Fast Construction of Robustness Degradation Function [1]

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Abstract

We develop a fast algorithm to construct the robustness degradation function, which describes quantitatively the relationship between the proportion of systems guaranteeing the robustness requirement and the radius of the uncertainty set. This function can be applied to predict whether a controller design based on an inexact mathematical model will perform satisfactorily when implemented on the true system.

1 Introduction

In recent years, there have been growing interest on the development of probabilistic methods for robustness analysis and design problems aimed at overcoming the computational complexity and the issue of conservatism of deterministic worst case framework [15, 16, 13, 12, 18, 2, 6, 5, 17, 9, 10, 7, 8, 20, 21, 14]. In the deterministic worst case framework, one is only interested in knowing if the robustness requirement is guaranteed for every value of the uncertainty. However, it should be borne in mind that the uncertainty set may include worst cases which never happen in reality. Instead of seeking the worst case guarantee, it is sometimes "acceptable" that the robustness requirement is satisfied for most of the cases. It has been demonstrated that the proportion of systems guaranteeing the robustness requirement can be close to 1 even if the radii of uncertainty set are much larger than the worst case deterministic robustness margin [2, 4, 5, 9, 17]. Therefore, it is of practical importance to construct a function which describes quantitatively the relationship between the proportion of systems guaranteeing the robustness requirement and the radius of uncertainty set. Such a function can serve as a guide for control engineers in evaluating the robustness of a control system once a controller design is completed. Such a function, referred as *robustness degradation function*, has been proposed by a number of researchers [2, 9]. For example, Barmish and Loaga [2] have constructed a curve of robustness margin amplification versus risk in a probabilistic setting. In a similar spirit, Calafiore, Dabbene and Tempo [9, 10] have constructed a probability degradation function in the context of real parametric and dynamic uncertainty.

In this paper, allowing the robustness analysis be performed in a distribution-free manner, we introduce the concept of *proportion* and adopt the assumption from classical robust control framework that uncertainty is deterministic and bounded. It follows naturally that the robustness of a system can be reasonably measured by the ratio of the volume (Lebesgue measure) of the set of uncertainty guaranteeing the robustness requirement to the overall set of uncertainty [18]. Evaluation of such a measure of robustness requires generating samples with uniform distribution over uncertainty sets such as a spectral normal ball or an l_p ball. The difficulty of generating such samples has been successfully resolved in [9, 10].

The conventional method for constructing the robustness functions is to perform, independently, a certain number of simulations for each value of the uncertainty radius and then plot the function. Although such a curve can be applied to evaluate the robustness of the control system, it may be computational expensive. This is especially true when many cycles of controller synthesis and robustness analysis are needed in the development of a high performance control system. Motivated by this situation, we focus on the machinery that

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can make the construction of such a function efficient. We have developed a Sample Reuse Algorithm that allows the simulations to be conducted in an iterative manner. The idea is to start simulation from the larger uncertainty set and save appropriate evaluations of the robust requirement for the use of later simulations on smaller uncertainty set. In this way the total number of simulations can be reduced significantly as compared to the conventional method.

Although our Sample Reuse Algorithm is derived from the worst case deterministic framework, the technique is also applicable when considering the random nature of the uncertainty. In such cases, the worst-case properties of uniform distribution given in the pioneering work [6, 2, 1] allow our algorithm to be applied to efficiently solve a wide variety of robustness analysis problems.

Finally, we would like to point out that, in additional to overcoming the NP hard barrier and the conservatism issue, an overwhelming advantage of this approach is that it provides solutions to many robustness problems with complicated robustness requirement and uncertainty bounding set for which there is no effective techniques in the classical deterministic robust control framework. For example, our algorithm can provide efficient solutions for robustness problems with time specifications and stability requirement and with the uncertainty bounding set taken as a spectral norm ball.

The organization of the paper is as follows. Section 2 gives the problem formulation. Section 3 presents our Sample Reuse Algorithm. Section 4 is the performance analysis of the algorithm. Section 5 applies the algorithm to examples.

2 Problem Formulation

We adopt the assumption, from the classical robust control framework, that the uncertainty is deterministic and bounded. We formulate a general robustness analysis problem as follows.

Let $\mathbf{P}$ denote a robustness requirement. The definition of $\mathbf{P}$ can be a fairly complicated combination of the following:

- Stability or $\mathcal{D}$-stability;
- H_∞ norm of the closed loop transfer function;
- Time specifications such as overshoot, rise time, settling time and steady state error.

Let $\mathcal{B}(r)$ denote the set of uncertainties with size smaller than r. In applications, we are usually dealing with uncertainty sets such as the following:

- l_p ball $\mathcal{B}^p(r) := \{\Delta \in \mathbf{R}^n : ||\Delta||_p \leq r\}$ where $||.|_p$ denotes the l_p norm and $p = 1, 2, \cdots, \infty$. In particular, $\mathcal{B}^\infty(r)$ denotes a box.
- Spectral norm ball $\mathcal{B}_\sigma(r) := \{\Delta \in \mathbf{\Delta} : \bar{\sigma}(\Delta) \leq r\}$ where $\mathbf{\Delta}$ is the class of allowable perturbations and $\mathbf{\Delta} := \{\text{blockdiag}[q_1 I_{r_1}, \cdots, q_s I_{r_s}, \Delta_1, \cdots, \Delta_c]\}$ where $q_i \in \mathbb{F}$, $i = 1, \cdots, s$ are scalar parameters with multiplicity $r_1, \cdots, r_s$ and $\Delta_i \in \mathbb{F}^{n_i \times m_i}$, $i = 1, \cdots, c$ are possibly repeated full blocks. Here $\mathbb{F}$ is either the complex field $\mathbf{C}$ or the real field $\mathbf{R}$.
- Homogeneous star-shaped bounding set $\mathcal{B}_H(r) := \{r(\Delta - \Delta_0) + \Delta_0 : \Delta \in Q\}$ where $Q \subset \mathbf{R}^n$ and $\Delta_0 \in Q$ (see [2] for a detailed illustration).

Throughout this paper, $\mathcal{B}(r)$ refers to any type of uncertainty set described above. Define a function $||.||$ such that $||X|| := \min\{r : X \in \mathcal{B}(r)\}$ for any X, i.e., $\mathcal{B}(||X||)$ includes X exactly in the boundary. By such definition, $||X|| = \min\{r : \frac{X - \Delta_0}{r} + \Delta_0 \in Q\}$, $||X|| = \bar{\sigma}(X)$, and $||X|| = ||X||_p$ in the context of homogeneous star-shaped bounding set, spectral norm ball and l_p ball respectively.

To allow the robustness analysis be performed in a distribution-free manner, we introduce the notion of *proportion* as follows. For any $\Delta \in \mathcal{B}(r)$ there is an associated system $G(\Delta)$. Define *proportion* $\mathbb{P}(r) := \frac{\text{vol}(\{\Delta \in \mathcal{B}(r):\, G(\Delta) \text{ guarantees } \mathbf{P}\})}{\text{vol}(\mathcal{B}(r))}$ with $\text{vol}(S) := \int_{q \in S} dq$, where the notion of dq is illustrated as follows:

- (I): If $q = [x_{rs}]_{n \times m}$ is a real matrix in $\mathbf{R}^{n \times m}$, then $dq = \prod_{r=1}^{n} \prod_{s=1}^{m} dx_{rs}$.
- (II): If $q = [x_{rs} + j y_{rs}]_{n \times m}$ is a complex matrix in $\mathbf{C}^{n \times m}$, then $dq = \prod_{r=1}^{n} \prod_{s=1}^{m} (dx_{rs} dy_{rs})$.
- (III): If $q \in \mathbf{\Delta}$, then $dq = (\prod_{i=1}^{s} dq_i)(\prod_{i=1}^{c} d\Delta_i)$ where the notion of dq_i and $d\Delta_i$ is defined by (I) and (II).

It follows that $\mathbb{P}(r)$ is a reasonable measure of the robustness of the system [9, 19]. In the worst case deterministic framework, we are only interested in knowing if $\mathbf{P}$ is guaranteed for every Δ. However, one should bear in mind that the uncertainty set in our model may include worst cases which never happen in reality. Thus, it would be "acceptable" in many applications if the robustness requirement $\mathbf{P}$ is satisfied for most of the cases. Hence, due to the inaccuracy of the model, we should also obtain the value of $\mathbb{P}(r)$ beyond the deterministic robustness margin.

Clearly, $\mathbb{P}(r)$ is deterministic in nature. However, we can resort to a probabilistic approach to evaluate $\mathbb{P}(r)$. To see this, one needs to observe that a random variable with *uniform distribution* over $\mathcal{B}(r)$, denoted by $\mathbf{\Delta}^u$, guarantees that $\Pr\{\mathbf{\Delta}^u \in S\} = \frac{\mathrm{vol}(S \bigcap \mathcal{B}(r))}{\mathrm{vol}(\mathcal{B}(r))}$ for any S, and thus $\mathbb{P}(r) = \Pr\{\, G(\mathbf{\Delta}^u)$ guarantees $\mathbf{P}\}$. It follows that a Monte Carlo method can be employed to estimate $\mathbb{P}(r)$ based on i.i.d. observations of $\mathbf{\Delta}^u$.

It is interesting to know how the function $\mathbb{P}(r)$ degrades with respect to r when r increases from a to b where $b > a \geq 0$. In a similar spirit, such a function has been proposed as *Confidence Degradation Function* in [2] and as *Probability Degradation Function* in [9, 10]. In this paper, we refer function $\mathbb{P}(.)$ as *robustness degradation function* for the following reasons. Firstly, we introduce confidence interval for assessing the accuracy of the estimate of $\mathbb{P}(r)$. To be useful, every numerical method should be associated with an assessment for the accuracy of the estimate. Monte Carlo simulation is no exception. To avoid confusion, we reserve the notion of "confidence" for the purpose of interval estimation. Secondly, we introduce the concept of *proportion* for measuring robustness, which has no probabilistic content. Thirdly, $\mathbb{P}(r)$ is a robustness measure and is usually decreasing with respect to r when $\mathbb{P}(r)$ is close to 1.

To construct such a function of practical importance, the conventional way is to grid the interval $[a, b]$ as $a = \rho_1 < \rho_2 < \cdots < \rho_l = b$ and estimate $\mathbb{P}(\rho_i)$ by conducting N i.i.d. sampling experiments for each ρ_i. In total, we need Nl samples. In the next section we show that the number of experiments can be significantly reduced.

3 Sample Reuse Algorithm

To improve efficiency, we shall make use of the following simple yet important observation.

Let q^* be an observation of a random variable with uniform distribution over $\mathcal{B}(r^*) \supseteq \mathcal{B}(r)$ such that $q^* \in \mathcal{B}(r)$. Then q^* can also be viewed as an observation of a random variable with uniform distribution over $\mathcal{B}(r)$.

In our algorithm, we flip the order of ρ_i by defining $r_i = \rho_{l+1-i}$ for $i = 1, 2, \cdots, l$. Thus, the direction of simulation is backward. Our algorithm is described as follows.

Sample Reuse Algorithm

Input: Sample size N, confidence parameter $\delta \in (0, 1)$ and uncertainty radii r_i, $i = 1, 2, \cdots, l$.

Output: Proportion estimate $\widehat{\mathbb{P}}_i$ and the related confidence interval for $i = 1, \cdots, l$. In the following, m_{i1} denotes the number of sampling experiments conducted at r_i and m_{i2} denotes the number of observations guaranteeing $\mathbf{P}$ during the m_{i1} sampling experiments.

Step 1. Let $M = [m_{ij}]_{l \times 2}$ be a zero matrix.

Step 2. For $i = 1$ to $i = l$ do the following:

- Let $r \leftarrow r_i$.
- While $m_{i1} < N$ do the following:
  - Generate uniform sample q^* from $\mathcal{B}(r)$. Evaluate the robustness requirement $\mathbf{P}$ for q^*.
  - Let $m_{s1} \leftarrow m_{s1} + 1$ for any s such that $r_s \geq ||q||$.
  - If robustness requirement $\mathbf{P}$ is satisfied for q^* then let $m_{s2} \leftarrow m_{s2} + 1$ for any s such that $r_s \geq ||q||$.
- Let $\widehat{\mathbb{P}}_i \leftarrow \frac{m_{i2}}{N}$ and construct the confidence interval of confidence level $100(1 - \delta)\%$.

It follows that q^* can be viewed as an observation of a random variable with uniform distribution over $\mathcal{B}(r_j)$ if and only if $r_j \geq ||q||$. Hence, if the robustness requirement $\mathbf{P}$ has been evaluated for $\mathcal{B}(r_i)$ at sample q^*, the result can be accepted without repeated evaluation of $\mathbf{P}$ for all $\mathcal{B}(r_j)$ such that $r_j \geq ||q||$. *Thus, sample reuse allows us to save both the sample generation and the evaluation of $\mathbf{P}$ for the sample. It is also interesting to point out that the samples collected for each r_i are i.i.d. and thus confidence interval can be rigorously constructed based on the evaluation of $\mathbf{P}$ for the samples.*

4 Sample Reuse Factor

Let $\mathbf{n}_i$ be the number of simulations required at r_i. Define *sample reuse factor* $\mathcal{F}_{reuse} := \frac{Nl}{\mathcal{E}[\sum_{i=1}^{l} \mathbf{n}_i]}$, where $\mathcal{E}(X)$ denotes the expectation of random variable X. Obviously, $\mathcal{F}_{reuse}$ measures the improvement of efficiency upon the conventional method. We demonstrate that the improvement can be significant in most applications.

Theorem 1 *The sample reuse factor* $\mathcal{F}_{reuse} = \frac{l}{l-\sum_{i=2}^{l}\left(\frac{r_i}{r_{i-1}}\right)^d}$ *where* $d = n$ *for* l_p *ball* $\mathcal{B}^p(r)$ *and homogeneous star-shaped bounding set* $\mathcal{B}_H(r)$*; and* $d = \sum_{i=1}^{s}\kappa(q_i) + \sum_{j=1}^{c}\kappa(\Delta_j)$ *for spectral norm ball* $\mathcal{B}_\sigma(r)$ *with* $\kappa(.)$ *defined so that* $\kappa(X) = 2[\min(m,n)]^2 + 2|m-n| - \min(m,n) + 1$ *if* $X \in \mathbf{C}^{n\times m}$ *and* $\kappa(X) = [\min(m,n)]^2 + |m-n|$ *if* $X \in \mathbf{R}^{n\times m}$.

For illustration purposes, we choose $r_i = b - \frac{(b-a)(i-1)}{l-1}$ for $i = 1, 2, \cdots, l$. By Theorem 1, $\mathcal{F}_{reuse} = \frac{l}{l-\sum_{i=2}^{l}\left(1-\frac{1}{\frac{l-1}{1-\frac{a}{b}}-i+2}\right)^d}$. Figure 1 shows that the improvement over the conventional approach are impressive.

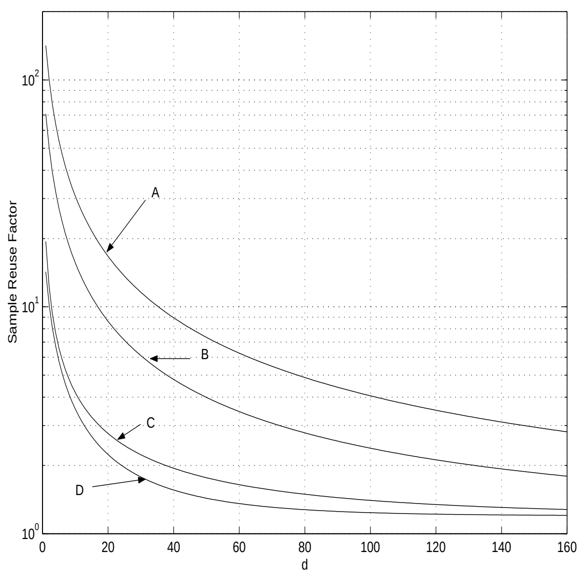

Figure 1: Performance Improvement ($A: l = 200,\ b = 2a$; $B: l = 100,\ b = 2a$; $C: l = 100,\ a = 0$; $D: l = 20,\ b = 2a$)

5 Illustrative Examples

In this section we demonstrate through examples the power of Sample Reuse Algorithm in solving a wide variety of complicated robustness analysis problems which are not tractable in the classical deterministic framework.

First, we consider an example which has been studied in [11] by a deterministic approach. The system is as shown in Figure 2.

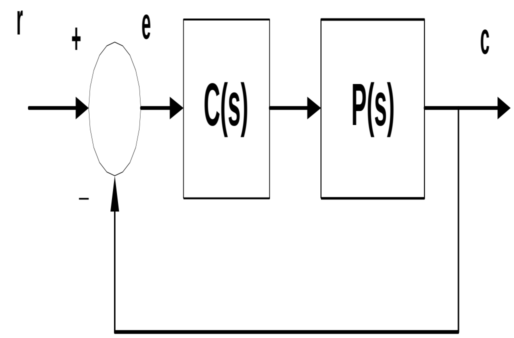

Figure 2: Uncertain System

The compensator is $C(s) = \frac{s+2}{s+10}$ and the plant is $P(s) = \frac{800(1+0.1\delta_1)}{s(s+4+0.2\delta_2)(s+6+0.3\delta_3)}$ with parametric uncertainty $\Delta = [\delta_1, \delta_2, \delta_3]^{\mathrm{T}}$. The nominal system is stable. The closed-loop roots of the nominal system are: $z_1 = -15.9178$, $z_2 = -1.8309$, $z_3 = -1.1256 + 7.3234i$, $z_4 = -1.1256 - 7.3234i$. The H_∞ norm of the nominal closed loop transfer function is $||T^0||_\infty = 2.78$. The peak value, rise time, settling time of step response of the nominal system, are respectively, $P^0_{peak} = 1.47$, $t^0_r = 0.185$, $t^0_s = 3.175$. In all of the following examples, we take $l = 100$. To guarantee that the absolute error of the estimate for the proportion is less than 0.01 with confidence level 99%, we choose $N = 26,492$ based on the well known Chernoff bound. Since the Chernoff bound is conservative, we also performed a post-experimental evaluation of the estimates by construct confidence intervals with confidence level 99%.

Figure 3 is the robustness degradation curve, with the robustness requirement $\mathbf{P}$ defined as stability and H_∞ norm $< 170\%\ ||T^0||_\infty$, and the uncertainty set defined as the ellipsoid $\mathcal{B}_2(r) := \{\Delta : ||\Delta||_2 \le r\}$.

Figure 4 is the robustness degradation curve with the robustness requirement $\mathbf{P}$ defined as $\mathcal{D}$-stability with the domain of poles defined as: Real part < -1.5, or fall within one of the two disks centered at z_3 and z_4 with radius 0.3. The uncertainty set is defined as the polytope $\mathcal{B}_H(r) := \left\{r\Delta + (1-r)\frac{\sum_{i=1}^{4}\Delta^i}{4} : \Delta \in \mathrm{conv}\{\Delta^1, \Delta^2, \Delta^3, \Delta^4\}\right\}$ where 'conv' denotes the convex hull of

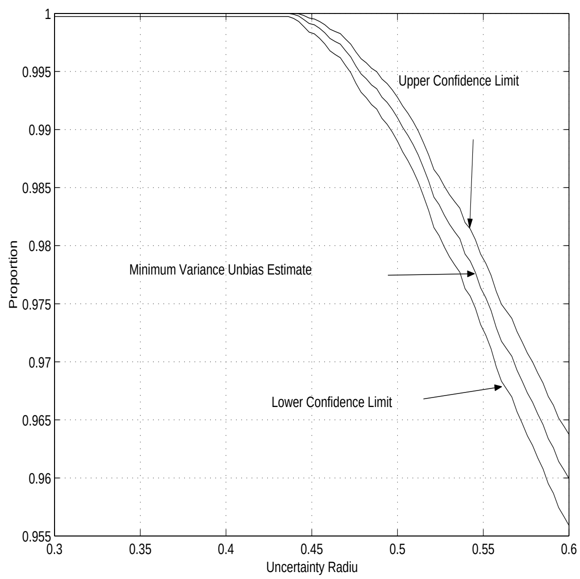

Figure 3: Robustness Degradation Curve (Reuse Factor = 43)

$\Delta^i = [\frac{1}{2}\sin(\frac{2i-1}{3}\pi), \quad \frac{1}{2}\cos(\frac{2i-1}{3}\pi), \quad -\frac{\sqrt{3}}{2}]^{\mathrm{T}}$ for $i = 1, 2, 3$ and $\Delta^4 = [0,\ 0,\ 1]^{\mathrm{T}}$.

Figure 5 is the robustness degradation curve for the case where the uncertainty set is $\mathcal{B}_\infty(r) := \{\Delta : \ ||\Delta||_\infty \leq r\}$, the robustness requirement $\mathbf{P}$ is : Stability, and rise time $t_r < 135\% \ t_r^0 = 0.25$, settling time $t_s < 110\% \ t_s^0 = 3.5$, overshoot $P_{peak} < 116\% \ P_{peak}^0 = 1.7$.

Finally, we consider the same example in [9] where the class of uncertainty is defined as

$$\boldsymbol{\Delta} := \{\mathrm{blockdiag}[q_1 I_5, \ q_2 I_5, \ \Delta_1]\}$$

where $\Delta_1 \in \mathbf{C}^{4\times 4}$ and I_5 denotes the identity matrix of 5×5. By Theorem 1, we have $d = 31$. Figure 6 shows the robustness degradation curve. An improvement (of efficiency) about 6 fold is achieved by our algorithm.

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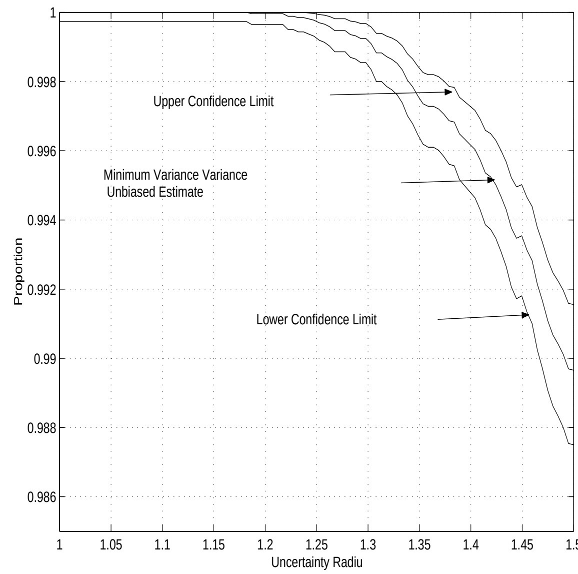

Figure 4: Robustness Degradation Curve (Reuse Factor = 49)

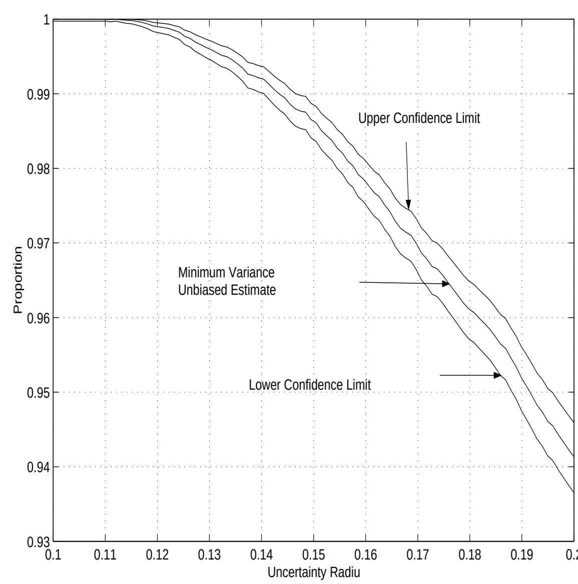

Figure 5: Robustness Degradation Curve (Reuse Factor = 38)

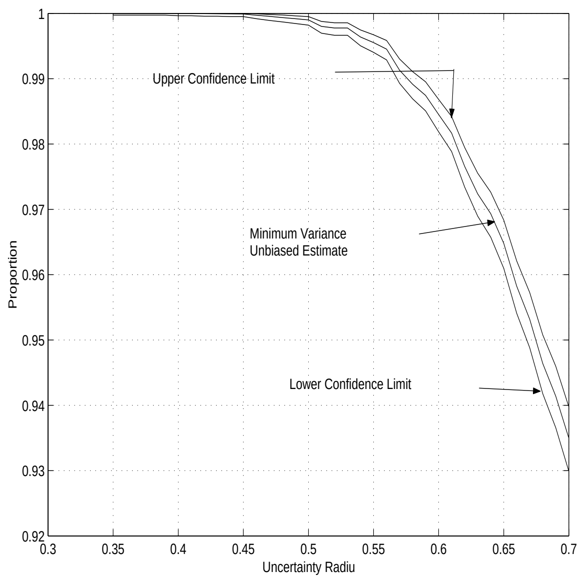

Figure 6: Robustness Degradation Curve (Reuse Factor = 6)

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