$Y(\hat{A}_q + \hat{B}_q R^{-1} D_q^* \hat{C}_q) + (\hat{A}_q + \hat{B}_q R^{-1} D_q^* \hat{C}_q)^* Y + Y \hat{B}_q R^{-1} \hat{B}_q^{\,*} Y + \hat{C}_q^{\,*} (I + D_q R^{-1} D_q^*) \hat{C}_q = 0 \quad (15)$$

where $R = \gamma^2 I - D_q^* D_q$. Since $Y > 0$, there exists a Cholesky factorization of $Y = T^* T$. Now T is invertible and can be used as a similarity transformation on Q

$$Q = \left[\begin{array}{c|c} T\hat{A}_q T^{-1} & T\hat{B}_q \\ \hline \hat{C}_q T^{-1} & D_q \end{array}\right] = \left[\begin{array}{c|c} A_q & B_q \\ \hline C_q & D_q \end{array}\right] \tag{16}$$

Thus, the Riccati equation in (15) becomes

$$A_q + A_q^* + B_q R^{-1} D_q^* C_q + C_q^* D_q R^{-1} B_q^* + B_q R^{-1} B_q^* + C_q^*(I + D_q R^{-1} D_q^*) C_q = 0 \tag{17}$$

Furthermore, A_q can be decomposed into a symmetric part A_{q_s} and a skew symmetric part A_{q_k} where

$$A_{q_s} = (A_q + A_q^*)/2 \qquad A_{q_k} = (A_q - A_q^*)/2 \tag{18}$$

Consequently, the skew symmetric part A_{q_k} disappears from (17) and the result in (13) is obtained.

□

Note that when $D_q = 0$, we have (Stein and Bosgra, 1991)

$$A_{q_s} = -\frac{1}{2}(B_q B_q^T/\gamma^2 + C_q^T C_q) \tag{19}$$

On the other hand, using the definition of the $\mathcal{H}_2$ norm given by (3) and (4), a parameterization for all $Q \in \mathcal{RH}_2$ follows (Campos and Zhou 2001).

Lemma 2 *Assume that $Q \in \mathcal{RH}_2$ has degree n_q, then Q can be represented in the following form*

$$\|Q\|_2^2 = Trace(B_q^T B_q)$$

with $Q = \left[\begin{array}{c|c} A_q & B_q \\ \hline C_q & 0 \end{array}\right]$ *and* $A_q = A_{q_s} + A_{q_k}$ *where* $A_{q_s} = -\frac{1}{2} C_q^T C_q$,

$$A_{q_k} = -A_{q_k}^T \in \mathcal{R}^{n_q \times n_q}, \quad B_q \in \mathcal{R}^{n_q \times p_2}, \quad C_q \in \mathcal{R}^{m_2 \times n_q}. \tag{20}$$

Proof: Assume that $Q = \left[\begin{array}{c|c} \hat{A}_q & \hat{B}_q \\ \hline \hat{C}_q & D_q \end{array}\right] \in \mathcal{RH}_2$ is a n_q^{th} order observable realization, then according to (3) and (4), $\exists Y > 0$ such that

$$A^T Y + Y A + C^T C = 0 \tag{21}$$

Since $Y > 0$, there exists a Cholesky factorization of $Y = T^T T$. Now T is invertible and can be used as a similarity transformation on Q

$$Q = \left[\begin{array}{c|c} T\hat{A}_q T^{-1} & T\hat{B}_q \\ \hline \hat{C}_q T^{-1} & 0 \end{array}\right] = \left[\begin{array}{c|c} A_q & B_q \\ \hline C_q & 0 \end{array}\right] \tag{22}$$

Thus, the Lyapunov equation in (21) becomes

$$A_q + A_q^T + C_q^T C_q = 0 \tag{23}$$

Moreover, A_q can be decomposed into a symmetric part A_{q_s} and a skew symmetric part A_{q_k}. Consequently, the skew symmetric part A_{q_k} disappears from (23) and it is obtained

$$\|Q\|_2^2 = Trace(B_q^T B_q) \qquad A_{q_s} = -\frac{1}{2} C_q^T C_q \tag{24}$$

□

Thus, if Q is constructed according with the previous lemma, then $\|Q\|_2 < \beta$ is equivalent to $Trace(B_q^T B_q) < \beta^2$.

5 Mixed $\mathcal{L}_1/\mathcal{H}_2/\mathcal{H}_\infty$ Performance Problems

Several mixed objective control problems will be presented in this section. Their solutions are critically dependent upon the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ controller state-space parameterizations presented in the previous sections. That is, the parameters of the transfer matrix Q will be used to optimize the performance indices and satisfy constraints. Note that since the controllers are based on the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ parameterizations the internal stability of the closed-loop is always guaranteed. Assume that the degree of Q is predefined to n_q, then according with the dimensions of the generalized plant (8), the number of variables of each component of Q is given by

$$A_{q_k} = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n_q} \\ -a_{12} & 0 & a_{23} & \cdots & a_{2n_q} \\ -a_{13} & -a_{23} & 0 & \cdots & a_{3n_q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{1n_q} & -a_{2n_q} & -a_{3n_q} & \cdots & 0 \end{bmatrix} \Rightarrow \frac{(n_q-1)n_q}{2} \tag{25}$$

$$B_q = \begin{bmatrix} b_{11} & \cdots & b_{1p_2} \\ \vdots & \ddots & \vdots \\ b_{n_q1} & \cdots & b_{n_qp_2} \end{bmatrix} \Rightarrow n_q \times p_2 \tag{26}$$

$$C_q = \begin{bmatrix} c_{11} & \cdots & c_{1n_q} \\ \vdots & \ddots & \vdots \\ c_{m_21} & \cdots & c_{m_2n_q} \end{bmatrix} \Rightarrow m_2 \times n_q \tag{27}$$

$$D_q = \begin{bmatrix} d_{11} & \cdots & d_{1p_2} \\ \vdots & \ddots & \vdots \\ d_{m_2 1} & \cdots & d_{m_2 p_2} \end{bmatrix} \quad \Rightarrow \quad m_2 \times p_2 \tag{28}$$

Since $Q \in \mathcal{RH}_\infty$ is an $m_2 \times p_2$ transfer matrix, the total number of variables in the optimization scheme will be $(n_q - 1)n_q/2 + n_q m_2 + n_q p_2 + m_2 p_2$. In order to solve the multi-objective problems, the elements of the state-space description of Q are aligned into a vector form

$$X = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{(n_q-1)n_q} & b_{11} & \cdots & b_{n_q p_2} & c_{11} & \cdots & c_{m_2 n_q} & d_{11} & \cdots & d_{m_2 p_2} \end{bmatrix}^T \tag{29}$$

Therefore, the proposed optimization problems will be solved with respect to the variable vector X. Note that if $Q \in \mathcal{RH}_2$, i.e. $D_q = 0$, the number of free variables reduces to $(n_q - 1)n_q/2 + n_q m_2 + n_q p_2$. During the optimization, an interval of variation will be set for all the parameters in X, i.e.

$$\begin{aligned} a_{min} &\le a_{ij} \le a_{max} \\ b_{min} &\le b_{ij} \le b_{max} \\ c_{min} &\le c_{ij} \le c_{max} \\ d_{min} &\le d_{ij} \le d_{max} \end{aligned} \tag{30}$$

However, not all the requirements in the elements of Q can be reflected into a range of variation for each parameter. Therefore, the following penalty functions are used in the optimization schemes to restrict the variations of some of the parameters and to enforce requirements on the transfer matrix Q

$$P(D, \gamma) = \begin{cases} M & \bar{\sigma}(D) \ge \gamma \\ 1 & otherwise \end{cases} \tag{31}$$

$$J(B, \gamma, \gamma_{opt}) = \begin{cases} N & Trace(B^T B) \ge \gamma^2 - \gamma_{opt}^2 \\ 1 & otherwise \end{cases} \tag{32}$$

where the matrices A, B and D are linked to the state-space realization of a system, $\gamma > \gamma_{opt} > 0$, M and N are constants $\gg 1$.

Consider a generalized feedback system described in figure 2 with

$$G_m = \left[\begin{array}{c|ccc} A & B_0 & B_1 & B_2 \\ \hline C_0 & D_{00} & D_{01} & D_{02} \\ C_1 & D_{10} & D_{11} & D_{12} \\ C_2 & D_{20} & D_{21} & D_{22} \end{array}\right], \quad \begin{bmatrix} z_0 \\ z \\ y \end{bmatrix} = G_m \begin{bmatrix} w_0 \\ w \\ u \end{bmatrix} \tag{33}$$

Let G_0 and G be defined as

$$G_0 = \left[\begin{array}{c|cc} A & B_0 & B_2 \\ \hline C_0 & D_{00} & D_{02} \\ C_2 & D_{20} & D_{22} \end{array}\right], \quad G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{34}$$

Then, the next closed-loop transfer matrices can be determined

$$T_{z_0w_0} = \mathcal{F}_l\left(G_0, K\right), \quad T_{zw} = \mathcal{F}_l\left(G, K\right) \tag{35}$$

5.1 Mixed $\mathcal{H}_\infty/\mathcal{H}_\infty$ Design (Robust $\mathcal{H}_\infty$ Performance)

The robust performance problem can be formulated as

$$\min_{stabilizing\ K} \|T_{z_0w_0}\|_\infty \quad s.t. \quad \|T_{zw}\|_\infty < \gamma \tag{36}$$

for some $\gamma > \gamma_\infty$ where

$$\gamma_\infty = \min_{stabilizing\ K} \|T_{zw}\|_\infty \tag{37}$$

In order to solve (36), the following numerical optimization is proposed

$$\min_{A_{q_k},B_q,C_q,D_q} P(D_q,\gamma) \cdot \|\mathcal{F}_l\left(G_0, \mathcal{F}_l\left(M_\infty, Q\right)\right)\|_\infty \tag{38}$$

where $Q = \left[\begin{array}{c|c} A_{q_k} + A_{q_s} & B_q \\ \hline C_q & D_q \end{array}\right]$, M_∞ is given by (10) and A_{q_s} by (13). Therefore, (A_{q_k}, B_q, C_q, D_q) are the free variables in the optimization scheme. The penalty function $P(\cdot,\cdot)$ is included to restrict the maximum singular value of D_q to be $< \gamma$, which is a necessary condition in the Bounded Real Lemma to have $\|Q\|_\infty < \gamma$. This condition could not be incorporated in the optimization directly since it is difficult to give an interval of variation for the elements of a matrix to have $\bar{\sigma} < \gamma$. So it was decided to introduce the penalty function in the cost function to detect violations of the condition and penalize these solutions. The result obtained from the optimization in (38) is finally used to construct the multi-objective controller as $K = \mathcal{F}_l\left(M_\infty, Q\right)$.

5.2 Mixed $\mathcal{H}_2/\mathcal{H}_\infty$

This mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design problem can be formulated as

$$\min_{stabilizing\ K} \|T_{z_0w_0}\|_2 \quad s.t. \quad \|T_{zw}\|_\infty < \gamma \tag{39}$$

for some $\gamma > \gamma_\infty$. In order to solve (39), the following numerical optimization is proposed

$$\min_{A_{q_k},B_q,C_q} \|\mathcal{F}_l\left(G_0, \mathcal{F}_l\left(M_\infty, Q\right)\right)\|_2 \tag{40}$$

where $Q = \left[\begin{array}{c|c} A_{q_k} + A_{q_s} & B_q \\ \hline C_q & 0 \end{array}\right]$ since now $Q \in \mathcal{RH}_2$, M_∞ is given by (10) and A_{q_s} by (13). Note that a penalty function is not needed since, Q has to be strictly proper, i.e. $D_q = 0$. Consequently, in this mixed problem, the free variables in the optimization are now (A_{q_k}, B_q, C_q).

5.3 Mixed $\mathcal{L}_1/\mathcal{H}_\infty$

The mixed $\mathcal{L}_1$ and $\mathcal{H}_\infty$ design problem can be stated as

$$\min_{stabilizing\ K} \|T_{z_0w_0}\|_1 \quad s.t. \quad \|T_{zw}\|_\infty < \gamma \tag{41}$$

for some $\gamma > \gamma_\infty$. In order to solve (41), the following numerical optimization is proposed

$$\min_{A_{q_k}, B_q, C_q, Dq} P(D_q, \gamma) \cdot \|\mathcal{F}_l\left(G_0, \mathcal{F}_l\left(M_\infty, Q\right)\right)\|_1 \tag{42}$$

where M_∞ is given by (10), A_{q_s} by (13) and $Q = \left[\begin{array}{c|c} A_{q_k} + A_{q_s} & B_q \\ \hline C_q & D_q \end{array}\right]$. Note that the penalty function $P(\cdot,\cdot)$ is needed again to restrict $\bar{\sigma}(D_q) < \gamma$ since $Q \in \mathcal{RH}_\infty$. Therefore, (A_{q_k}, B_q, C_q, D_q) are the free variables in the optimization scheme.

5.4 $\mathcal{L}_1/\mathcal{H}_2$ Design

The mixed $\mathcal{L}_1/\mathcal{H}_2$ problem is defined as:

$$\min_{stabilizing\ K} \|T_{z_0w_0}\|_1 \quad s.t. \quad \|T_{zw}\|_2 < \gamma \tag{43}$$

for some $\gamma > \gamma_{opt}$ where

$$\gamma_{opt} = \min_{stabilizing\ K} \|T_{zw}\|_2 \tag{44}$$

In order to solve (43), the following numerical optimization is proposed

$$\min_{A_{q_k}, B_q, C_q, Dq} J(B_q, \gamma, \gamma_{opt}) \cdot \|\mathcal{F}_l\left(G_0, \mathcal{F}_l\left(M_2, Q\right)\right)\|_1 \tag{45}$$

where $A_{q_s} = -C_q^T C_q/2$ and $Q = \left[\begin{array}{c|c} A_{q_k} + A_{q_s} & B_q \\ \hline C_q & 0 \end{array}\right]$. Therefore, (A_{q_k}, B_q, C_q) are the free variables in the optimization scheme with $\|Q\|_2^2 < \gamma^2 - \gamma_{opt}^2$. The penalty function $J(\cdot,\cdot,\cdot)$ is introduced here to enforce the restriction on the norm of Q, i.e. limit the value of the elements of B_q to satisfy $Trace(B_q^T B_q) < \gamma^2 - \gamma_{opt}^2$. Note that this condition cannot be translated into any pattern selection for the elements of B_q. Therefore, it was chosen to include the penalty function in the cost function to penalize any combination of parameters that violate it.

The optimization problems just presented in (38), (40), (42) and (45) are generally nonconvex and present intrinsically multi-modal characteristics. Therefore, it is natural to think that these problems are not practical to solve. Nevertheless, evolution optimization has turned out to be quite useful to solve these complicated optimization problems. Moreover, if it is used in conjunction with a well-known gradient-based optimization technique, a powerful optimization scheme is obtained that can effectively solve the proposed problems.

6 Optimization Scheme

In the first attempt to solve the mixed design problems (38), (40), (42) and (45), standard nonlinear optimization techniques such as quasi-newton method and conjugate gradient (Grace 1992, Nocedal and Wright 1999) were applied. However, convergence was always limited to the initial conditions. The algorithms constantly got caught in local minima. A change of direction was obviously needed. Thus, the evolutionary algorithms came as viable solution to solve these problems. However, the gradient-based approaches have nice properties that should not be forgotten. Therefore, it was decided to use a gradient-based algorithm in conjunction with an evolutionary optimization such as genetic or evolution algorithms. In this way, the genetic/evolution algorithm was applied first to perform a global search in the parameters space and find a minimum solution. Next, a local search was conducted to obtain the optimal solution. This two-stage optimization outperformed the use of each one of the algorithms alone.

In the first stage, a genetic/evolution algorithm was chosen to solve (38), (40), (42) and (45). However, a natural inspired algorithm such as simulated annealing could have also been used. For the second stage, a quasi-newton plus linear search (Nocedal and Wright 1999) scheme was used to perform the local search. Thus, the best solution coming from the genetic/evolution algorithm was used as a starting point for the gradient-based search. A brief description of the genetic algorithm used in the paper is presented next. The gradient-based optimization was carried out by using the Optimization Toolbox (Grace 1992) of MATLAB©.

6.1 Evolution Algorithm

Consider a function that has to be optimized with m inputs and one output. The output of this function is referred as its fitness. The idea is now to adjust the input parameters in order to find an optimum in the fitness. One combination of m input parameters is called an individual. A group of n individuals is a population. The idea is to start with a randomly selected initial population (mutation), creating a group of children out of their parents. The fitness of the children is now evaluated and compared with their parent's fitness, and the best of both are selected to be the next generation of parents. This procedure will go on until an optimum is found, or a given termination criterion is fulfilled.

Following these ideas, the evolution algorithm EVAOCP (Evolution Algorithm for Optimization of Continuous Parameters) is proposed for the optimization process

1. Initialize parameters of the evolution algorithm.
2. Select initial population (this can be a random selection, a guess or a result of a previous experiment).
3. Check if the conditions for termination of algorithm are satisfied: optimality, max. # of iterations or no-progress
   - **Y**ES : Set the best values obtained during the optimization process.
   - **N**O : Continue with Evolution Alg.
4. Adjust step (mutation range) according with progress achieved.
5. Create children from parents set.
   (a) Check mutation factor.
   (b) Determine new step.
   (c) Generate children by adding a random perturbation of variance 'step' to parents.
   (d) Check that children satisfy the parameters bounds.
6. Evaluate children fitness.
7. Compares children and parents performance, keep the best of both.
8. Compute progress velocity according with # of children better than parents.
9. Select the best solutions to judge optimality.
10. Go back to 3.

6.2 Genetic Algorithm

In a genetic algorithm, the optimized parameters are arranged in combination sets that are called chromosomes. An interval of variation is assigned for each parameter in the optimization search. Thus, bounds for the elements in a chromosome are defined a priori. To start, an initial combination of chromosomes is used, usually chosen randomly; after the initial evaluation a predefined number of the best chromosomes is held, population. This population is going to be kept constant after each generation. From the population, a set of chromosomes is chosen, parents, to generate a new breed of chromosomes, children. The parents chromosomes are chosen based on a proportional selection according with their fitness. The new children are created by a linear combination of two parents, crossover. Note that this is a special type of crossover operator for a real parameters representation. From the set of parents and children, random perturbations are introduced with a predefined rate, mutation. The combined set, parents and children, is evaluated and a new population is selected. The cycle process, generation, is repeated until an ending condition, no progress or convergence, is satisfied.

The algorithm called GAOCP (Genetic Algorithm for Optimization of Continuous Parameters) was coded in MATLAB© and it presents the following steps:

1. Initialize GAOCP parameters,
2. Generate initial population,
3. Evaluate cost function,
4. Select population,
5. Check progress velocity,
6. Check for convergence,
7. Parents are selected from the population,
8. New generation (children) is created from parents,
9. Mutation is introduced randomly,
10. Go back to step 3.

7 Numerical Examples

In this section, the proposed mixed performance problems will be illustrated through some numerical examples. Due to space limit, we shall only include two examples here. In all these examples, the first stage of the optimization scheme was carried out by a genetic/evolution algorithm. The second stage was carried out by a quasi-newton plus linear search scheme (Grace 1992).

So far, there is no direct computation of the $\mathcal{L}_1$ norm except its mathematical definition (6). Consequently, in order to approach the $\mathcal{L}_1/\mathcal{H}_\infty$ (42) or $\mathcal{L}_1/\mathcal{H}_2$ (45) designs, the norm computation was approximated. Note that the impulse response of each element of (1) is given by

$$g_{ij}(t) = C_i e^{At} B_j + D_{ij}\delta(t) \tag{46}$$

where $B = [B_1 \dots B_m]$, $C = [C_1 \dots C_p]^T$ and $D = [D_{ij}]$. Hence, the infinite integration of (7) can be approximated according with the poles of $G_{ij}(s)$, estimating the time such that $g_{ij}(t)$ is below certain percentage of its peak value and performing the integration in that interval of time.

The following numerical examples were computed on a PC Pentium III at 933 MHz. The constants M in (31) was given the value 1×10^6.

7.1 Example 1

This example is taken from Baeyens and Khargonekar (1994). The multi-objective problem is a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design with the generalized plant G_m given by

$$G_m = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_0 & 0 & D_0 \\ C_1 & 0 & D_1 \\ C_2 & D_2 & 0 \end{array}\right] \tag{47}$$

where the description of the matrices $(A, B_1, B_2, C_0, C_1, C_2)$ is given in (Baeyens and Khargonekar 1994). This is a special case of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems formulated in the previous sections with $w_0 = w$.

Thus the mixed problem is formulated as

$$\min_{K\,stabilizing} \|T_{z_0 w}\|_2 \quad s.t. \quad \|T_{zw}\|_\infty < \gamma \tag{48}$$

The optimization scheme in (40) was used to obtain the mixed controller. In order to judge the performance of the proposed scheme, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimization was also carried out using the *LMI Toolbox* (Gahinet *et al* 1995) of MATLAB©. In this example, the results reported for the proposed method (40) are obtained by using an evolution algorithm (EVAOCP) in the first stage of the optimization. The results were almost identical if a genetic algorithm was used instead. The example was run for two values of γ: 1.6 and 2.0. Note that the closed-loop performance $\|T_{zw}\|_\infty < \gamma$ is always guaranteed by the proposed optimization algorithm (40).

The computational effort and performance for different order Qs was investigated for $\gamma = 1.6$. The algorithm was evaluated 5 consecutive times for the same order of Q. Table 1 presents these results. The mean value and standard deviation (sdt) for each respective order of the floating point operations (flops), computation time, and resulting performance $\|T_{z_0w}\|_2$ are presented. Therefore, it is seen that there is no significant performance improvement by choosing a Q of order higher than 2^{nd}. Also, it is clear that the algorithm reached almost constantly the same performance level ($\|\cdot\|_2$) for each order, since the standard deviation is close to zero for all cases. It is also evident that the computational cost is increased by raising the Q order. Figure 3 shows the flops required during the optimization as a function of the Q order.

Table 2 summarize the results obtained for $\gamma = 1.6$ and 2.0. A 2^{nd} order Q is needed to provide these results. Thus, the optimization was carried out for 5 parameters (Q is strictly proper). In the first stage, the $\mathcal{H}_2$ cost was 23.04 and 4.72 for $\gamma = 1.6$ and 2.0 respectively. In the second stage, these values were reduced further to 22.875 and 4.53. Figure 4 presents the evolution of the cost function for $\gamma = 2.0$ during the first stage.

7.2 Example 2

This example is taken from Sznaier and Blanchini (1994). The mixed $\mathcal{L}_1/\mathcal{H}_\infty$ design is to solve the following minimization

$$\min_{K stabilizing} \|T_{z_0w}\|_1 \quad s.t. \quad \|T_{zw}\|_\infty < \gamma \tag{49}$$

The generalized plant G is given by

$$G = \begin{bmatrix} \begin{bmatrix} 0 \\ W_2 \end{bmatrix} & \begin{bmatrix} W_1 \\ W_2 P \end{bmatrix} \\ W_2 & W_2 P \\ -1 & -P \end{bmatrix} \tag{50}$$

where W_1, W_2 and P are defined in (Sznaier and Blanchini 1994). The value of γ was set to 2.6 as in the original paper. Consequently, the optimization proposed in (42) was applied in order to compute the mixed controller. For this example, a genetic algorithm (GAOCP) was used in the first stage of the optimization. However, no clear advantages or disadvantages are shown using either algorithm.

Similarly to the first example, the computational effort and performance for different order Qs was first investigated ($\gamma = 2.6$). Table 3 presents these results varying the Q order among 1^{st} and 3^{rd}. No significant performance improvements were observed by using a Q of higher order. The algorithm was run again 5 consecutive times for the same order of Q. Analyzing table 3, it is observed that the algorithm reached almost constantly the same performance level ($\|\cdot\|_1$) for any order, since the standard deviation is always close to zero. However, the computational cost varied drastically for all cases. In addition, the computational time and flops are constantly increased by raising the Q order. Figure 5 shows the flops required during the optimization as a function of the Q order.

Table 4 summarize these result with a 2^{nd} order Q. Hence, a 6 parameter optimization was carried out (Q is now a proper transfer matrix). After the first stage optimization, the optimal cost was 4.1216. The second stage of the optimization reduced this value to 4.04 (i.e. $\|T_{z_0w}\|_1 = 4.04$). Furthermore, the value of the $\mathcal{H}_\infty$ norm is $\|T_{zw}\|_\infty = 2.5476 < 2.6$. On average, 307.77 sec. and 6.66×10^9 flops were needed to reach a solution. In this case, the computation of the $\mathcal{L}_1$ norm slowed down the optimization scheme. However, an improvement of $\mathcal{L}_1$ performance is seen compared with the result by Sznaier and Blanchini (1994). In figure 6, the evolution of the cost function is presented for the first stage of the optimization scheme.

8 Conclusions

Optimization schemes are presented to solve the multi-objective design problems. Parameterizations of norm bounded $\mathcal{H}_\infty$ and $\mathcal{H}_2$ functions were used to limit the number of variables and restrict the optimizations. This step reduces the complexity of the mixed synthesis problem and guarantees the closed-loop $\mathcal{H}_\infty$ or $\mathcal{H}_2$ performance according with the selected parameterization. Thus, the search for the appropriate Q parameter looks to minimize another performance measure. The resulting optimizations with this method are highly nonlinear and present multi-modal characteristics. For this reason, a two-stage algorithm was used in the optimization process. Numerical examples show the success of the optimization schemes to design mixed controllers. However, it is not possible to establish the best achievable performance with these techniques and this issue has to be explored iteratively. It is interesting to note that only low order Qs were needed in the numerical examples. Thus, the orders of resulting controllers are comparable to those of the generalized plants. In all the benchmark examples, the performance was always improved compared to previous results published in the literature. It should be pointed out that more complex mixed problems can also be treated in the same framework without any additional difficulty.

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Q	Flops		Computation Time		$\|T_{z_0w}\|_2$	
order	mean	std	mean	std	mean	std
1^{st}	1.38×10^8	3.33×10^5	19.12	0.38	23.10	0.00
2^{nd}	1.68×10^8	7.96×10^7	23.24	0.41	22.88	0.00
3^{rd}	2.91×10^8	2.86×10^6	28.48	1.70	22.87	0.00
4^{th}	4.33×10^8	2.49×10^7	36.02	2.22	22.85	0.01
5^{th}	6.72×10^8	7.62×10^6	47.43	0.22	22.84	0.01

Table 1: Computation effort and performance for $\gamma = 1.6$ in example 1.

γ	$\Vert T_{z_0w}\Vert_2$ Baeyens and Khargonekar (1994)	LMI toolbox	proposed optimization (2^{nd} order Q)
1.6	28.13	37.20	22.88
2.0	5.49	7.45	4.53

Table 2: Closed-loop performance of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controllers in example 1.

Q	Flops		Computation Time		$\|T_{z_0w}\|_1$	
order	mean	std	mean	std	mean	std
1^{st}	5.08×10^9	1.40×10^9	265.51	73.44	4.43	0.00
2^{nd}	6.66×10^9	3.23×10^9	307.77	149.42	4.04	0.03
3^{rd}	8.08×10^8	2.23×10^9	334.72	92.38	4.03	0.04

Table 3: Computation performance for $\gamma = 2.6$ in example 2.

γ	$\Vert T_{z_0 w}\Vert_1$	
	Sznaier and Blanchini (1994)	proposed optimization (2^{nd} order Q)
2.6	4.82	4.04

Table 4: Closed-loop performance of the mixed $\mathcal{L}_1/\mathcal{H}_\infty$ controllers in example 2.

Captions to Figures

Figure 1 LFT representation.

Figure 2 Multiobjective LFT form.

Figure 3 Computation effort (flops) as a function of the Q order for example 1 and $\gamma = 1.6$.

Figure 4 Evolution of cost function during the first stage of the optimization: example 1 and $\gamma = 2.0$.

Figure 5 Computation effort (flops) as a function of the Q order for example 2 and $\gamma = 2.6$.

Figure 6 Evolution of cost function during the first stage of the optimization: example 2.

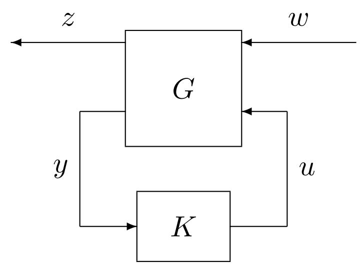

Figure 1: LFT representation.

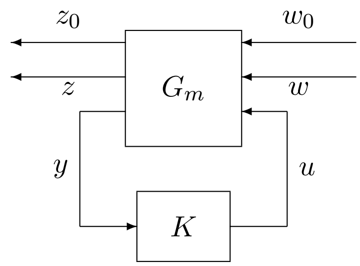

Figure 2: Multiobjective LFT form.

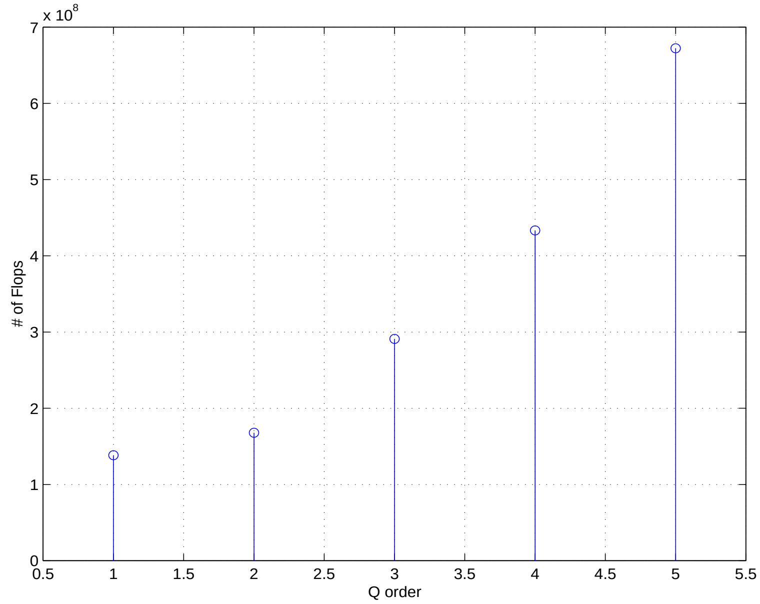

Figure 3: Computation effort (flops) as a function of the Q order for example 1 and $\gamma = 1.6$.

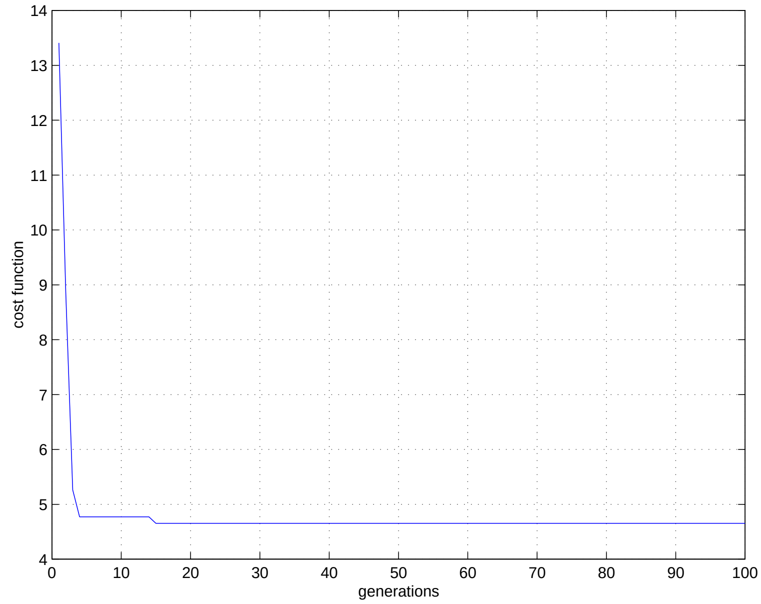

Figure 4: Evolution of cost function during the first stage of the optimization: example 1 and $\gamma = 2.0$.

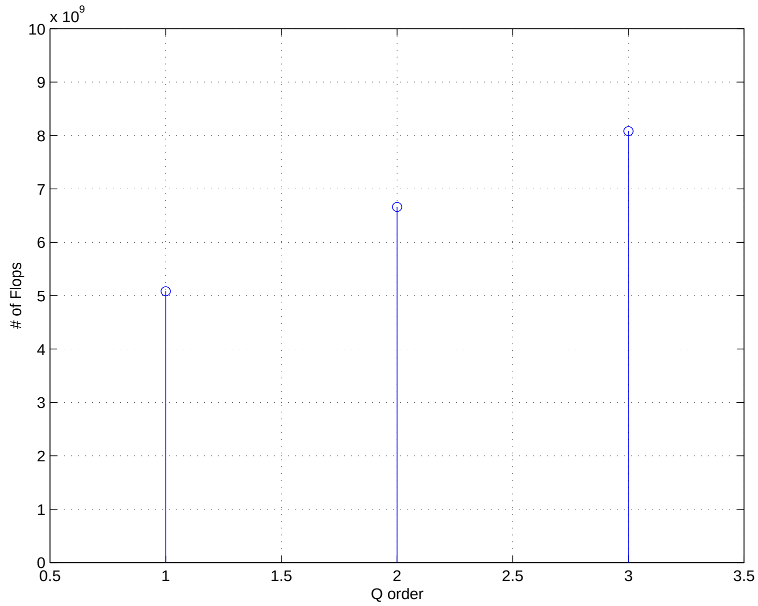

Figure 5: Computation effort (flops) as a function of the Q order for example 2 and $\gamma = 2.6$.

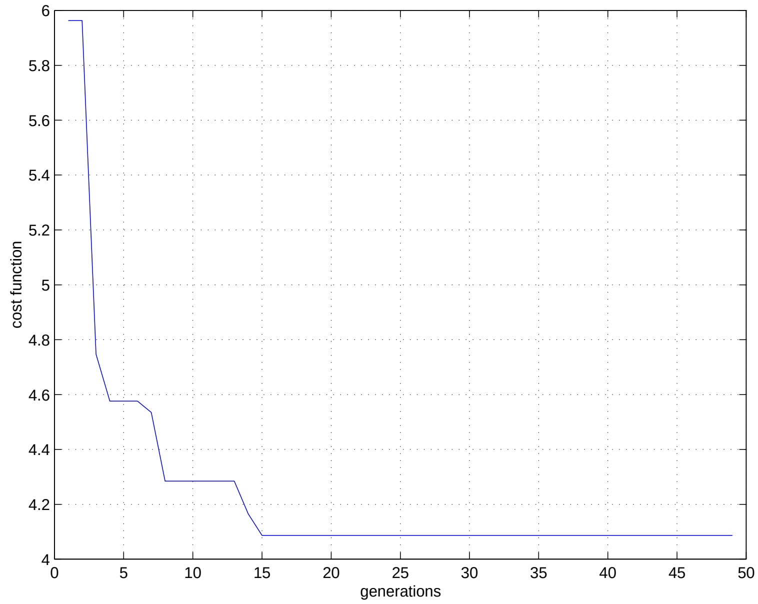

Figure 6: Evolution of cost function during the first stage of the optimization: example 2.