H_∞ AND H_2 STRONG STABILIZATION BY NUMERICAL OPTIMIZATION *

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Abstract: This paper presents a numerical approach for designing stable MIMO H_∞ and H_2 controllers. The proposed technique searches for a norm constrained stable transfer matrix Q in the H_∞ and H_2 suboptimal controller parameterizations so that the final controllers are stable. The norm constrained $Q's$ are explicitly parameterized for any fixed order and the H_2 and H_∞ strong stabilization problems are then converted to nonlinear constrained optimization problems. A numerical search is carried out by a two-stage optimization scheme in order to reach an optimal solution. Examples show the performance improvements of these algorithms over methods already presented in the literature.

Keywords: Strong Stabilization, Controller Parameterization, H_∞ Control, H_2 Control, Optimization.

1. INTRODUCTION

A necessary and sufficient condition for the existence of a stable stabilizing (*strong stabilizing*) controller for a plant is the so-called parity interlacing property (p.i.p.) (Vidyasagar (1985)). Procedures for designing strong stabilizing controllers are outlined in Vidyasagar (1985). It is well-known that the controllers obtained from $\mathcal{H}_2$/LQG optimal control theory are not guaranteed to be stable. In Corrado et al. (1997), Ganesh and Pearson (1989), and Halevi (1994), this problem has been analyzed and alternative algorithms has been proposed. Similarly, the H_∞ strong stability problem has also been addressed by Özbay (1995). Since the H_∞ suboptimal controller is in general not unique, it is reasonable to expect that even if the H_∞ central controller is unstable, there might still be a stable controller that could satisfy the H_∞ norm bound when the p.i.p. condition is satisfied. In Choi and Chung (2000), and Zeren and Özbay (2000), an approach for designing stable H_∞ controllers has been suggested based on the parameterization of all suboptimal H_∞ controllers. This approach converts conservatively the stable H_∞ controller design problem into another 2-block standard H_∞ problem. Recently, Campos and Zhou (2001) suggested a method to alleviate this conservativeness. In general, all these analytical methods do not guarantee solutions for the H_2 or H_∞ strong stabilization problems, and they require an iterative search in order to reach a satisfactory solution.

In recent years, evolutionary schemes such as genetic algorithms have been extensively used to solve nonlinear constrained optimization problems. These algorithms are usually applied to complex optimization problems where multi-local minima can restrict global convergence. Evolutionary schemes are inspired by the natural selection criteria where the stronger organisms are likely to survive after generations.

Genetic algorithms present two main characteristics: a multi-directional (random) search and an information exchange among best solutions. Pioneering work in this field is due to Holland Holland (1975) who first proposed the basic principles of genetic algorithms. Applications of genetic algorithms to control and signal processing have been reported in literature: digital IIR filter design (Man, Tang and Kwong (1999)), adaptive recursive filtering (White and Flockton (1997)), active noise control (Tang et al. (1995)), systems model reduction (Li et al. (1997)), etc.

In this paper, new approaches for designing stable MIMO H_∞ and H_2 controllers will be proposed. In these approaches, a numerical search using a two-stage optimization algorithm is carried out for norm constrained stable transfer matrices Q in the H_∞ and H_2 suboptimal controller parameterizations. The norm constrained $Q's$ are explicitly parameterized for any fixed order. Examples show the performance improvements of these algorithms over methods already presented in the literature.

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The rest of the paper is organized as follows. First, the notation used in the paper is presented in Section 2. Next, in Section 3 a description of the H_∞ strong stability problem and the equivalent optimization problem are outlined. Similarly, the characterization of the H_2 strong stabilization problem is detailed in Section 4. Section 5 describes the two-stage optimization scheme. Section 6 presents numerical examples and in Section 7 some conclusions are drawn.

2. NOTATION

Define a transfer function $G(s)$ by its state-space realization (A, B, C, D). The H_2 norm is defined as

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} Trace[G^*(j\omega)G(j\omega)]d\omega$$

An alternative characterization can be obtained by using the state space representation of G. For a stable G, we have

$$\|G\|_2^2 = Trace(B^*QB) = Trace(CPC^*) \qquad (1)$$

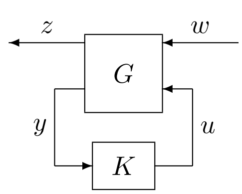
where P and Q are the observability and controllability Gramians. The H_∞ norm is defined as:

$$\|G\|_\infty = \sup_{Re(s)>0} \bar{\sigma}[G(s)] = \sup_{\omega\in\mathcal{R}} \bar{\sigma}[G(j\omega)].$$

Consider that the closed-loop system is described in LFT form, where the generalized plant G and controller K are assumed to be real-rational and proper. The dimensions of G are given by $z(t) \in \mathcal{R}^{p_1}$, $y(t) \in \mathcal{R}^{p_2}$, $w(t) \in \mathcal{R}^{m_1}$, $u(t) \in \mathcal{R}^{m_2}$, and $x(t) \in \mathcal{R}^n$. Then, G is partitioned accordingly

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \qquad (2)$$

and the transfer function from w to z is given by $T_{zw} = \mathcal{F}_l(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$ where $\mathcal{F}_l(\cdot,\cdot)$ is called a lower linear fractional transformation.



3. H_∞ STRONG STABILIZATION

3.1 H_∞ Suboptimal Parameterization

Assume that the generalized plant G is given by:

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \qquad (3)$$

with some suitable assumptions. It has been proved by Doyle et al. (1989), that all controllers $K(s)$ satisfying a *suboptimal* H_∞ restriction (i.e. given $\gamma > 0$, a stabilizing controller $K(s)$ satisfies $\|\mathcal{F}_l(G, K)\|_\infty < \gamma$) can be parameterized by $Q \in RH_\infty$, $\|Q\|_\infty < \gamma$ such that $K = \mathcal{F}_l(M_\infty, Q)$

$$M_\infty = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right] = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \qquad (4)$$

where M_∞ is constructed from the solutions of two Riccati equations (Doyle et al. (1989), and Zhou and Doyle (1998)). Based on the partition of the generalized plant G, it is clear that M_∞ and Q are $(p_2+m_2)\times(p_2+m_2)$ and $m_2 \times p_2$ transfer matrices respectively.

3.2 H_∞ Strong Stabilization Problem

A *strong stabilization* problem is to design a controller $K \in RH_\infty$ such that the closed-loop is internally stabilized and some performance specifications are satisfied. Consequently, the H_∞ Strong Stabilization problem is now defined as: *given* $\gamma > 0$, *find a stabilizing controller* $K \in RH_\infty$ *such that* $\|\mathcal{F}_l(G, K)\|_\infty < \gamma$. It can be seen from the parameterization of all suboptimal H_∞ controllers that in order to find a stable stabilizing K, it is enough to find a $Q \in RH_\infty$ with $\|Q\|_\infty < \gamma$ such that Q stabilizes M_∞. It is not hard to see that this can be achieved if and only if Q stabilizes $M_{22} = (\hat{A}, \hat{B}_2, \hat{C}_2, \hat{D}_{22})$. This gives the next result.

Lemma 1: *Assume that a solution to the suboptimal H_∞ control problem exists for a given $\gamma > 0$, i.e., $\exists M_\infty$ such that $\|\mathcal{F}_l(G, K)\|_\infty < \gamma$ and $K = \mathcal{F}_l(M_\infty, Q)$ where $Q \in RH_\infty$ and $\|Q\|_\infty < \gamma$. Then, the H_∞ Strong Stabilization is solvable if and only if $\exists Q = (A_q, B_q, C_q, D_q)$ of some suitable order, with $\|Q\|_\infty < \gamma$, such that A_k is Hurwitz where*

$$A_k = \begin{bmatrix} \hat{A} + \hat{B}_2R^{-1}D_q\hat{C}_2 & \hat{B}_2R^{-1}C_q \\ B_q\hat{R}^{-1}\hat{C}_2 & A_q + B_q\hat{R}^{-1}\hat{D}_{22}C_q \end{bmatrix}$$

$$R = I - D_q\hat{D}_{22} \text{ and } \hat{R} = I - \hat{D}_{22}D_q$$

3.3 H_∞ State-Space Parameterization

In Steinbuch and Bosgra (1991), a parameterization for an H_∞ norm bounded strictly proper and stable transfer matrix is presented. Now, the result is extended to the proper case in the following lemma.

Lemma 2: *Let $\gamma > 0$ and let Q be a stable transfer matrix of degree n_q and $\|Q\|_\infty < \gamma$, then Q can be represented as $Q = (A_q, B_q, C_q, D_q)$ with $A_q = A_{q_k} + A_{q_s}$, for some $A_{q_k} = -A_{q_k}^* \in \mathcal{R}^{n_q\times n_q}$, $B_q \in \mathcal{R}^{n_q\times p_2}$, $C_q \in \mathcal{R}^{m_2\times n_q}$, $D_q \in \mathcal{R}^{m_2\times p_2}$, and*

$$\begin{aligned} A_{q_s} &= \frac{1}{2}\{-B_qR^{-1}D_q^*C_q - C_q^*D_qR^{-1}B_q^* \qquad (5) \\ &\quad -B_qR^{-1}B_q^* - C_q^*(I + D_qR^{-1}D_q^*)C_q\} \\ \bar{\sigma}(D_q) &< \gamma \qquad (6) \end{aligned}$$

where $R = \gamma^2 I - D_q^* D_q$.

Note that when $D_q = 0$, we have (Steinbuch and Bosgra (1991))

$$A_{q_s} = -\frac{1}{2}(B_q B_q^*/\gamma^2 + C_q^* C_q) \quad (7)$$

Using the previous parameterization, the H_∞ strong stabilization problem can be converted to an optimization problem where A_{q_k}, B_q, C_q and D_q are free parameters.

3.4 Optimization Problem

The corresponding optimization problem is then defined as

$$l_{opt} = \min_{A_{q_k}, B_q, C_q, D_q} P(D_q) * e^{\max\{real[\lambda(A_k)]\}} \quad (8)$$

where

$$P(D_q) = \begin{cases} C & \bar{\sigma}(D_q) \geq \gamma \\ 1 & otherwise \end{cases}$$

and

$$A_k = \begin{bmatrix} \hat{A} + \hat{B}_2 R^{-1} D_q \hat{C}_2 & \hat{B}_2 R^{-1} C_q \\ B_q \hat{R}^{-1} \hat{C}_2 & A_q + B_q \hat{R}^{-1} \hat{D}_{22} C_q \end{bmatrix}$$

where $\lambda(\cdot)$ denote the eigenvalues of the corresponding matrix, A_q is constructed following (5) and $C \gg 1$ is a chosen constant. Note that the proposed optimization problem has a positive cost function for any combination of parameters and the H_∞ strong stabilization problem is solved if $l_{opt} < 1$. In the optimization algorithm, predefined ranges of variations for the elements of A_{q_k}, B_q and C_q were first established. These intervals were chosen according with the maximum and minimum elements of the corresponding generalized plant G. Note that it is usually hard to establish a range of variation for the elements of D_q in order to satisfy (6). Therefore, it was decided to limit each element of D_q to be $< \gamma$ and include a penalty function $P(\cdot)$ to penalize the combinations that violate (6).
Assume that the degree of Q is predefined to n_q, then according with the dimensions of the generalized plant (2) in the original H_∞ problem, the number of variables of each component of Q is given by

$$A_{q_k} = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n_q} \\ -a_{12} & 0 & \cdots & a_{2n_q} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n_q} & -a_{2n_q} & \cdots & 0 \end{bmatrix} \Rightarrow \frac{(n_q - 1)n_q}{2}$$

$$B_q = \begin{bmatrix} b_{11} & \cdots & b_{1p_2} \\ \vdots & \ddots & \vdots \\ b_{n_q 1} & \cdots & b_{n_q p_2} \end{bmatrix} \Rightarrow n_q \times p_2$$

$$C_q = \begin{bmatrix} c_{11} & \cdots & c_{1n_q} \\ \vdots & \ddots & \vdots \\ c_{m_2 1} & \cdots & c_{m_2 n_q} \end{bmatrix} \Rightarrow m_2 \times n_q$$

$$D_q = \begin{bmatrix} d_{11} & \cdots & d_{1p_2} \\ \vdots & \ddots & \vdots \\ d_{m_2 1} & \cdots & d_{m_2 p_2} \end{bmatrix} \Rightarrow m_2 \times p_2$$

Since Q is an $m_2 \times p_2$ proper transfer matrix, the total number of variables in the optimization scheme will be $(n_q - 1)n_q/2 + n_q m_2 + n_q p_2 + m_2 p_2$. In order to implement the optimization in a systematic scheme, the variables are aligned into a vector format

$$X = [a_{11}\ a_{12}\ \cdots\ a_{(n_q-1)n_q}\ b_{11}\ \cdots\ b_{n_q p_2}\ c_{11}\ \cdots\ c_{m_2 n_q}\ d_{11}\ \cdots\ d_{m_2 p_2}]^T \quad (9)$$

4. H_2 STRONG STABILIZATION

Assume that the generalized plant G has the following realization

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \quad (10)$$

with some appropriate assumptions and dimensions. Similar to the H_∞ case, the H_2 *strong stabilization* problem is defined as: *find a proper, real-rational and stable controller K which stabilizes G internally and minimizes the H_2 norm of the transfer matrix T_{zw} from w to z.*

4.1 H_2 Parameterization

The solution to the standard H_2 problem can be characterized in terms of the Youla parameterization of all admissible controllers: $K = \mathcal{F}_l(M_2, Q)$. In this formulation, $Q \in RH_2$ and M_2 is given by

$$M_2 = \left[\begin{array}{c|cc} \hat{A}_2 & -L_2 & B_2 \\ \hline F_2 & 0 & I \\ -C_2 & I & 0 \end{array}\right] \quad (11)$$

where $\hat{A}_2 = A + B_2 F_2 + L_2 C_2$, see Zhou and Doyle (1998). So, the closed-loop H_2 norm is given by

$$\|T_{zw}\|_2^2 = \|G_c B_1\|_2^2 + \|F_2 G_f\|_2^2 + \|Q\|_2^2$$

where the transfer matrices G_c and G_f are defined in Zhou and Doyle (1998). Consequently, the H_2 problem can be viewed as

$$\min_{Q \in RH_2} \|Q\|_2$$

Obviously, $Q = 0$ represents the optimal solution to the standard H_2 problem. As in the H_∞ parameterization, M_2 and Q are $(p_2 + m_2) \times (p_2 + m_2)$ and $m_2 \times p_2$ transfer matrices respectively.
Using the previous formulation of the H_2 problem, the strong stabilization problem can be stated as

$$\min_{Q \in RH_2} \|Q\|_2$$

such that

$$\begin{bmatrix} \hat{A}_2 & B_2 C_q \\ -B_q C_2 & A_q \end{bmatrix} \ is\ Hurwitz \quad (12)$$

where $Q = (A_q, B_q, C_q, 0)$. The condition (12) is needed since Q must stabilize M_2. Now, using the definition of the H_2 norm given by (1), a parameterization for all $Q \in RH_2$ follows.

Lemma 3: *Assume that $Q \in RH_2$ has degree n_q, then Q can be represented in the following form*

$$\|Q\|_2^2 = Trace(B_q^* B_q)$$

with $Q = (A_q, B_q, C_q, 0)$ and $A_q = A_{q_s} + A_{q_k}$ where $A_{q_s} = -\frac{1}{2} C_q^* C_q$,

$$\begin{aligned} A_{q_k} &= -A_{q_k}^* \in \mathcal{R}^{n_q \times n_q} \\ B_q &\in \mathcal{R}^{n_q \times p_2} \\ C_q &\in \mathcal{R}^{m_2 \times n_q} \end{aligned}$$

Using the previous result, the H_2 strong stabilization problem is now formulated as

$$\min_{A_{q_k}, B_q, C_q} Trace(B_q^* B_q) \tag{13}$$

such that

$$\begin{bmatrix} \hat{A}_2 & B_2 C_q \\ -B_q C_2 & -\frac{1}{2} C_q^* C_q + A_{q_k} \end{bmatrix} \; is\ Hurwitz$$

4.2 Optimization Problem

Comparing to the H_∞ formulation, the H_2 strong stabilization problem involves more constraints since H_2 minimization is now incorporated in addition to the controller stability. So, the optimization scheme becomes more complex. In order to solve (13), the next optimization scheme is proposed:

$$l_{opt} = \min_{A_{q_k}, B_q, C_q} Trace(B_q^* B_q) + J(A_k) \tag{14}$$

where

$$J(A_k) = \begin{cases} M e^{\max\{real[\lambda(A_k)]\}} & \max\{real[\lambda(A_k)]\} \geq 0 \\ 0 & otherwise \end{cases}$$

and

$$A_k = \begin{bmatrix} \hat{A}_2 & B_2 C_q \\ -B_q C_2 & -\frac{1}{2} C_q^* C_q + A_{q_k} \end{bmatrix}$$

Here, the constant M must be chosen sufficiently large such that $M > Trace(B_q^* B_q)$ all the time. Therefore, $l_{opt} < M$ means that the strong stabilization condition was satisfied. In this formulation, a discontinuous penalty function $J(\cdot)$ is introduced in order to enforce the strong stabilization restriction. Similar to the H_∞ case, intervals of variations were set for the elements of the matrices A_{q_k}, B_q and C_q. These intervals were chosen according with the elements of the corresponding matrices in the generalized plant G. The number of variables to optimize will be $(n_q - 1)n_q/2 + n_q m_2 + n_q p_2$ which is lower than in the H_∞ case since Q is now strictly proper.

The optimization problems just introduced in (8) and (14) are generally nonconvex and present intrinsically multi-modal characteristics. Therefore, it is natural to think that these problems are not practical to solve. Nevertheless, genetic optimization has turned out to be quite useful to solve these complicated optimization problems. Moreover, if it is used in conjunction with a well-known gradient-based optimization technique, a powerful optimization scheme is obtained that can effectively solve the proposed problems.

5. OPTIMIZATION SCHEME

In the first attempt to solve the strong stabilization problems (8) and (14), standard nonlinear optimization techniques such as quasi-newton method and conjugate gradient (Grace (1992)) were applied. However, convergence was always limited to the initial conditions. The algorithms constantly got caught in local minima. A change of direction was obviously needed. Thus, the genetic algorithms came as viable solution to solve these problems. However, the gradient-based approaches have nice properties that should not be forgotten. Therefore, it was decided to use a gradient-based algorithm in conjunction with a genetic algorithms. In this way, the genetic algorithm was applied first to perform a global search in the parameters space and find a minimum solution. Next, a local search was conducted to obtain the optimal solution. This two-stage optimization outperformed the use of each one of the algorithms alone.

For the first stage, the algorithm called GAOCP (Genetic Algorithm for Optimization of Continuous Parameters) was coded in MATLAB. However, any other type of evolutionary scheme such as *evolution algorithm* and *genetic programming*, or natural inspired algorithm such as *simulated annealing* could have been used instead. For the second stage, a quasi-newton plus linear search (Nocedal and Wright (1999)) scheme was used to perform the local search. The gradient-based optimization was carried out by using the *Optimization Toolbox* (Grace (1992)) of MATLAB.

6. NUMERICAL EXAMPLES

The solutions to the optimization problems (8) and (14) for the following numerical examples were computed with a Sun Microsystems Ultra 5 work station. The constants C and M in (8) and (14) were given the values 1×10^3 and 1×10^6 respectively.

6.1 Example 1 (H_∞ and H_2 Strong Stabilization)

The benchmark SISO example used by Campos and Zhou (2001), and Zeren and Özbay (1999) was examined. The generalized plant is described thoroughly in Campos and Zhou (2001). The parameter $\beta > 0$ was introduced to include some penalty on the control signal. The generalized plant is non-minimum phase and it has a double pole at the origin, but it satisfies the p.i.p. condition, so it could be strongly stabilized. The system is SISO therefore the optimal H_∞ controller is unique. Consequently, it will not be possible to achieve γ_{opt} with a stable controller. The optimal H_∞ performance is 0.2324 and 0.1415, and the corresponding central controllers are unstable with complex right half plane poles located at $0.045 \pm 1.83j$ and $0.114 \pm 1.97j$ for $\beta = 0.1$ and 0.01 respectively.

TABLE 1. THE SMALLEST γ ACHIEVED H_∞ STRONG STABILIZATION FOR EXAMPLE 1.

β	(14)	Campos and Zhou (2001)
0.1 $\gamma_{opt} = 0.232$	$\gamma = 0.233$	$\gamma = 0.237$
0.01 $\gamma_{opt} = 0.142$	$\gamma = 0.145$	$\gamma = 0.151$

The optimization scheme in (8) was carried out by the combined GAOCP and gradient-based approach. The results are presented in Table 1. A Q of 2^{nd} order was enough to achieve strong stabilization. Nevertheless, a higher order Q was also tested but no improvements in the performance were seen. Thus, the optimization was computed for 6 variables. Note that the achieved closed-loop performance improved the results in (Campos and Zhou (2001)). For the case $\beta = 0.1$, the optimization took an average of 96.03 sec. and 3.355×10^8 flops, and 60.37 sec. and 1.929×10^8 flops for $\beta = 0.01$. In Figure 1, the evolution of the cost function during the first-stage optimization is presented for $\beta = 0.1$. After the first stage, the minimum performance was 1.0019. In the next stage, it was possible to reduce the cost function further and find a stable controller. Hence, the optimum performance was 0.9931 (i.e. a stable controller). The GACOP algorithm itself could not achieve the results in Table 1 and in general needs a longer time to reach a satisfactory solution.

If the generalized plant is taken with $\beta = 0.001$, the H_2 optimal controller is unstable with complex RHP poles located at $0.138 \pm 1.79j$. Thus, the optimization in (14) was employed to obtain the best H_2 controller that satisfied the strong stability condition. Table 2 presents the results. A Q of 1^{st} order was not able to achieve strong stabilization. However, a Q of degree $\geq 2^{nd}$ was able to keep the controller stable and have an H_2 performance closer to the optimal. An average of 31.84 sec and 1.70×10^8 flops were needed to reach a solution. After the first stage (GAOCP), the best performance was 2.39×10^3. Finally, after the local search (gradient approach), the minimum cost was reduced to 1.88×10^3.

TABLE 2. CLOSED-LOOP PERFORMANCE FOR H_2 STRONG STABILIZATION: EXAMPLE 1.

		Degree of Q			
	Optimal	1^{st}	2^{nd}	3^{rd}	4^{th}
$\|T_{zw}\|_2$	0.109	unstable	0.118	0.118	0.127

It should be intuitive that as the order of Q is increased the H_2 performance of the closed-loop system should improve or at least should not be deteriorated. However, the results in Table 2 show the opposite. Note that for Q of 4^{th} order, the number of parameters in the optimization is 14. In this case, the parameters space is very large and it is highly probable that the optimization could get trapped in a local minima. In order to verify this analysis, the population size and mutation rate are increased in the GAOCP algorithm. Then, for Q of 4^{th} order the optimal performance is now 0.1179 which is consistent with the other two cases in Table 2. As it is expected, the algorithm now takes longer time and more flops to reach a solution (76.51 sec and 3.99×10^3 respectively). However, the increase in computation time does not improve drastically the best results already obtained with a Q of 2^{nd} order. In summary, the optimization problem becomes very complex when more than 10 parameters are involved and the probability to reach only a local minimum increases. The same observations can be made for other examples to be presented later.

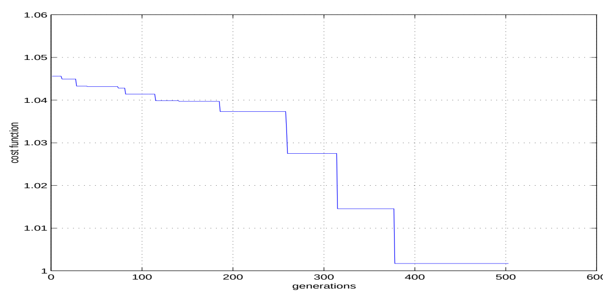


FIG. 1. Evolution of Cost Function During Optimization for H_∞ Design Example 1 and $\beta = 0.1$.

6.2 Example 2 (H_2 Strong Stabilization)

A benchmark problem in H_2 strong stabilization was taken from Ganesh and Pearson (1989). The realization of the generalized plant G is presented completely in Ganesh and Pearson (1989) or Corrado et al. (1997). The generalized plant (2^{nd} order) is stable, so the p.i.p. is obviously satisfied. The optimal H_2 performance is 493.8 and the controller has a unstable pole at 18.7. In Ganesh and Pearson (1989), the optimum H_2 performance with a stable controller (4^{th} order) was computed, 622.73. The resulting optimal controller has two poles at the origin, but if the stability boundary is moved back to $s = -0.5$ (i.e. suboptimal controller), the H_2 cost is now 628.40. In Corrado et al. (1997) new results were reported for this example. The closed-loop H_2 cost was 627.31 and 622.30 for a second and fourth order controllers respectively. Running the H_2 optimization algorithm (14), the results in Table 3 were obtained. An average of 26.59 sec and 1.94×10^7 flops were needed to reach a solution. In this example, a Q of 1^{st} order (i.e. 3^{rd} order controller) achieved the smallest closed-loop H_2 norm while keeping a stable overall controller. The result obtained is then into $< 1\%$ error of the optimal performance in Corrado et al. (1997).

TABLE 3. CLOSED-LOOP PERFORMANCE FOR H_2 STRONG STABILIZATION: EXAMPLE 2.

	Corrado et al. (1997)	Degree of Q 1^{st}	2^{nd}
$\|T_{zw}\|_2$	622.20	627.36	627.42

6.3 Example 3 (H_∞ Strong Stabilization)

In order to demonstrate the extension of the proposed algorithm to an MIMO problem, the aircraft *tracking problem* presented in Özbay (1995) is used. The system has two output measurements and one control input. The realization of the generalized plant G is analyzed and described in Özbay (1995).

The optimal H_∞ performance is 1.72 and the corresponding controller has a right half plane pole located at 2.81. In Özbay (1995), a constant Q, $Q = [-1.99\ 0]$, was able to strongly stabilize the system with $\gamma = 2$. However, for $Q = [-1.98\ 0]$ the controller is not stable anymore. The algorithm in (8) was then started trying to improve the performance $\|T_{zw}\|_\infty < 2$. However, the optimization proved to be very complex; for $1.8 < \gamma < 1.99$ the optimization algorithm sometimes was able to reach a stable controller. Nevertheless, as the value of γ approaches to 1.99 the probability raises. The degree of Q was varied from 0^{th} to 5^{th} but there was no improvement in this pattern. Nevertheless, for $\gamma \geq 1.99$ a stable controller was always reached with a Q of 2^{nd} order. Thus, a total of 9 parameters were optimized. The algorithm (8) took in average 38.05 sec and 7.069×10^7 flops to find a stable controller for $\gamma = 1.99$. The first optimization (GAOCP) was able to reach a cost of 1.0012 and the local search reached finally a stable controller. Thus, the optimum cost was $= 0.9970$.

7. CONCLUSIONS

Optimization schemes were presented to solve the H_∞ and H_2 strong stabilization problems. The resulting schemes are highly nonlinear and present multi-modal characteristics. A two-stage algorithm was used in the optimization process. Numerical examples show the success of the optimization schemes to design stable controllers for their corresponding problems. The controllers achieved closed-loop performance close to the optimal. Only low order Q were needed in the numerical examples. Thus, the order of resulting controllers is comparable to the generalized plants. In all the benchmark examples, the performance was improved or comparable to previous results published in literature.

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