# Are the Tradeoffs between Performance and Robustness Intrinsic for Feedback Systems? *

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**Abstract**

In this paper, we demonstrate through a simple example the effectiveness of GIMC architecture in the design of robust and high performance feedback controllers. We show that the design tradeoffs between performance and robustness for a feedback system depend on the controller architecture.

**Keywords:** Robustness, Robust Performance, Design Tradeoff, Generalized Internal Model Control

## 1 Introduction

It is commonly known in the control community that there are intrinsic tradeoffs between achievable performance and robustness for a given control architecture, see for example [3, 6, 7, 10, 11] for some detailed analyses and discussions. In other words, in order to achieve certain performance, one must sacrifices some robustness properties of the control systems and vise versa once the control architecture is chosen. For example, a high performance controller designed for a nominal model may have very little robustness against the model uncertainties and external disturbances. For this reason, worst-case robust control design techniques such as $H_\infty$ control, $L_1$ control, $\mu$ synthesis, etc, have gained popularity in the last twenty years or so, see for example, [1, 4, 5, 8, 9, 10, 11] and references therein. Unfortunately, it is well recognized in the robust control community that the robustness of the closed-loop system design is usually achieved at the expense of performance. In particular, it is well known that the robust control design techniques such as $H_\infty$ optimization, $L_1$ optimization, and $\mu$ synthesis usually result in controllers that are robust to model uncertainties and external disturbances but may have very poor nominal performance. This is not hard to understand since most robust control design techniques are based on the worst possible scenarios which may never occur in a particular control system. Thus such controllers are not very desirable in many applications. Nevertheless, the ability to be able to control the system under the worst-case scenario is also very important in many applications and hence it is desirable to have design techniques

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that can achieve the same level of robustness when there are model uncertainties and external disturbances while at the same time perform well when there is no or little model uncertainties and external disturbances. The control architecture proposed by this author in [12], which is called Generalized Internal Model Control (GIMC), seems to be a good candidate for achieving this objective. In other words, the tradeoffs between robustness and performance in a feedback control system depend very much on the control architecture. It is our intention in this paper to demonstrate this design technique through a simple example so that application engineers may make appropriate decisions in their applications as what the most effective techniques may be applied.

The paper is organized as follows. Section 2 introduces the GIMC architecture. Section 3 uses a simple example to demonstrate the GIMC control design technique and its effectiveness. Some concluding remarks are given in Section 3.

The notations used in the paper are standard. $H_\infty$ denotes the Banach space of bounded analytic functions with the $\infty$ norm of a scalar function defined as $\|F\|_\infty = \sup_\omega |F(j\omega)|$ for any $F \in H_\infty$.

## 2 Generalized Internal Model Control Structure

Consider a standard feedback configuration shown in Figure 1 where $\tilde{G}$ is a linear time invariant plant and $K$ is a linear time invariant controller. It is well understood that the model $\tilde{G}$ is in general not perfectly known. What one actually knows is a nominal model $G$. Now assume that $K_0$ is a stabilizing controller for the nominal plant $G$ and assume $G$ and $K_0$ have the following stable coprime factorizations

$$K_0 = \tilde{V}^{-1}\tilde{U}, \quad G = \tilde{M}^{-1}\tilde{N}.$$

Then it is well known [9, 10, 11] that every stabilizing controller for $G$ can be written in the following form:

$$K = (\tilde{V} - Q\tilde{N})^{-1}(\tilde{U} + Q\tilde{M})$$

for some $Q \in H_\infty$ such that $\det(\tilde{V}(\infty) - Q(\infty)\tilde{N}(\infty)) \neq 0$.

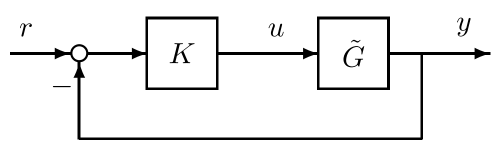


Figure 1: Standard Feedback Configuration

It is proposed in [12] that this controller can be implemented as shown in Figure 2. This controller architecture is called the Generalized Internal Model Control (GIMC) in [12] due to the similarity with the well-known Internal Model Control (IMC), see [8]. Note that the feedback diagram in Figure 2 is not equivalent to the diagram in Figure 1 since the reference signal $r$ enters into the system from a different location. Nevertheless, the internal stability of the system is not changed since the transfer function from $y$ to $u$ is $-K$ and is not changed. Thus this controller implementation also stabilizes internally the feedback system with plant $G$ for any $Q \in H_\infty$ such that $\det(\tilde{V}(\infty) - Q(\infty)\tilde{N}(\infty)) \neq 0$.

The distinct feature of this controller implementation is that the inner loop feedback signal $f$ is always zero, i.e., $f = 0$, if the plant model is perfect, i.e., if $\tilde{G} = G$. The inner loop is only active when there is a model uncertainty or other sources of uncertainties such as disturbances and sensor

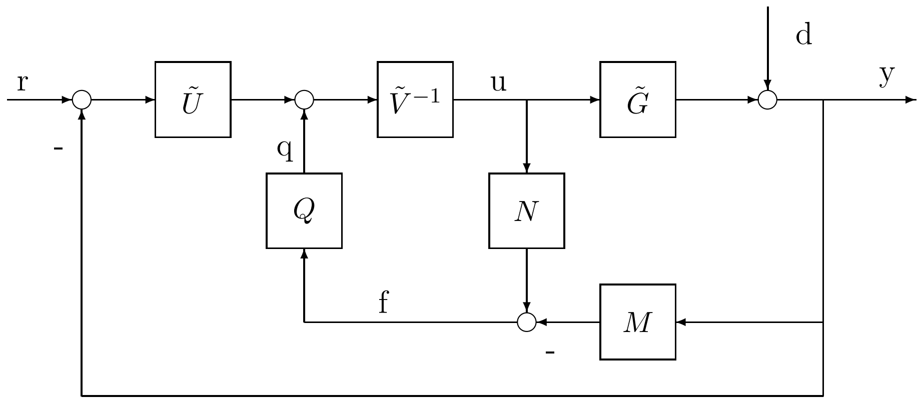


Figure 2: Generalized Internal Model Control Structure

noises. Hence $Q$ can be designed to robustify the feedback systems. Thus this controller design architecture has a clear separation between performance and robustness.

**Controller Design:** A high performance robust system can be designed in two steps: (a) Design $K_0 = \tilde{V}^{-1}\tilde{U}$ to satisfy the system performance specifications with a nominal plant model $G$; (b) Design $Q$ to satisfy the system robustness requirements. Note that the controller $Q$ will not affect the system nominal performance.

In the case when $\tilde{U}$ is minimum phase, then without loss of generality, we can take $Q = \tilde{U}\hat{Q}$ for some stable $\hat{Q}$. Then the controller can be written as

$$K = (I - K_0\hat{Q}\tilde{N})^{-1}(K_0 + K_0\hat{Q}\tilde{M})$$

and the GIMC structure becomes the one shown in Figure 3.

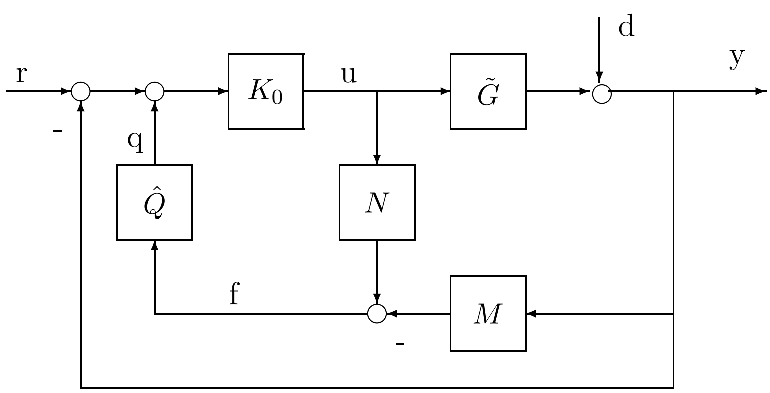


Figure 3: An Alternative GIMC Implementation

# 3 A Design Example

To demonstrate the effectiveness of the GIMC architecture, we shall take a simple example from *$\mu$-Analysis and Synthesis Toolbox* [1]. The nominal plant is given by

$$G = \frac{1}{s-1}.$$

The true plant is known to be in a multiplicative set

$$\mathcal{M}(G, W_u) := \left\{ G(1 + \Delta W_u) : \max_{\omega} |\Delta(j\omega)| \leq 1 \right\}$$

with

$$W_u = \frac{\frac{1}{4}\left(\frac{1}{2}s + 1\right)}{\frac{1}{32}s + 1}$$

and $\Delta$ can be any transfer function such that $G(1+\Delta W_u)$ and $G$ have the same number of unstable poles. The block diagram of this control system is shown in Figure 4.

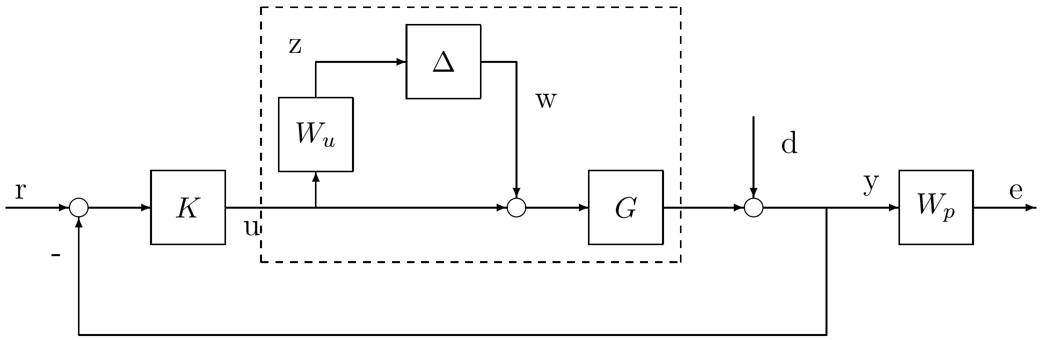


Figure 4: Uncertain Feedback Control System

The feedback system can be put in a LFT form as shown in Figure 5 with

$$P = \begin{bmatrix} 0 & 0 & W_u \\ W_p G & W_p & W_p G \\ G & I & G \end{bmatrix}.$$

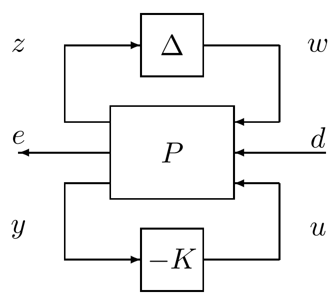


Figure 5: LFT form of the feedback system

As stated in [1], the performance objective of this control system is to keep the closed-loop system stable and have output disturbance rejection up to 0.6 rad/sec, with at least 100 : 1 disturbance rejection at DC for all possible models in the set.

The design objective can be approximately be represented as a weighted $H_\infty$ norm constraint on the sensitivity function $T_{ed}$:

$$\|T_{ed}\|_\infty = \left\| \frac{W_p}{1 + \tilde{G}K} \right\|_\infty \leq 1$$

for all $\tilde{G} \in \mathcal{M}(G, W_u)$ with the weighting function

$$W_p = \frac{\frac{1}{4}s + 0.6}{s + 0.006}.$$

Let $M$ be the transfer matrix from $(w, d)$ to $(z, e)$,

$$\begin{bmatrix} z \\ e \end{bmatrix} = M(s) \begin{bmatrix} w \\ d \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} w \\ d \end{bmatrix}$$

Then the nominal performance (i.e., when $\Delta = 0$) can be evaluated by the transfer function

$$T_{ed}^0 := T_{ed}|_{\Delta=0} = M_{22} = \frac{W_p}{1 + GK}$$

and the robust stability margin can be evaluated by the transfer function

$$T_{zw} = M_{11} = \frac{W_u GK}{1 + GK}.$$

Finally, the robust performance condition

$$\|T_{ed}\|_\infty = \left\| \frac{W_p}{1 + \tilde{G}K} \right\|_\infty \leq 1$$

is satisfied if and only if the structured singular value

$$\mu_{\Delta_P}(M(j\omega)) \leq 1, \quad \forall \omega$$

where $\Delta_P = \mathrm{diag}(\Delta, \Delta_f)$.

Two PI controllers are designed in [1] for this uncertain system

$$K_1 = \frac{10(0.9s + 1)}{s}, \qquad K_2 = \frac{2.8s + 1}{s}.$$

The frequency responses of $T_{ed}^0$ for both controllers are shown in Figure 6 and it is clear that the nominal performance criteria are satisfied by both controllers since $|T_{ed}^0(j\omega)| < 1$ for all frequencies. Moreover, the plot also shows that $K_1$ has much better nominal performance than $K_2$ does. Similarly, the frequency responses of $T_{zw}$ shown in Figure 7 indicate that the robust stability condition, $\|T_{zw}\|_\infty < 1$, is satisfied by both controllers with $K_2$ having much large robust stability margin than $K_1$ does.

On the other hand, the structured singular value plots in Figure 8 show that the robust performance is satisfied with $K_2$ but is not satisfied with $K_1$.

To evaluate the time domain behavior of the control system with both controllers, ten plants including the nominal and two "worst-case" plants in the set $\mathcal{M}(G, W_u)$ are used for performance evaluation in [1] and they are given by

$$G = \frac{1}{s-1}, \qquad G_1 = \frac{1}{s-1}\frac{6.1}{s+6.1}$$

$$G_2 = \frac{1.425}{s-1.425}, \qquad G_3 = \frac{0.67}{s-0.67}$$

$$G_4 = \frac{1}{s-1}\frac{-0.07s+1}{0.07s+1}, \qquad G_5 = \frac{1}{s-1}\frac{70^2}{s^2 + 2 \cdot 0.15 \cdot 70s + 70^2}$$

$$G_6 = \frac{1}{s-1}\frac{70^2}{s^2 + 2 \cdot 5.6 \cdot 70s + 70^2}, \qquad G_7 = \frac{1}{s-1}\left(\frac{50}{s+50}\right)^6$$

$$G_{wc1} = \frac{1}{s-1}\frac{-2.9621(s-9.837)(s+0.76892)}{(s+32)(s+0.56119)}, \qquad G_{wc2} = \frac{1}{s-1}\frac{s^2 + 3.6722s + 34.848}{(s+7.2408)(s+32)}$$

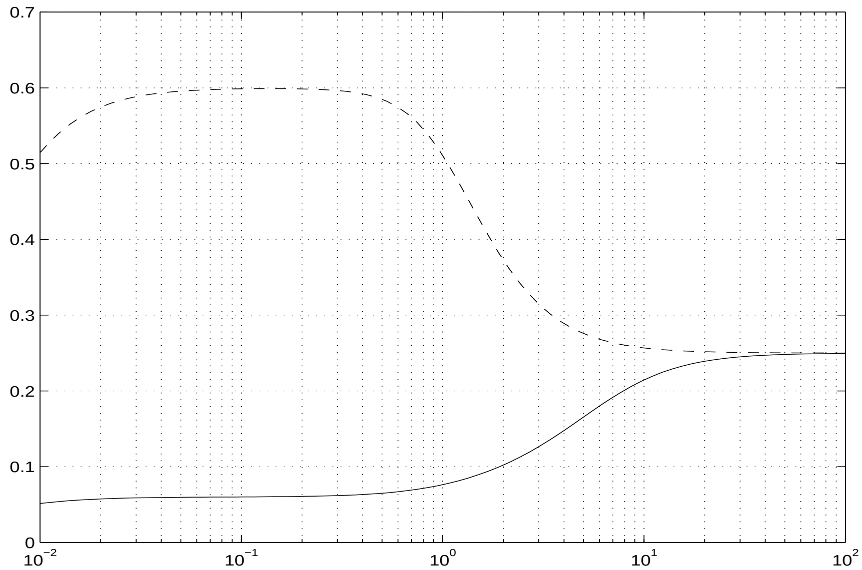


Figure 6: Frequency Responses of $T_{ed}^0$ for Nominal Performance: $K_1$ (solid) and $K_2$ (dashed)

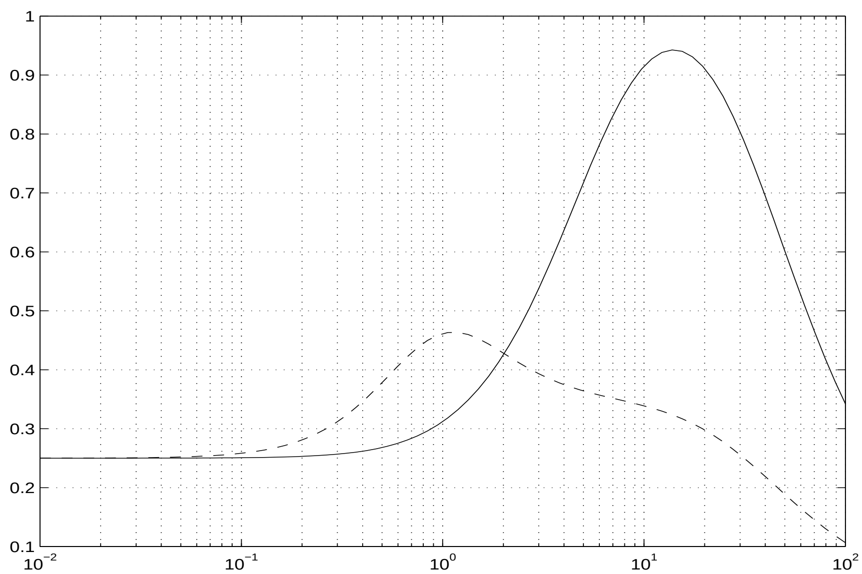


Figure 7: Frequency Responses of $T_{zw}$ for Robust Stability: $K_1$ (solid) and $K_2$ (dashed)

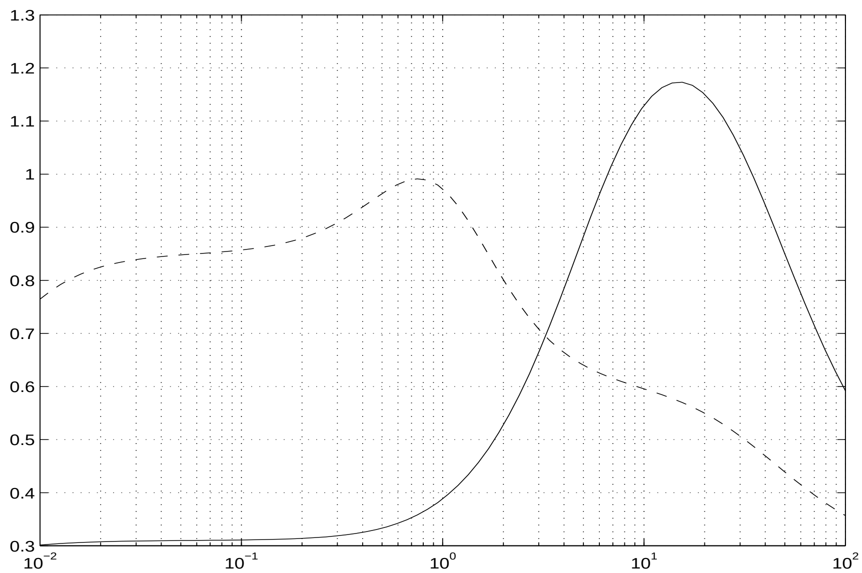


Figure 8: Frequency Responses of $\mu_{\Delta_P}(M(j\omega))$ for Robust Performance: $K_1$ (solid) and $K_2$ (dashed)

The step responses of the closed-loop system with $K_1$ and $K_2$ implemented as in Figure 1 are shown respectively in Figure 9 and Figure 10.

It is clear from the simulations that the controller $K_1$ gives very good and fast response for the nominal system but it performs very badly for some perturbed plants. On the other hand, the controller $K_2$ shows a very robust performance with respect to model uncertainties but the responses are very slow and the closed-loop system performs poorly in the nominal case as well as in the perturbed case.

From the above analysis, it is very desirable to have a controller that will take advantages of good performance of $K_1$ in the nominal case and good robustness of $K_2$ in the perturbed cases. The GIMC architecture is a good candidate for achieving this objective.

Let $G = N/M$ be a stable factorization with

$$N = \frac{1}{s+1}, \quad M = \frac{s-1}{s+1}.$$

Then it is easy to show that

$$K_2 = \frac{K_1(1+\hat{Q}M)}{1-K_1\hat{Q}N}$$

with

$$\hat{Q}(s) = -\frac{0.1s(6.2s+1)(s+1)}{(0.9s+1)(s^2+1.8s+1)}.$$

Hence the controller $K_2$ can be implemented using a GIMC structure as shown in Figure 11. Note that the transfer function from $y$ to $u$ is $-K_2$. Thus the robustness properties of the closed-loop system are the same as the controller $K_2$ is implemented in the standard feedback framework.

The step responses of the nominal system with $K_1$, $K_2$, and the GIMC implementation are shown in Figure 12. The step responses show clearly that the control system with $K_1$ and GIMC have the same nominal responses and are much better than that due to $K_2$. Figure 13 shows the

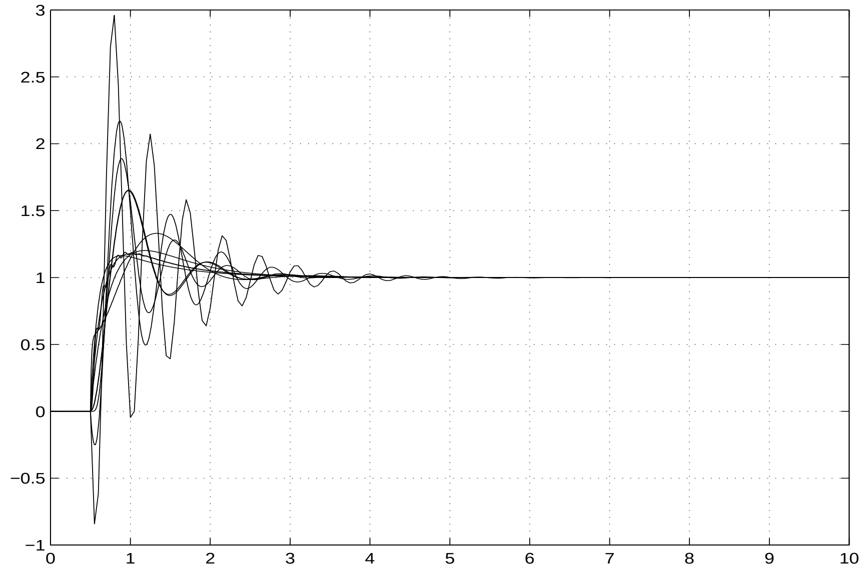


Figure 9: Step Response with $K_1$ and Various Plants for Standard Feedback Implementation

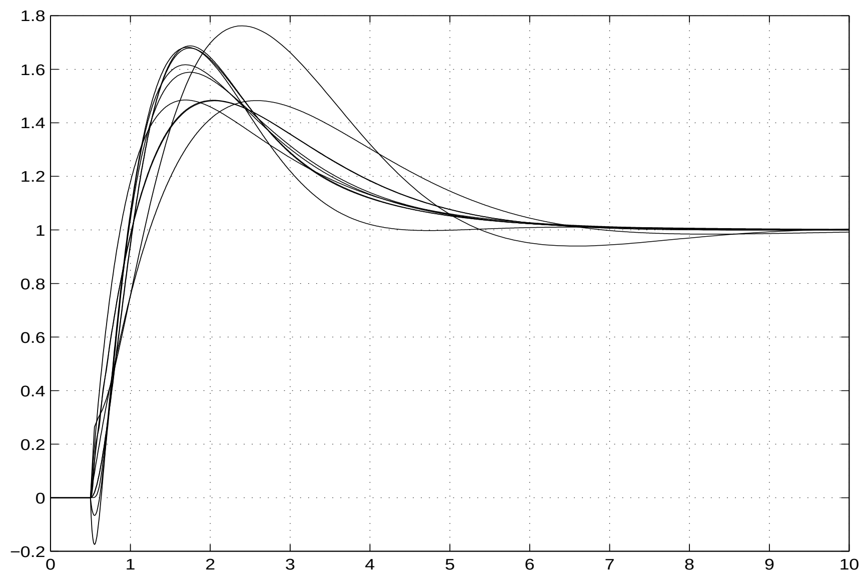


Figure 10: Step Response with $K_2$ and Various Plants for Standard Feedback Implementation

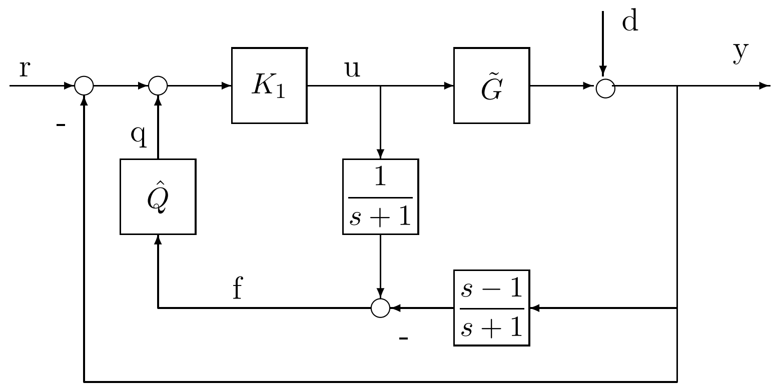


Figure 11: GIMC Implementation of $K_2$

step responses of the closed-loop system with the nominal model ($G$) as well as various perturbed models ($G_1, \ldots, G_7$, $G_{wc1}$, and $G_{wc2}$) when $K_2$ is implemented using the GIMC structure as shown in Figure 11.

For comparison, the step responses for the "worst-case" plants with the controller $K_1$, $K_2$, and the GIMC implementation of $K_2$ are shown in Figures 14 and 15. It is clear from the simulations that the GIMC implementation delivers good nominal response as well as superb robust performance.

# 4 Conclusions

We have shown through a simple first order example the effectiveness of the GIMC architecture. The price for achieving such a high performance and robust controller is the complexity of the controller. In that regard, controller approximation method may be applied as suggested in [12]. The application of GIMC in fault tolerant control is discussed in [2].

# References

[1] G. Balas, J. C. Doyle, K. Glover, A. Packard, and R. Smith, *$\mu$-Analysis and Synthesis Toolbox*, MUSYN Inc. and The MathWorks, Inc., 1991.

[2] D. U. Campos-Delgado and K. Zhou, "Fault tolerant control of a gyroscope system," *American Control Conference*, Arlington, VA, pp. 2688-2693, June 25-27, 2001.

[3] J. Chen, "Sensitivity integral relations and design trade-offs in linear multivariable feedback systems," *IEEE Trans. Auto. Contr.*, Vol. 40, No. 10, pp. 1700-1716, 1995.

[4] M. Dahleh and I.J. Diaz-Bobillo, *Control of Uncertain Systems: A Linear Programming Approach*, Prentice Hall, 1995.

[5] J. C. Doyle, K. Glover, P.P. Khargonekar, B.A. Francis, "State-space solutions to standard $\mathcal{H}_\infty$ and $H_\infty$ control problems," *IEEE Trans. Auto. Control*, vol. AC-34, No. 8, pp. 831-847, 1989.

[6] J. S. Freudenberg and D. P. Looze, *Frequency Domain Properties of Scalar and Multivariable Feedback Systems*, Lecture Notes in Contr. and Info. Science, Vol. 104, Springer-Verlag, Berlin, 1988.

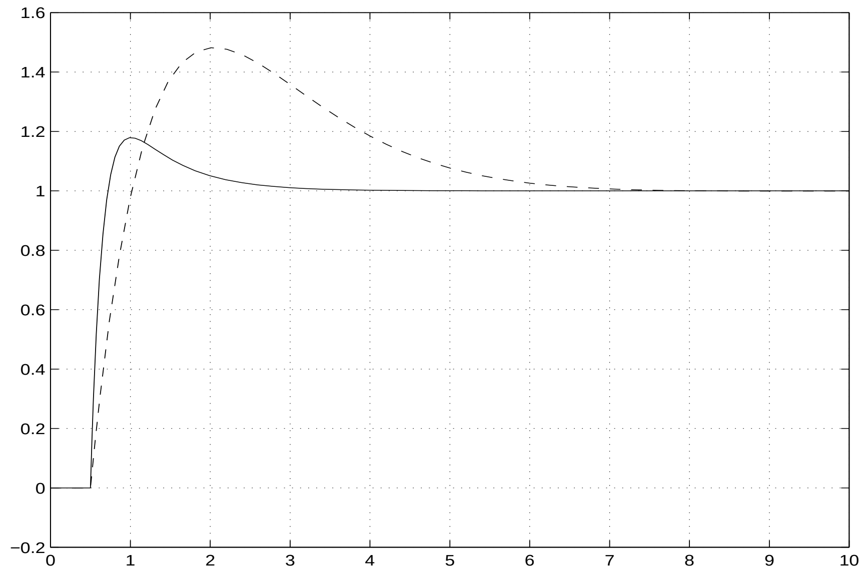


Figure 12: Step Responses of the Nominal $G$: $K_1$ (solid), $K_2$ (dashed), and GIMC (solid)

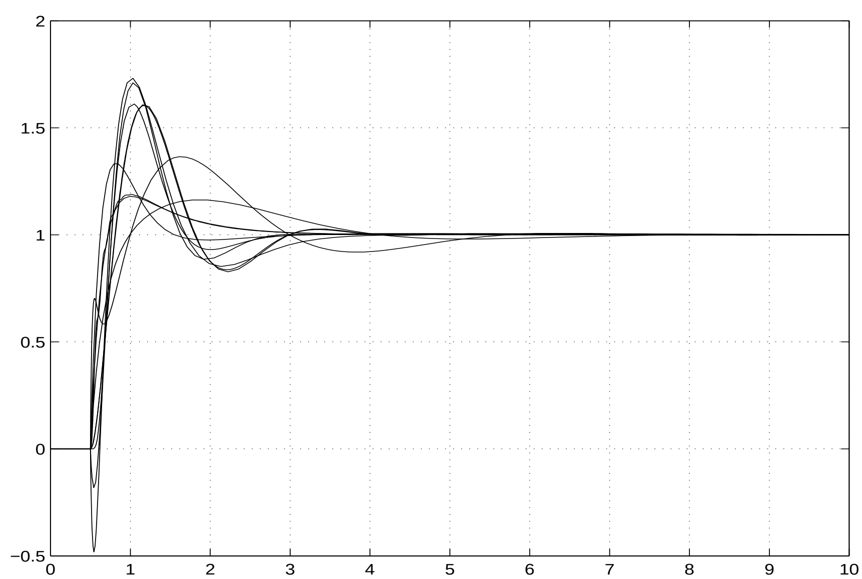


Figure 13: Step Responses of the Closed-loop System with $K_2$ Implemented Using the GIMC Structure Under Various Perturbations

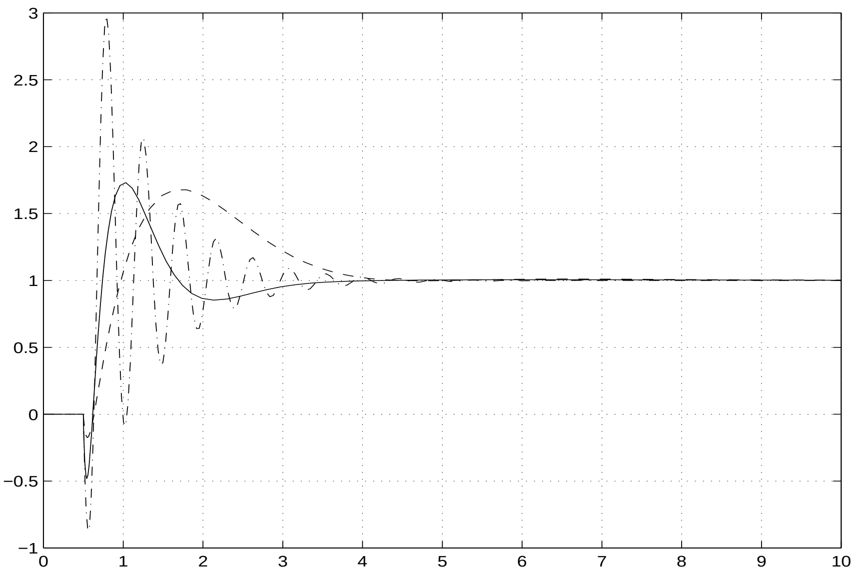


Figure 14: Step Responses of $G_{wc1}$: $K_1$ (dash-dot), $K_2$ (dashed), and GIMC (solid)

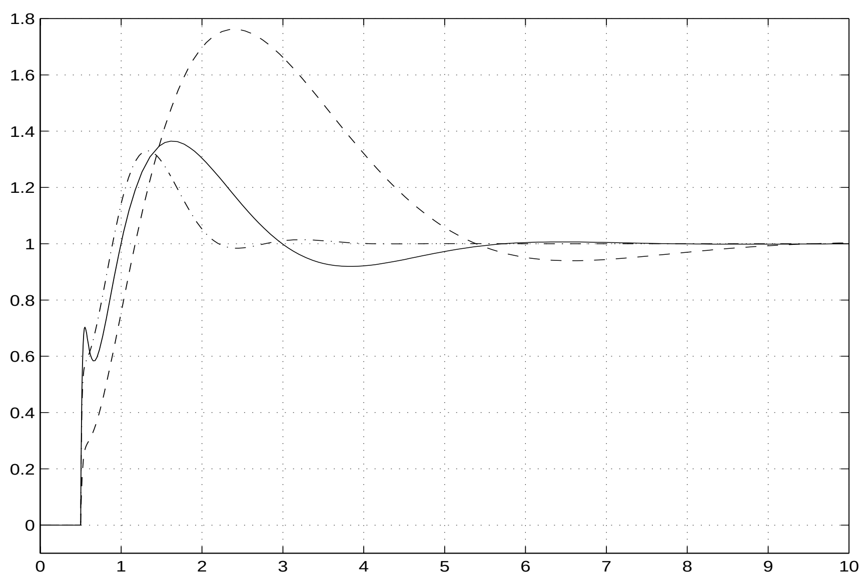


Figure 15: Step Responses of $G_{wc2}$: $K_1$ (dash-dot), $K_2$ (dashed), and GIMC (solid)

[7] I. M. Horowitz, *Synthesis of Feedback Systems*, Academic Press, London, 1963.

[8] M. Morari and E. Zafiriou, *Robust Process Control*. Prentice Hall, Englewood Cliffs, New Jersey, 1989.

[9] M. Vidyasagar, *Control System synthesis*, The MIT Press, Cambridge, 1985.

[10] K. Zhou and J. C. Doyle, *Essentials of Robust Control*, Prentice Hall, Upper Saddle River, New Jersey, 1998.

[11] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*, Prentice Hall, Upper Saddle River, New Jersey, 1996.

[12] K. Zhou and Z. Ren, "A New Controller Architecture for High Performance, Robust and Fault Tolerant Control," *IEEE Transactions on Automatic Control*, Vol. 46, No.10, pp. 1613-1618, 2001.