Proceedings of SPIE Conference on Signal and Data Processing of Small Targets,
Orlando, FL, USA, April 2002. Paper 4728-60

A Survey of Maneuvering Target Tracking—Part IV: Decision-Based Methods *

X. Rong Li and **Vesselin P. Jilkov**
Department of Electrical Engineering
University of New Orleans
New Orleans, LA 70148, USA
504-280-7416 (phone), 504-280-3950 (fax), xli@uno.edu, vjilkov@uno.edu

Abstract

This is the fourth part of a series of papers that provide a comprehensive survey of techniques for tracking maneuvering targets without addressing the so-called measurement-origin uncertainty. Part I [1] and Part II [2] deal with target motion models. Part III [3] covers the measurement models and the associated techniques. This part surveys tracking techniques that are based on decisions regarding target maneuver. Three classes of techniques are identified and described: equivalent noise, input detection and estimation, and switching model. Maneuver detection methods are also included.

Key Words: Target Tracking, Adaptive Filtering, Maneuver Detection, Survey

1 Introduction

This is the fourth part of a series of papers that provide a comprehensive survey of the techniques for tracking maneuvering targets without addressing the so-called measurement-origin uncertainty. Part I [1] and Part II [2] deal with general target motion models and ballistic target motion models, respectively. Part III [3] covers measurement models, including measurement model-based techniques, used in target tracking.

In the history of development of maneuvering target tracking (MTT) techniques, single model based adaptive Kalman filtering free of decision came into existence first. Decision-based techniques appeared next. This was followed by multiple-model algorithms, which have become quite popular. More recently, nonlinear filtering techniques, such as sampling based algorithms, have been gaining moment.

This part surveys decision-based techniques for MTT, that is, techniques in which a key component is explicit decisions on target maneuver. In subsequent parts, multiple-model approach, exact and approximate nonlinear filters, and sampling based algorithms will be surveyed; performance analysis and evaluation as well as applications will be addressed. A summary part will also be provided.

There are numerous methods for adaptive estimation and filtering, decision making, and nonlinear filtering in the literature. Only those that have been proposed for, applied to, or possess substantial potentials for MTT are included in this survey. On the other hand, it is our intention to cast problems and techniques in a slightly wider context than most previous treatments so as to make more clear the forest rather than just trees.

In target tracking, the actual measurement system is typically nonlinear, as described in Part III. In this part, however, we mainly focus on linear measurement systems for simplicity. This simplification has the following justification. Not only have the techniques that handle nonlinear measurements been covered in Part III, they are also to a large extent independent of the MTT techniques surveyed here, which focus on the uncertainty in the target motion due to possible maneuvers.

As stated repeatedly in the previous parts, we appreciate receiving comments and missing material that should be included in this part. While we may not be able to respond to each correspondence, information received will be considered seriously for the refinement of this part for its final publication in a journal or book.

The rest of the paper is organized as follows. Sec. 2 briefly describes the methods considered in this part as a whole. Sec. 3 surveys maneuver detectors. Secs. 4, 5, and 6 cover three different classes of methods, referred to as equivalent noise, input detection and estimation, and switching model, respectively. Concluding remarks are given in the final section.

* Research supported by ONR grant N00014-00-1-0677, NSF grant ECS-9734285, and NASA/LEQSF grant (2001-4)-01.

2 Decision-Based Approach to Maneuvering Target Tracking

In the decision-based approach, target tracking as a hybrid estimation problem involving both estimation and decision is solved by combining estimation with explicit, hard decision. This approach is one of the most natural for MTT. It is covered with varying degrees in several books on target tracking [4, 5, 6, 7, 8, 9, 10].

This approach to MTT distinguishes itself from other approaches in that the adaptation in estimating the target state is directed by decisions regarding target maneuver, in particular, its onset and termination. This decision-directed adaptation may take different forms. Most of these techniques amount to using two types of filters, one with a narrow bandwidth (e.g., low gain) for precision tracking in normal situations and the other with a wide bandwidth (e.g., high gain) for effective tracking during target maneuvers. In this way, it aims at achieving good tracking performance in both situations rather than a compromise. These filters may be based on the same or different models. When a single model is used in the linear case, such adaptive techniques are traditionally considered as part of the so-called *adaptive Kalman filtering*. While more than one model may be used, only one is in effect at one time.

Decision-based techniques for MTT developed so far fall into three classes, referred to as *equivalent noise*, *input detection and estimation*, and *switching model* approaches and described in Secs. 4, 5, and 6, respectively.

We first survey techniques for maneuver detection developed so far in the next section.

3 Maneuver Detection

Although the ultimate goal of MTT is estimation of the target state, in the decision-based approaches, estimation is directed by decision regarding maneuvers. This makes reliable and timely decision the key in these approaches.

The fundamental questions here are: "Is the target maneuvering?" In other words, whether the target is maneuvering is crucial information here. Answering this question is a decision problem, which can be formulated as a hypothesis testing problem

$$H_0 : \text{The target is not maneuvering;} \quad H_1 : \text{ The target is maneuvering}$$

Many solution techniques are available in statistics for such problems.

Both maneuver onset and termination represent a change in the target motion pattern. This change exhibits itself more or less in our observations of the target. Detection of maneuver onset and termination thus amounts to detecting a change in the observations — a random process. This is known as **change-point detection** in statistics[1]. It has a very large body of literature that includes abundant results (see, e.g., [11, 12, 13, 14, 15, 16, 17, 18, 19] and references therein). Unfortunately, this treasure has been largely overlooked by the tracking community partly because most of it is not easily accessible by engineering-oriented researchers. However, it could certainly facilitate development and design of better maneuver detectors.

Two other fundamental questions are: "When did the target start maneuvering?" and "When did it stop maneuvering?" In other words, it is important to infer the *onset time* and *termination time* of a maneuver. The determination of maneuver onset and termination times can be cast either as an estimation or decision problem. Estimation and decision are twins. They both aim at inferring an unknown quantity using available information. Their basic difference is that decision is the selection from a discrete (often finite) set of candidates, while all possible outcomes of estimation form a continuum. In the continuous-time case, it would be more natural to formulate the determination of onset and termination times as an estimation problem, but a decision framework appears to be more appropriate for the discrete-time case.

In maneuver detection, the focus is detection of maneuver onset, rather than maneuver termination. The two main reasons for this are level of difficulty and the consequence of an incorrect decision. In general, it is more difficult to detect maneuver termination than maneuver onset because nonmaneuver is a well-defined motion pattern — straight and level motion at a constant velocity — while maneuver essentially includes all other motion patterns. For instance, a maneuver model has a larger covariance of measurement residuals than a nonmaneuver model due to the fact that the latter has a larger state vector and assumes more motion uncertainty than the former. Fortunately, timely detection of maneuver termination is usually not as important as that of maneuver onset because tracking a maneuvering target assuming it is not maneuvering may have a serious consequence (e.g., track loss), while tracking a nonmaneuvering target assuming it is maneuvering usually only suffer minor performance degradation.

[1] Some people prefer "change detection."

3.1 Chi-Square Test Based

Most maneuver detectors *used* in MTT are (true, quasi, or pseudo) chi-square significance test based. They employ a statistic that is truly or approximately chi-square distributed under H_0 for maneuver onset detection or under H_1 for maneuver termination detection. Assume ϵ is (approximately) chi-square distributed with n degrees of freedom (denoted as $\epsilon \sim \chi^2_n$) under H_0. Then a chi-square test based maneuver detector will declare detection of a maneuver if

$$\epsilon > \lambda = \chi^2_n(\alpha) \tag{1}$$

where $1 - \alpha$ is the level of confidence, which should be set quite high (e.g., 95% or 99%). Note that $\epsilon \leq \lambda$ does not imply absence of a maneuver.

It is well known that $\| y - \bar{y} \|^2_{\Sigma^{-1}} = (y - \bar{y})'\Sigma^{-1}(y - \bar{y})$ is χ^2_n distributed for any n-dimensional Gaussian random vector $y \sim \mathcal{N}(\bar{y}, \Sigma)$. In this sense, chi-square test provides a check of the goodness of fit to judge if y indeed has the assumed distribution (or if this statistical distance between y and $\bar{y}$ matches the distribution). Chi-square test is perhaps the most popular statistical test because of its simplicity, even though it is not necessarily optimal in any sense. Rigorously speaking, the validity of a chi-square test relies on the assumption that individual terms are Gaussian and independent, which is not necessarily valid in practice. Nevertheless, chi-square tests are commonly used in these situations.

In maneuver detection, two popular choices for y are measurement residual $\tilde{z}$ and input estimate $\hat{u}$.

Residual based. In this case, *normalized residual squared* $\epsilon_k = \tilde{z}'_k S_k^{-1} \tilde{z}_k$ is used, where $\tilde{z}_k = z_k - \hat{z}_{k|k-1}$ is the measurement residual and $S_k = \text{cov}(\tilde{z}_k)$. Its *moving sum* ϵ^s_k over a sliding window $[k - s + 1, k]$ of length s as well as *fading-memory sum* ϵ^ρ_k are

$$\epsilon^s_k = \sum_{j=k-s+1}^{k} \epsilon_j, \quad \epsilon^\rho_k = \sum_{j=1}^{k} \rho^{k-j}\epsilon_j = \rho\epsilon^\rho_{k-1} + \epsilon_k, \quad 0 < \rho < 1, \quad s_\rho = \frac{1}{1-\rho} \tag{2}$$

where s_ρ is the effective window length of the fading-memory sum. Under the linear-Gaussian assumption and H_0, residual sequence $\langle \tilde{z}_k \rangle$ is zero-mean, Gaussian, and white. Then ϵ_k and ϵ^s_k are chi-square distributed with n_z and sn_z degrees of freedom, respectively (i.e., $\epsilon_k \sim \chi^2_{n_z}$, $\epsilon^s_k \sim \chi^2_{sn_z}$), where $n_z = \dim(z)$. As a weighted sum of i.i.d. Gaussian variables, ϵ^ρ_k is not chi-square distributed, but by moment matching it can be approximately treated as a scaled version of a chi-square variable, that is

$$\epsilon^\rho_k \sim \frac{1}{1+\rho}\chi^2_{n_\rho} \quad \text{with} \quad n_\rho = n_z \frac{1+\rho}{1-\rho} \tag{3}$$

Consequently, (1) can be used to detect maneuver onset, where $\epsilon = \epsilon_k$, ϵ^s_k, or ϵ^ρ_k. Note that a χ^2_n variable has mean n and variance $2n$, and thus ϵ^s_k/s or ϵ^ρ_k/s_ρ (i.e., each term in the sum) becomes less random as the window length s (or s_ρ) becomes larger, which however often implies a longer detection delay. The detection threshold λ can be obtained from their respective distributions.

As already mentioned, chi-square tests based on residuals have been a fairly standard tool for maneuver detection. Its applications are too numerous to list. A sample can be found in [20, 4, 21, 5, 6, 9, 22, 7, 10] and references therein.

Input estimate (IE) based. If the target is not maneuvering, its control input (e.g., acceleration or its increment) u is zero, and thus any estimate of the input that is linear in the measurement residuals[2] under the linear-Gaussian assumption is zero-mean and Gaussian. As a result, $\epsilon = \hat{u}'\Sigma^{-1}\hat{u}$ is $\chi^2_{n_u}$ distributed under H_0, where $\Sigma = \text{cov}(\hat{u})$ and $n_u = \dim(u)$. Consequently, (1) can be used to detect maneuver onset. Mainly because of their simplicity, the IE-based chi-square tests are present in many IE-based algorithms [23, 24, 25, 26, 27, 28]. The test can also be used to detect maneuver termination [20, 10]. It can be used based on a moving sum or fading-memory sum as well, where $\epsilon_k = \hat{u}'_k \Sigma_k^{-1} \hat{u}_k$. However, the terms that form the sum are usually not independent and thus the sum is usually not really chi-square distributed. A rigorous analytical determination of the corresponding detection probability P_D is virtually impossible since it depends on the generally unknown input. Evaluation of P_D could be done by simulation.

3.2 Generalized Likelihood Ratio Test Based

Let u be the input that is responsible for maneuver and let n be the maneuver onset time so that $u_\kappa = 0$ for $\kappa < n$ and $u_\kappa \neq 0$ for $\kappa \geq n$ over the time window $[k - s, k)$. Consider the following maneuver hypotheses in terms of input value

$$H_0 \quad : \quad u_\kappa = 0 \;\; \text{for all } \kappa \in [k-s, k) \tag{4}$$

$$H_1(u, n) \quad : \quad u_n = u_{n+1} = \cdots = u_{k-1} = u \neq 0 \quad \text{for some } n \in [k-s, k) \tag{5}$$

[2]This is true for almost all input estimates developed.

where the input level u and the maneuver onset time n are unknown.

The likelihood ratio of H_1 vs. H_0 with given u and n is

$$\Lambda(u,n) = \frac{f\left(z_s^k | H_1(u,n)\right)}{f\left(z_s^k | H_0\right)} \tag{6}$$

where z_s^k stands for set of measurements $\{z_{k-s}, \ldots, z_k\}$. Many optimal solutions of the above hypothesis testing problem are based on this likelihood ratio, which is however unknown due to its dependence on u and n. In such a case, a general principle widely used is to replace the unknown likelihood functions by their maxima over the unknown parameters (u,n); that is, replace $f\left(z_s^k|H_1(u,n)\right)$ with $f\left(z_s^k|H_1(\hat{u},\hat{n})\right) = \max_{(u,n)} f\left(z_s^k|H_1(u,n)\right)$, where $(\hat{u},\hat{n}) = \arg\max_{(u,n)} f\left(z_s^k|H_1(u,n)\right)$ is the maximum likelihood estimate of (u,n). In essence, this principle replaces an unknown likelihood ratio with its most probable likelihood ratio, which does make sense. The resulting likelihood ratio is known as the *generalized likelihood ratio*. Then the **generalized likelihood ratio** (**GLR**) test compares this ratio or its equivalent with a threshold.

In the context of maneuver detection, the *joint* maximum likelihood estimate $(\hat{u},\hat{n})$ is found in two steps as follows. Denote by $J(u,n) = \log\Lambda(u,n)$ the log-likelihood ratio. First, find $\hat{u}(n) = \arg\max_u J(u,n)$ as the input estimate given onset time n and then

$$(\hat{u},\hat{n}) = \arg\max_{[\hat{u}(n),n]} J[\hat{u}(n),n] \tag{7}$$

The main reason for this two-step approach is the ease at finding $\hat{u}(n)$. Then the **GLR maneuver detector** declares detection of a maneuver if the generalized log-likelihood ratio

$$J(\hat{u},\hat{n}) = \log\frac{f\left(z_s^k|H_1(\hat{u},\hat{n})\right)}{f\left(z_s^k|H_0\right)} \tag{8}$$

exceeds a properly chosen threshold. In this case, the GLR estimates of the input and onset time $\hat{u}$ and $\hat{n}$ obtained by (7) are validated (see Sec. 5.1).

As shown in Sec. 5.1, under the linear-Gaussian assumption, $\hat{u}(n) = \Sigma(n)e(n)$ is easily obtainable (in fact, it is the least-squares estimate of u given n), where $\Sigma(n) = \text{MSE}[\hat{u}(n)]$ is the mean-square error matrix of $\hat{u}(n)$ and $e(n)$ is given later by (43); further, it can be easily verified that

$$J[\hat{u}(n),n] = -\frac{1}{2}\sum_\kappa (\tilde{z}_\kappa^*)' S_\kappa^{-1} \tilde{z}_\kappa^* + \frac{1}{2}\Delta J[\hat{u}(n),n] \tag{9}$$

where $\tilde{z}_\kappa^*$ is the residual at time κ under H_0 and $S_\kappa = \text{cov}(\tilde{z}_\kappa^*)$. The increment ΔJ due to the unknown input is given by

$$\Delta J[\hat{u}(n),n] = \hat{u}(n)'[\Sigma(n)]^{-1}\hat{u}(n) = e(n)'\Sigma(n)e(n) \tag{10}$$

Note, however, that $\hat{u} \neq \Sigma(\hat{n})e(\hat{n})$. As such, the above GLR test does not lead to the following maneuver detector

$$\hat{u}'[\Sigma(\hat{n})]^{-1}\hat{u} > \lambda \tag{11}$$

or (not equivalently)

$$e(\hat{n})'\Sigma(\hat{n})e(\hat{n}) > \lambda \tag{12}$$

These two detectors are nevertheless used in some algorithms. Note that (11) is in general not a chi-square test since $\hat{u}$ given by (7) is not necessarily linear in the residuals. Also, implementation of the double maximization (7) over an window $[k-s,k)$ requires input estimators running for each $\kappa \in [k-s,k)$.

Development and applications. The above GLR detector was proposed in [29] for fault detection and used in many MTT algorithms. More details were given in [29, 19]. Prior to [29], a GLR-based maneuver detector was proposed in [30] in a less general setting. The maneuver detection is based on the GLR test for detecting a maneuver-induced bias in the constant-velocity (CV) filter's residual sequence $\langle\tilde{z}_k^*\rangle$. This bias is modeled as $b_k(n) = h_k(n)Tu$, where $h_k(n) = (k-n)^2 T$, k is the current time, n is the maneuver onset time, T is the sampling period, and u is an unknown constant related to the maneuver input magnitude. It was presented therein that $\hat{u}_k(n) = \Sigma_k(n)e_k(n)$ with

$$[\Sigma_k(n)]^{-1} = \sum_{\kappa=n+1}^{k} (h_\kappa(n))^2, \qquad e_k(n) = \sum_{\kappa=n+1}^{k} h_\kappa(n)\tilde{z}_\kappa^*$$

The proposed GLR bias (maneuver) detector over the window $[k-s,\, k)$ is

$$\max_{k-s\le n<k} e_k(n)'\Sigma_k(n)e_k(n) > \lambda \tag{13}$$

and the estimate of the maneuver onset is $\hat{n} = \arg\max_{k-s\le n<k} e_k(n)'\Sigma_k(n)e_k(n)$. To reduce the computational burden of the algorithm, an approximate detector has also been developed therein. Application of the GLR-based maneuver detection in a 1D tracking filter, discussed in Sec. 5.3.5, can also be found in [31].

As discussed in [19, 32], while providing an appealing analytical framework for change detection, the GLR method has its major drawbacks in the heuristic choosing of decision threshold and heavy computational burden.

3.3 Other Detectors

Marginalized likelihood ratio test. The *marginalized likelihood ratio* (MLR) method, proposed recently in [33], appears to be more efficient than the GLR test assuming more prior information. Its basic idea is to obtain the *marginal* ML estimate $\hat{n}$ that has the maximum likelihood for an *average* u**,** rather than using the joint MLE $(\hat{u}, \hat{n})$, as given by (7). In essence, MLR test checks the ratio of *average* likelihoods, as opposed to the ratio of *most probable* likelihoods in the GLR test. The hypotheis testing problem for H_0 vs. H_1 is formulated with respect to the *marginalized log-likelihood ratio* (MLR)

$$J(n) = \log \frac{f\left(z_s^k | H_1(n)\right)}{p\left(z_s^k|H_0\right)} = \log \frac{E\left[f\left(z_s^k|H_1(u,n)\right)\right]}{p\left(z_s^k|H_0\right)} \tag{14}$$

where

$$f\left(z_s^k|H_1(n)\right) = E\left[f\left(z_s^k|H_1(u,n)\right)\right] = \int f\left(z_s^k|H_1(u,n)\right) f(u)\, du \tag{15}$$

The test is

$$J(\hat{n}) > 0 \tag{16}$$

where $J(\hat{n}) = \max_n J(n)$ is the maximum MLR.

In this formulation the input u is considered as a random variable, in contrast to the GLR method where it is assumed a deterministic constant. The prior of u can be chosen, for example, as diffuse uniform (noninformative). The input level is eliminated by averaging over all possible levels. Clearly the crucial problem of threshold determination of the GLR test is circumvented in the MLR formulation. Under some condition and with a special choice of the GLR threshold, both tests coincide. Fairly efficient algorithms for estimating n were also presented in [33]. The MLR test is also more robust than the GLR test to unknown noise levels.

Gaussian significance test based. In this detector, a maneuver is declared if a component $\hat{u}_i$ of input estimate (assumed to be Gaussian distributed) is statistically significant, that is, $\hat{u}_i/\Sigma_i^{1/2} > \lambda$, where $\Sigma_i = \text{var}(\hat{u}_i)$ and the threshold λ is determined from the standard Gaussian distribution. It is used in, e.g., [34, 21, 4, 7, 28].

CUSUM based. The popular cumulative sum (CUSUM) algorithm [35, 36, 17, 19] can be applied to maneuver detection with an input estimate $\hat{u}$ as follows: Declare a maneuver if $S_k - \min_{k-s\le\kappa\le k} S_\kappa \ge \lambda$, where

$$S_k = \sum_{\kappa=k-s}^{k} \log\left(f\left(\tilde{z}_\kappa|H_1(\hat{u},n)\right)/f\left(\tilde{z}_\kappa|H_0\right)\right)$$

is the cumulative sum of log-likelihood ratios. The rationale behind is the observation that S_k generally goes down with time in the absence of maneuver, but goes up during maneuver, and thus the maneuver onset time corresponds roughly to the time S_k reached its minimum. In the linear Gaussian case, S_k is simply a sum of normalized residual squared $\epsilon_k = \tilde{z}_k' S_k^{-1}\tilde{z}_k$. A maneuver detector for 2D tracking was developed in [37]. It uses the normalized residual $S_k^{-1/2}\tilde{z}_k$ with its scalar measure

$$\eta_k = \frac{1}{\sqrt{n_z}} \mathbf{1}_{n_z}' S_k^{-1}\tilde{z}_k \quad \text{with} \quad \mathbf{1}_{n_z} = (1, 1, \ldots, 1)'$$

which has the standard Gaussian distribution if the residual sequence $\tilde{z}_k$ is Gaussian and white. η_k and other possible distance measures were discussed in [19] and relevant references therein. This detector is decoupled from input estimation and computationally more efficient than the standard detector. The use of fading-memory sum, known as geometric moving avarage in statistics, is well established in tracking, but it is only one of a wide variety of choices available for change-point detection. A successful use of a CUSUM maneuver detector was reported recently in [19].

SPRT based. The celebrated sequential probability ratio test (SPRT) [38, 39, 17] can be applied to maneuver detection with an input estimate $\hat{u}$ as follows: Declare maneuver if $J(\hat{u}, n) \leq B$; declare no maneuver if $J(\hat{u}, n) \geq A$; otherwise no decision and continue to test using more measurements (i.e., $k := k + 1$). Here A and B are two thresholds, which can approximately set to $B = \beta / (1 - \alpha)$ and $A = (1 - \beta) / \alpha$ for given $\alpha = P\{\text{false detection}\}$ and $\beta = P\{\text{miss detection}\}$, and $J(\hat{u}, n) = \log \left(f \left(\tilde{z}_s^k | H_1(\hat{u}, n) \right) / f \left(\tilde{z}_s^k | H_0 \right) \right)$ are log-likelihood ratio, which can be computed recursively by using the residuals of the Kalman filters (KFs) matched to $H_1(\hat{u}, n)$ for maneuver onset time $n = k - s, k - s + 1, \ldots, k - 1$. Detection of changes in process noise covariance Q by SPRT was given in [40]. More recent results for change-point detection along the line of SPRT and *quickest detection* can be found in [41, 11, 42, 43, 44]. Some of them have been applied to fault detection, but not to maneuver detection to our knowledge, except that a quickest maneuver detector was given in [45] (with a subsequent correction).

4 Equivalent-Noise Approach

Almost all types of target motion can be described by the following state-space model

$$x_{k+1} = f_k(x_k, u_k, w_k) \tag{17}$$

where x is the state, u is the control input, and w is the process noise.

In the equivalent-noise approach, the basic assumption is that the maneuver effect can be modeled by (part of) a white or colored noise process sufficiently well. In other words, it is assumed that the above equation that describes target motions can be simplified to

$$x_{k+1} = f(x_k, w_k^*)$$

with an adequate accuracy, where w^* is **equivalent noise** that quantifies the error of this model in describing the target motions, in particular, maneuvers. Of course, the statistics (e.g., the mean and covariance) of this noise w^*, nonstationary in general, are not known. Valid or not, this fundamental assumption *converts the problem of MTT to that of state estimation in the presence of nonstationary process noise with unknown statistics*. Here lies the basic idea of the approach.

Numerous techniques have been developed for such state estimation problems in stochastic systems research over the past several decades, in particular, from late 1960's to early 1980's. Almost all of them are limited to linear systems (so is our description below), that is, assume that the system dynamics can be described by

$$x_{k+1} = F_k x_k + \Gamma_k w_k^* \tag{18}$$

with noise w^* of unknown statistics (e.g., mean $\bar{w}$ and covariance Q). Traditionally, its state estimation using a linear system of observations in white noise is considered as an essential part of what is known as *adaptive Kalman filtering*.

In MTT, the equivalent noise w^* has been assumed to be either white or colored. The best-known example of the former is the popular (nearly) constant-acceleration (CA) models, while the best-known representative of the latter is the Singer model. All models in the entire Sec. 4 of Part I [1] belong to this class. In particular, they are based on theories of white noise, Markov processes, and semi-Markov processes. These models have been surveyed and, in our opinion, adequately described in Part I [1], which will not be repeated here.

In order for this equivalent-noise approach to be effective for MTT, a basic requirement is that it must be able to respond quickly enough to maneuver onset and termination. In other words, this approach has a *modeling* side, as described in Sec. 4 of Part I, and an *adaptation* side, which is almost always guided by decisions concerning maneuvers. In the remaining of this section, we will focus on the adaptation side. More specifically, we will connect this MTT approach with adaptive Kalman filtering techniques, point out their major differences, and briefly describe adaptation schemes developed particularly for this approach to MTT. That is why this equivalent-noise approach is covered in this part dedicated to decision-based approaches.

There exists a large body of literature on adaptive Kalman filtering. A well-known early publication is [46]. While many of the techniques in this area have been developed for linear state estimation with unknown input, which is the topic of the next section, more of the proposed techniques are directed towards the case with unknown noise statistics. The equivalent noise w^* is assumed white in almost all these adaptive Kalman filtering techniques. For white noise, mean and covariance are clearly the two most important components of its statistics, and are in fact the only components considered in most cases, which is often justified by the Gaussian assumption on the noise, supported by central limit theorems.

There are generally two classes of adaptive Kalman filtering techniques for linear state estimation in white noise with unknown statistics. The first class, referred to as *noise identification*, explicitly identifies the noise statistics in real time and state estimation is done using the identified noise statistics; the other class, referred to as *adaptive gain* below, accounts for the effect of the uncertainty in the noise statistics on state estimation indirectly in the filter gain without explicit identification of the noise statistics.

Relatively fewer techniques have been developed for gain adaptation (see, e.g., [47, 48]). [49] proposed adaptation of the filter gain based on the deviation of the measurement residuals from orthogonality. More recent publications include [50], which also proposed a gain adaptive KF for linear model with an unknown Q (and measurement noise covariance R). On the other hand, abundant results are available for noise identification and particularly identification of covariances, which were surveyed in [48, 51, 52], with many references. Four groups of methods were identified [48]: (a) Bayesian approach, where Bayes' rule is used to update the prior distribution of noise statistics by measurements (see, e.g., [53]); (b) maximum likelihood estimation, where noise statistics are estimated by maximizing a (log)likelihood function of them (see, e.g., [54]); (c) correlation methods, where noise statistics are related to and then determined by the (sample) autocorrelation of the measurement (residual) sequence (see, e.g., [47, 55]); and (d) covariance matching, where noise statistics are estimated by matching between theoretical and sample covariances (see, e.g., [56]). A more detailed description of these methods was included in [48], while the brief review included in [57, 58] emphasizes on the more recent results.

Some of these adaptive techniques have been introduced or implemented in the context of MTT [59]. We mention here several examples: [60] implemented the algorithm of [47] for online identification of Q in an EKF for tracking a maneuvering reentry vehicle; similarly, also for maneuvering reentry, [61] used the procedure of [56] for noise adaptation; [62, 63] used a least squares update of the process noise based on a fading-memory sum of the residuals; similar ideas of limited and weighted limited memory covariance adaptation were implemented in [64] and [65], where the latter developed and applied an EKF with adaptive noise estimation for a short range, air-to-air, maneuvering target interception scenario; [66] proposed a simple adaptive algorithm for unknown (slowly varying or piecewise constant) Q; and [67] provided an accurate procedure for adaptive computation of Q in an EKF for ballistic target tracking.

As stated before, a fast response to maneuvers is essential for the equivalent-noise approach to work well. This is much more difficult to achieve than in the case where noise statistics vary not so quickly. As elaborated in [58], however, most adaptive Kalman filtering techniques are not particularly suitable to handle fast-varying unknown noise statistics. In fact, most of them are valid only for stationary noise or noise with slowly varying statistics. There are only a few exceptions, including those presented in [68, 69, 70, 58], which are based on the use of multiple models — the topic of a forthcoming part of this survey. In our opinion, it is this distinctive requirement of MTT that has prevented effective application of many adaptive Kalman filtering techniques in MTT. As a matter of fact, common practice with this equivalent-noise approach to MTT is to tune Q offline in advance. For example, a heuristic relationship $Q_k = F_k Q_{k-1} F_k'$ was used for the adaptation of Q in a practical implementation of a ballistic tracking filter [71].

A popular technique here is **noise-level adjustment** [56, 46, 9], [10]. It is assumed here that the effect of a maneuver on state estimation can be accounted for by increasing the process noise level (covariance Q): (a) scale up Q by a fudge factor ϕ: $Q_{k-1} := \phi_k Q_{k-1}$, or (b) switch to a pre-specified higher noise covariance: $Q_{k-1} = Q_2 > Q_1$ to yield a larger covariance for the measurement residual:

$$S_k = H_k(F_{k-1}P_{k-1|k-1}F_{k-1}' + Q_{k-1})H_k' + R_k$$

where Q_1 is the covariance of the process noise without maneuver. The upward and downward adjustments are initiated by a detection of maneuver onset and termination, respectively. Usually the simple chi-square test based on a fading-memory sum or sliding window of normalized measurement residual squares, as described in Sec. 3.1, is used, although many other maneuver detection techniques described in Sec. 3 can also be used. Note that for this simple test, downward detection is more difficult than upward detection. A similar ad hoc implementation of noise-level adjustment for MTT can be found in [34].

Another major difference between conventional adaptive Kalman filtering and the requirements for MTT is the following. While adaptive Kalman filtering normally deals with white noise models, colored noise models (as well as white noise models) are widely used in MTT, as described in Part I. This difference has not been emphasized probably because Markovian colored noise models can always be converted to white noise models by, e.g., state augmentation.

5 Input Detection and Estimation

Let the dynamics of a maneuvering target be given by (17). The basic idea of this approach is to explicitly estimate the unknown control input u_k and then estimate the state using the estimated input $\hat{u}_k$, although it may be more accurate (but less tractable) to estimate the state and input jointly. Compared with the equivalent-noise approach, which does not rely on explicit estimation of the unknown input, this approach is more direct and appears more appealing in general when the input is indeed estimable. We shall refer to this approach as **input detection and estimation** (IDE), although the simpler term *input* *estimation* may be more preferable, which unfortunately is usually associated with a particular method in this approach due to historical reasons.

Since almost all techniques in this approach only deal with linear dynamics, we will consider only the following linear system

$$x_{k+1} = F_k x_k + \mathsf{u}_k + \Gamma_k w_k \tag{19}$$
$$z_k = H_k x_k + v_k \tag{20}$$

where $\mathsf{u}_k = G_k u_k$ is the *input level* and $F_k = F_{\text{CV}}$ usually (see Part I). In general, it is assumed that $\mathsf{u}_k \neq 0$ when the target is maneuvering at time k and $\mathsf{u}_k = 0$ when the target is not maneuvering. Only a linear measurement system is considered here, as justified in Sec. 1.

With this linear system, the MTT problem becomes that of state estimation with unknown input. As for the case with unknown noise statistics, this problem also belongs to *adaptive Kalman filtering*. Unlike the unknown noise statistics case, where few adaptive Kalman filtering techniques are directly applicable to MTT, many adaptive Kalman filtering techniques for unknown input are applicable to MTT directly.

Clearly, the key to this IDE approach is the estimation of the input process $\langle \mathsf{u}_k \rangle$. There are three main uncertainties associated with $\langle \mathsf{u}_k \rangle$: (a) unknown input level $G_k u_k$ that may or may not vary during maneuver, (b) unknown maneuver onset time n, and (c) unknown maneuver termination time m. In other words,

$$\langle \mathsf{u}_k \rangle = \{\ldots, 0, \ldots, 0, \mathsf{u}_n, \mathsf{u}_{n+1}, \ldots, \mathsf{u}_{m-1}, 0, \ldots, 0, \ldots\}$$

In general, this IDE approach has the following three essential ingredients: (a) estimation of input; (b) state estimation using the estimated input; and (c) detection of maneuver onset and termination. The maneuver detection component is needed because otherwise the estimated input may be statistically insignificant, leading to inferior performance of the state estimation.

5.1 Estimation of Input

The description below follows [72].

Let $\{\hat{x}_k, P_k\}$ be the unified notation for both prediction $\{\hat{x}_{k|k-1}, P_{k|k-1}\}$ and update $\{\hat{x}_{k|k}, P_{k|k}\}$. Similarly, let $\tilde{z}_k = z_k - H_k \hat{x}_k - \bar{v}_k$ and $S_k = \text{cov}(\tilde{z}_k)$ be the corresponding measurement residuals and covariances, respectively. Consider now two KFs $\mathcal{F}_* = \{\hat{x}_k^*, P_k^*\}$ and $\mathcal{F}_\# = \{\hat{x}_k^\#, P_k^\#\}$. $\mathcal{F}_*$ assumes that input terms are all zero, while the hypothetical KF $\mathcal{F}_\#$ uses perfect knowledge of the nonrandom input. Note that $\mathcal{F}_*$ and $\mathcal{F}_\#$ have the same covariances (e.g., $P_k^* = P_k^\#$ and $S_k^* = S_k^\#$) and filter gain (i.e., $K_k^* = K_k^\#$) since the input terms are assumed nonrandom. It is straightforward to show by elementary manipulations that [72]

$$\hat{x}_k^\# = \hat{x}_k^* + \Delta\hat{x}_k \tag{21}$$

where

$$\Delta\hat{x}_{k|k-1} = \hat{x}_{k|k-1}^\# - \hat{x}_{k|k-1}^* = \tilde{\mathbf{G}}_k \mathbf{u}_k \tag{22}$$
$$\Delta\hat{x}_{k|k} = \hat{x}_{k|k}^\# - \hat{x}_{k|k}^* = U_k \tilde{\mathbf{G}}_k \mathbf{u}_k \tag{23}$$
$$\mathbf{u} = [\mathsf{u}_n', \mathsf{u}_{n+1}', \ldots, \mathsf{u}_{m-1}']', \quad \mathbf{u}_k = T_k \mathbf{u}$$
$$\tilde{\mathbf{G}}_k = \begin{cases} 0 & k \le n \\ \mathbf{G}_k & k = n+1, n+2, \ldots, m \\ L_{k,m}\mathbf{G}_m & k > m \end{cases}$$
$$\mathbf{G}_k = [L_{k,n+1}, L_{k,n+2}, \ldots, L_{k,k}], \quad n < k \le m$$
$$L_{j,j} = I, \qquad L_{k,i} = L_{k-1}L_{k-2}\cdots L_i, \ k > i, \quad L_{k,i} = 0, k < i$$
$$L_j = F_j U_j, \quad U_j = I - K_j^* H_j$$

T_k is a matrix such that $\mathbf{u}_k = T_k \mathbf{u}$ is an empty vector (i.e., $T_k = [\ ]$ is an empty matrix without an element) if $k \le n$, $\mathbf{u}_k = \mathbf{u}$ (i.e., $T_k = I$) if $k \ge m$, otherwise $\mathbf{u}_k = [\mathsf{u}_n', \mathsf{u}_{n+1}', \ldots, \mathsf{u}_{k-1}']'$ (i.e., $T_k = [\text{diag}(I, \ldots, I), 0]$ is a matrix with $(k-n) \times (m-n)$ blocks). Note that $\mathbf{G}_k \mathbf{u}_k = \sum_{i=n}^{k-1} L_{k,i+1} \mathsf{u}_i$, $k = n+1, n+2, \ldots, m$. Note that matrix $\tilde{\mathbf{G}}_k$ relates the input $\mathbf{u}_k$ to its contribution $\Delta\hat{x}_{k|k-1}$ to state prediction. Such a linear relation exists because the system is linear.

It thus follows that

$$\tilde{z}_k^* = \Delta\tilde{z}_k + \tilde{z}_k^\# \tag{24}$$

where $\Delta\tilde{z}_k = H_k\Delta\hat{x}_k$. Since $\tilde{z}_k^*$ is easily obtainable while $\tilde{z}_k^{\#}$ is actually not obtainable, (24) is in the form of $y_k = h_k(\mathbf{u}_k) + v_k$ (with $y_k = \tilde{z}_k^*$ and $v_k = \tilde{z}_k^{\#}$) and thus can be viewed as an observation model of $\mathbf{u}_k$. Note that (a) measurements $y_k = \tilde{z}_k^*$ are linear in the input $\mathbf{u}_k$; (b) under the linear-Gaussian assumption of the KF, the measurement noise $v_k = \tilde{z}_{k|k-1}^{\#}$ is zero mean and white with covariance $S_{k|k-1}^*$. This equation is the basis of several IDE methods, described below.

Assume that observations over the time window $[k-s, k]$ are given. Under the linear-Gaussian assumption of the KF, it follows from (24) that the log-likelihood function of $(\mathbf{u}_k, n, m)$ is given by

$$\begin{aligned}\log f(\tilde{\mathbf{z}}_k^*|\mathbf{u}_k, n, m) &= -\frac{1}{2}\sum_{\kappa=k-s}^{k}(\tilde{z}_\kappa^* - H_\kappa\tilde{\mathbf{G}}_\kappa\mathbf{u}_k)'(S_\kappa^*)^{-1}(\tilde{z}_\kappa^* - H_\kappa\tilde{\mathbf{G}}_\kappa\mathbf{u}_k) \\ &= -\frac{1}{2}(\tilde{\mathbf{z}}_k^* - \mathbf{H}_k\mathbf{u}_k)'\mathbf{S}_k^{-1}(\tilde{\mathbf{z}}_k^* - \mathbf{H}_k\mathbf{u}_k) \qquad (25)\end{aligned}$$

where $\tilde{\mathbf{z}}_k^* = [(\tilde{z}_{k-s}^*)', (\tilde{z}_{k-s+1}^*)', \ldots, (\tilde{z}_k^*)']'$, $\mathbf{S}_k = \text{diag}[S_{k-s}^*, S_{k-s+1}^*, \ldots, S_k^*]$, and

$$\mathbf{H}_k = [(H_{k-s}\tilde{\mathbf{G}}_{k-s})', (H_{k-s+1}\tilde{\mathbf{G}}_{k-s+1})', \ldots, (H_k\tilde{\mathbf{G}}_k)']'$$

To be precise the prediction version (e.g., $\tilde{\mathbf{z}}_k^* = \tilde{\mathbf{z}}_{k|k-1}^*$) should be used.

Consequently, the input levels and maneuver onset and termination times can be estimated by the maximum likelihood method as

$$(\hat{\mathbf{u}}_k, \hat{n}, \hat{m})^{\text{ML}} = \arg\max_{\mathbf{u},n,m} \log f(\tilde{z}^*|\mathbf{u}_k, n, m) \qquad (26)$$

In view of (24), it is also reasonable to estimate $(\mathbf{u}_k, n, m)$ by the following weighted least-squares method directly without the linear-Gaussian assumption:

$$(\hat{\mathbf{u}}_k, \hat{n}, \hat{m})^{\text{LS}} = \arg\min_{\mathbf{u}_k,n,m} J_k(\mathbf{u}_k, n, m) \qquad (27)$$

where

$$J_k(\mathbf{u}_k, n, m) = (\tilde{\mathbf{z}}_k^* - \mathbf{H}_k\mathbf{u}_k)'\mathbf{S}_k^{-1}(\tilde{\mathbf{z}}_k^* - \mathbf{H}_k\mathbf{u}_k) \qquad (28)$$

Under the linear-Gaussian assumption, $J_k(\mathbf{u}_k, n, m) = -2\log f(\tilde{\mathbf{z}}_k^*|\mathbf{u}_k, n, m)$ and thus $(\hat{\mathbf{u}}_k, \hat{n}, \hat{m})^{\text{LS}} = (\hat{\mathbf{u}}_k, \hat{n}, \hat{m})^{\text{ML}}$.

The solution of the above ML or LS problem for $\mathbf{u}_k$ given (n, m) is

$$\hat{\mathbf{u}}_k(n, m) = \Sigma_k(n, m)e_k(n, m) \qquad (29)$$

where

$$\begin{aligned}\Sigma_k(n, m) &= \text{MSE}[\hat{\mathbf{u}}_k(n, m)] = (\mathbf{H}_k'\mathbf{S}_k^{-1}\mathbf{H}_k)^{-1} \qquad (30) \\ e_k(n, m) &= \mathbf{H}_k'\mathbf{S}_k^{-1}\tilde{\mathbf{z}}_k^* \qquad (31)\end{aligned}$$

Note that given (n, m), a linear unbiased estimator of $\mathbf{u}_k$ using data $\tilde{\mathbf{z}}_k^* = \mathbf{H}_k\mathbf{u}_k + \tilde{\mathbf{z}}_k^{\#}$ with known $E[\tilde{\mathbf{z}}_k^{\#}]$ and cov$(\tilde{\mathbf{z}}_k^{\#})$ exists if and only if $\mathbf{H}_k$ has full column rank (i.e., $\det(\mathbf{H}_k'\mathbf{H}_k) \neq 0$) [73]. This formulation reveals the condition under which a linear unbiased estimator of $\mathbf{u}_k$ exists. See [72] for more details. Under the linear-Gaussian assumption, $\hat{\mathbf{u}}_k(n, m)$ using the prediction version (e.g., $\tilde{z}_k^* = \tilde{z}_{k|k-1}^*$) is unbiased and Gaussian distributed with covariance $\Sigma_k(n, m)$: $\hat{\mathbf{u}}_k(n, m) \sim \mathcal{N}[\mathbf{u}, \Sigma_k(n, m)]$. This provides a justification of the simple chi-square test for maneuver detection, described in Sec. 3.1.

In the literature, the maneuver termination time is never included in the problem formulation. The only exception known to the authors is [72], from which the above formulation follows. While this has little impact on the peak error following immediately a maneuver onset, it is significant in reducing the peak error due to maneuver termination.

5.2 State Estimation

(21) suggests that given an estimate $\hat{\mathbf{u}}_k$ of the input $\mathbf{u}_k$, the state can be estimated simply by

$$\hat{x}_k = \hat{x}_k^* + \Delta\hat{x}_k|_{(\mathbf{u}_k,n,m)=(\hat{\mathbf{u}}_k,\hat{n},\hat{m})}$$

that is,

$$\begin{aligned}\hat{x}_{k|k-1} &= \hat{x}_{k|k-1}^* + \hat{\mathbf{G}}_k\hat{\mathbf{u}}_k \qquad (32) \\ \hat{x}_{k|k} &= \hat{x}_{k|k}^* + U_k\hat{\mathbf{G}}_k\hat{\mathbf{u}}_k \qquad (33)\end{aligned}$$

where $\hat{\mathbf{G}}_k = \tilde{\mathbf{G}}_k|_{(n,m)=(\hat{n},\hat{m})}$.

It follows from (21) and the above equation that $\hat{x}^{\#}_{k|k-1} - \hat{x}_{k|k-1} = \tilde{\mathbf{G}}_k \mathbf{u}_k - \hat{\mathbf{G}}_k \hat{\mathbf{u}}_k \approx \hat{\mathbf{G}}_k(\mathbf{u}_k - \hat{\mathbf{u}}_k)$, and thus

$$x_k - \hat{x}_{k|k-1} \approx x_k - \hat{x}^{\#}_{k|k-1} + \hat{\mathbf{G}}_k(\mathbf{u}_k - \hat{\mathbf{u}}_k)$$

Since $P^{\#}_{k|k-1} = \text{MSE}(\hat{x}^{\#}_{k|k-1}) = E[(x_k - \hat{x}^{\#}_{k|k-1})(x_k - \hat{x}^{\#}_{k|k-1})'] = P^*_{k|k-1}$, we have

$$P_{k|k-1} = \text{MSE}(\hat{x}_{k|k-1}) \approx P^*_{k|k-1} + \hat{\mathbf{G}}_k \Sigma_k \hat{\mathbf{G}}'_k + C_{\tilde{x}^{\#}\tilde{\mathbf{u}}}\hat{\mathbf{G}}'_k + \hat{\mathbf{G}}_k C'_{\tilde{x}^{\#}\tilde{\mathbf{u}}} \tag{34}$$

where $C_{\tilde{x}^{\#}\tilde{\mathbf{u}}} = E[(x_k - \hat{x}^{\#}_{k|k-1})(\mathbf{u}_k - \hat{\mathbf{u}}_k)']$. Note that the maneuver onset and termination times affect $\hat{x}_k$ and P_k only through $\hat{\mathbf{u}}_k$ and $\Sigma_k = \text{MSE}(\hat{\mathbf{u}}_k)$. It was argued in [23] that $C_{\tilde{x}^{\#}\tilde{\mathbf{u}}}\hat{\mathbf{G}}'_k = 0$ because $\hat{\mathbf{u}}_k$ is a linear combination of uncorrelated measurement residuals. In fact, $\hat{\mathbf{u}}_k$ is indeed a linear combination of measurement residuals $\tilde{z}^*_k$, but they are not uncorrelated even under the linear-Gaussian assumption; furthermore, even if they are, we could not conclude $C_{\tilde{x}^{\#}\tilde{\mathbf{u}}}\hat{\mathbf{G}}'_k = 0$ in general. General formulas of $C_{\tilde{x}^{\#}\tilde{\mathbf{u}}}\hat{\mathbf{G}}'_k$ are given in [72]. Clearly, ignoring $C_{\tilde{x}^{\#}\tilde{\mathbf{u}}}\hat{\mathbf{G}}'_k$ leads to

$$P_{k|k-1} \approx P^*_{k|k-1} + \hat{\mathbf{G}}_k \Sigma_k \hat{\mathbf{G}}'_k \tag{35}$$

Consequently, it is thus seen that the state estimation can simply be done approximately by a correction in the estimate and error covariance. This is often referred to as **state estimate correction**. It should be noted that the correction equations (32) and (35) are approximate since they do not account for the error arising from estimation of (n, m).

5.3 Various Algorithms

While a general formulation and solution of the problem of estimating the input level and the effective interval is given in Sec. 5.1, most IDE algorithms developed are under some simplifying assumptions. More specifically, they are based on one or more of the following assumptions.

(**A**) Constant-input assumption: The unknown input level is constant during a maneuver (i.e., over $[n, m-1]$), that is, $\mathsf{u}_n = \mathsf{u}_{n+1} = \cdots = \mathsf{u}_{m-1} = \mathsf{u}$.

(**B**) Maneuver-duration assumption: A maneuver may terminate only after it is detected at time k; that is, maneuver duration is always larger than maneuver detection delay.

(**C**) Constant-delay assumption: A maneuver starting at time n may only be detected at time $n+s$; that is, a maneuver detected at time k always started at time $k-s$ (i.e., nonzero input starting at $k-s-1$).

Clearly, all these assumptions are quite restrictive for realistic applications. It appears that Assumption (C) is most restrictive while Assumption (B) is probably least restrictive.

With Assumption (A), we have $\tilde{\mathbf{G}}_k \mathbf{u}_k = \tilde{G}_k \mathsf{u}$ and $\mathbf{H}_k \mathbf{u}_k = \tilde{H}_k \mathsf{u}$, where $\tilde{G}_k = 0$ if $k \le n$, $\tilde{G}_k = L_{k,m} G_{m,n}$ if $k \ge m$, otherwise $\tilde{G}_k = G_{k,n}$ with

$$G_{k,n} = \sum_{i=n}^{k-1} L_{k,i+1} \tag{36}$$

and

$$\tilde{H}_k = [\bar{H}'_{k-s}, \bar{H}'_{k-s+1}, \ldots, \bar{H}'_k]' = [(H_{k-s}\tilde{G}_{k-s})', (H_{k-s+1}\tilde{G}_{k-s+1})', \ldots, (H_k\tilde{G}_k)']'$$

As such, $\hat{\mathbf{u}}_k(n, m)$ becomes

$$\hat{\mathsf{u}}_k = \Sigma_k e_k \tag{37}$$

$$\Sigma_k^{-1} = \tilde{H}'_k \mathbf{S}_k^{-1} \tilde{H}_k = \sum_{i=k-s}^{k} \bar{H}'_i S_i^{-1} \bar{H}_i \tag{38}$$

$$e_k = \tilde{H}'_k \mathbf{S}_k^{-1} \tilde{\mathbf{z}}^*_k = \sum_{i=k-s}^{k} \bar{H}'_i S_i^{-1} \tilde{z}^*_i \tag{39}$$

With Assumptions (A) and (B) and the assumption that a maneuver is detected at k, we have $\tilde{\mathbf{G}}_k \mathbf{u}_k = \mathbf{G}_k \mathbf{u}_k = G_{k,n}\mathsf{u}$ if $k > n$ and $\tilde{\mathbf{G}}_k \mathbf{u}_k = 0$ otherwise, and $\mathbf{H}_k \mathbf{u}_k = \tilde{H}_k \mathsf{u}$, where if $n > k-s$ then

$$\tilde{H}_k = [\bar{H}'_{k-s}, \bar{H}'_{k-s+1}, \ldots, \bar{H}'_k]' = [0, \ldots, 0, (H_{n+1}G_{n+1,n})', (H_{n+2}G_{n+2,n})', \ldots, (H_k G_{k,n})']' \tag{40}$$

As such, $\hat{\mathbf{u}}_k(n,m)$ becomes $\hat{\mathsf{u}}_k = \Sigma_k e_k$, where Σ_k and e_k are given by (38)–(39) with $\bar{H}_i$ given by (40), that is,

$$\hat{\mathsf{u}}_k = \Sigma_k e_k \tag{41}$$

$$\Sigma_k^{-1} = \sum_{i=k-s}^{k} \bar{H}_i' S_i^{-1} \bar{H}_i = \sum_{i=n+1}^{k} (H_i G_{i,n})' S_i^{-1} (H_i G_{i,n}) \tag{42}$$

$$e_k = \sum_{i=k-s}^{k} \bar{H}_i' S_i^{-1} \tilde{z}_i^* = \sum_{i=n+1}^{k} (H_i G_{i,n})' S_i^{-1} \tilde{z}_i^* \tag{43}$$

With Assumptions (A)–(C) and the assumption that a maneuver is detected at k, we have $\tilde{\mathbf{G}}_k \mathbf{u}_k = \mathbf{G}_k \mathbf{u}_k = G_{k,n}\mathsf{u}$ and $\mathbf{H}_k \mathbf{u}_k = \tilde{H}_k \mathsf{u}$, where

$$\tilde{H}_k = [\bar{H}_{k-s}', \bar{H}_{k-s+1}', \ldots, \bar{H}_k']' = [(H_{n+1}G_{n+1,n})', (H_{n+2}G_{n+2,n})', \ldots, (H_k G_{k,n})']' \tag{44}$$

As such, $\hat{\mathbf{u}}_k(n,m)$ becomes $\hat{\mathsf{u}}_k = \Sigma_k e_k$, where Σ_k and e_k are given by (38)–(39) with $\bar{H}_i = H_i G_{i,n}$.

5.3.1 Generalized Likelihood Ratio Algorithms

In this algorithm, proposed in [74, 75] with Assumptions (A) and (B), input estimates $\hat{\mathsf{u}}_k(n)$ for all possible maneuver onset time n are computed and the one that maximizes the log-likelihood $\log f(\tilde{\mathbf{z}}_k^* | \hat{\mathsf{u}}, n)$ is taken to be the input estimate and the corresponding n as the onset time estimate $\hat{n}$. The maneuver detection in this algorithm is done based on the generalized likelihood ratio (GLR) test, hence the name **GLR algorithm**. Specifically, this algorithm for MTT consists of the following.

- *Input estimation.* For each $n = k-s, \ldots, k-1$, obtain the MLE $\hat{\mathsf{u}}_k(n) = \arg\max_{\mathsf{u}} \log f(\tilde{\mathbf{z}}_k^* | \mathsf{u}, n)$, which is given by (41)–(43) under the linear-Gaussian assumption. Then $\hat{\mathsf{u}}_k(\hat{n}) = \arg\max_{k-s \le n < k} \log f(\tilde{\mathbf{z}}_k^* | \hat{\mathsf{u}}_k(n), n)$.
- *Onset time estimation.* Obtain the MLE

$$\hat{n} = \arg\max_{k-s \le n < k} \log f(\tilde{\mathbf{z}}_k^* | \hat{\mathsf{u}}_k(n), n) = \arg\max_{k-s \le n < k} \Delta J_k(\hat{\mathsf{u}}_k(n), n) \tag{45}$$

  which is given by (10) under the linear-Gaussian assumption.
- *Maneuver detection.* A maneuver is declared if

$$\hat{\mathsf{u}}_k(\hat{n})' \Sigma_k(\hat{n})^{-1} \hat{\mathsf{u}}_k(\hat{n}) > \lambda \tag{46}$$

- *State estimate correction.* Use (32) and (35) if a maneuver is declared.

Note that since $\max_{(\mathsf{u},n)} \Delta J_k(\mathsf{u}, n) = \Delta J_k(\hat{\mathsf{u}}, \hat{n})$ under the linear-Gaussian assumption a rigorous GLR detector should use $\Delta J_k(\hat{\mathsf{u}}, \hat{n}) > \lambda'$, which in general differs from (46). Note also that $\hat{\mathsf{u}}_k(\hat{n})' \Sigma_k(\hat{n})^{-1} \hat{\mathsf{u}}_k(\hat{n}) = e_k(\hat{n})' \Sigma_k(\hat{n}) e_k(\hat{n})$ does not hold in general, although it is not uncommonly used.

In order to avoid a possible overcompensation, which for instance may arise from detecting the same jump repeatedly, an ad hoc technique was suggested to reinitialize the GLR after state estimate correction. This will also make it possible to detect a sequence of successive jump inputs to the system [75]. Another issue is the choice of the window length, which is a trade-off among input estimation accuracy, detection delay, and computational load.

5.3.2 Recursive GLR Algorithms

A recursive form of the above GLR algorithm was proposed in [74], [75] to save computation. A more general form was given in [76] as follows: for given n,

$$\Phi_k = F_{k-1}\Phi_{k-1} + G_{k-1,n} \tag{47}$$

$$\bar{H}_k = H_k (\Phi_k - F_{k-1}\Psi_{k-1}) \tag{48}$$

$$\Psi_k = F_{k-1}\Psi_{k-1} + K_k \bar{H}_k \tag{49}$$

$$\Sigma_k^{-1} = \Sigma_{k-1}^{-1} + \bar{H}_k' S_k^{-1} \bar{H}_k \tag{50}$$

$$e_k = e_{k-1} + \bar{H}_k' S_k^{-1} \tilde{z}_k^* \tag{51}$$

$$\hat{\mathsf{u}}_k = \Sigma_k e_k \tag{52}$$

The first three equations compute $\bar{H}_k$ recursively. (50)–(52) are a well-known recursive form of (42)–(43).

Tracking application. The above GLR algorithm was illustrated in [74, 75] via a simple example of 1D MTT using a KF with a CV model for a jump maneuver scenario. It was later applied to a more realistic 2D MTT scenario in a generic surface-to-air engagement in [76], which involves high nonlinearities in both target maneuver and measurement model, and the simulation results presented demonstrate working capabilities of the algorithm. Another application employing GLR technique was reported in [77] for MTT for homing missile guidance. The main problem encountered arises from the sudden and high magnitude jumps in the target lateral acceleration, modeled as a first-order Gauss-Markov process with an additional unknown constant-bias term (see Part I). This model was considered within the framework of the 2D general curvilinear motion model (see Part I) with polar measurements. A bank of EKFs was used to estimate the bias as part of the target state. The bank is formed by starting at each instant in a window a new EKF with an initial estimate from the nominal filter and a large initial bias covariance to allow the EKF to track the bias near its onset time. The most likely EKF is selected according to (45) and the corresponding time instant is taken to be the maneuver onset time. Then the test (46) is used to verify maneuver detection. If maneuver is declared, the most likely filter becomes the nominal and the whole process is restarted. A probabilistic weighted sum of the filters in the bank (i.e., MM approach) was also implemented as an alternative. This same idea was also considered in [31].

5.3.3 Degenerated Kalman Filter for Input

It is well known that a recursive LS estimator of an unknown but constant parameter can be viewed as a special case of the KF of the parameter. Since in the linear Gaussian case the above recursive GLR algorithm is in fact a recursive LS estimator in the information form, (50)–(52) may be viewed as the information form of the KF for estimating unknown constant u using measurement model (24). This recognition, made explicit in [78], enables us to utilize the power of the KF to handle the case with a certain special time-varying unknown input by treating the input as the state of a linear system with known dynamics but unknown onset time. The resulting algorithm [78] is a straightforward implementation of the GLR method with a KF incorporated for input estimation. It generalizes the earlier decoupled bias correction KF of [79], which solves the pure estimation problem (without addressing the problem of bias detection) in the presence of time-varying unknown bias, but without considering the uncertainty in the bias onset time.

It should be noted that the KF used here is in fact degenerated because the dynamics equation for the input is nonrandom. It appears more natural to assume the input as random and apply a truly KF here. It turns out that by so doing we arrive at an algorithm, known as variable state dimension, discussed in Sec. 6.1. However, a fundamental dilemma to represent maneuver by the input is that potentially time-varying unknown input is better modeled as nonrandom, but far more prior knowledge than what is usually available is needed to have a meaningful model of its dynamics other than the nearly constant one $u_{k+1} = u_k + w_k$.

The following standard covariance version of the KF was presented in [78]

$$\Sigma_k = \Sigma_{k-1} - \Sigma_{k-1}\bar{H}_k'\left(\bar{H}_k\Sigma_{k-1}\bar{H}_k' + S_k\right)^{-1}\bar{H}_k\Sigma_{k-1} \tag{53}$$

$$\hat{\mathsf{u}}_k = \hat{\mathsf{u}}_{k-1} + \Sigma_{k-1}\bar{H}_k'S_k^{-1}\left(\tilde{z}_k^* - \bar{H}_k\hat{\mathsf{u}}_{k-1}\right) \tag{54}$$

which is computationally more efficient than the information form when $\dim(z) < \dim(x)$.

5.3.4 Input Estimation Algorithm

The input-estimation (IE) algorithm was first developed for MTT in [23] using a simplified batch LS formulation of (28) that corresponds to Assumptions (A)–(C). It consists of three steps:

- *Input estimation.* Given by (37)–(39).
- *Maneuver detection.* A maneuver is declared if the estimated input is deemed statistically significant, that is, if

$$\hat{\mathsf{u}}_k'\Sigma_k^{-1}\hat{\mathsf{u}}_k \geq \lambda \tag{55}$$

  as described in Sec. 3.1.
- *State estimate correction.* Use (32) and (35) if a maneuver is declared.

This algorithm was implemented in [23] for a MTT problem using a planar CV target model in Cartesian coordinates (see Part I) with direct Cartesian position measurements, where the unknown input represents acceleration along each coordinate.

Although this algorithm is attractive in several aspects, it suffers from two major deficiencies, which stem from assuming *constant input* (Assumption A) and *known onset time* (Assumption C). These assumptions are rather unrealistic in typical target tracking applications, and lead to undesirable performance [80, 9]. Another complication arises from the linearity assumption of the measurement model, which is rarely true in practice, and thus additional work (e.g., unbiased measurement conversion, as discussed in details in Part III) is needed, giving rise to additional errors.

To overcome these deficiencies and improve the performance of the original algorithm, a number of extensions and enhancements have been proposed in the literature.

Enhancement by decoupled maneuver detector. The original IE algorithm developed in [23] does input estimation before maneuver detection because the detector (55) uses input estimates. Since a maneuver is less likely to occur than the CV motion and the batch LS based input estimator is computational demanding, this implies a huge waste in computation. To improve computational efficiency, it would make much better sense if the input is estimated only after a maneuver is declared since the latter requires much less computation. Such an IE algorithm was proposed in [37] with comparable performance to the original algorithm. The maneuver detection proposed therein amounts to checking if the actual multiple-step measurement residual matches its Kalman prediction based on a zero-input model. This detector was reported in [37] to have a higher detection probability than the residual-based detector (Sec. 3.1), at the same level of false alarm. Theoretically the same input estimator and state estimate corrector are used as in the original IE algorithm, except that a recursive form of the input estimator for a simplified 1D problem was also given in [37]. Unlike the original IE algorithm, however, the input estimation and maneuver detection here are not directly coupled.

5.3.5 Enhanced Input Estimation Algorithm

It was proposed in [31] to enhance the original IE algorithm by relaxing Assumption (C) of a constant delay of maneuver detection. The idea is to consider at time k a number of hypotheses regarding the maneuver onset time n, (e.g., all possible $n = 1, 2, \ldots, k-1$, or some subset) and evaluate their likelihoods by running in parallel the respective input estimators as in the original IE algorithm. Then either the probabilistically weighted average of all input estimates or the most likely input estimate is taken to be the final estimate of the input. (Likewise for the estimate of the maneuver onset time.) The latter is methodologically equivalent to the GLR algorithm presented above. The difference lies in the concrete implementation for the chosen target/measurement model. Maneuver detection and state estimate correction are done exactly the same as in the original IE algorithm. We will refer to the resulting algorithm as the *enhanced IE* (EIE) algorithm.

The development in [31] was done for a *one-dimensional* tracking problem, resulting in an easily implementable recursive formulation of the algorithm in closed form. There are two subtleties in the problem setting considered. First, a three-state CA model is chosen as the basic maneuver-free model, where the input stands for a *jump* in the acceleration level, rather than the acceleration itself. Second, the input is nonzero only at $k = n = m - 1$ (i.e., an impulse) since the input represents a jump in the acceleration. These subtleties greatly simplify the development of the recursive implementation.

This enhancement accounts for the uncertainty in the maneuver onset time and optimizes the size of the data batch needed for input estimation. Thus it is capable of coping with the "detection delay problem" of the original IE algorithm. The cost is a possibly large increase in input estimation so as to "analyze the past" with respect to maneuver onset time. This makes the development of a recursive version essential. The recursive input estimator is theoretically equivalent to the original batch input estimator, as shown in [81], but computationally more efficient. This however was achieved primarily due to the simplicity of the particular 1D problem setting considered. For more general and more realistic problems, further development is needed. The use of a CA model in principle leads to a degradation of estimation accuracy during a nonmaneuvering motion as compared to the CV model-based KF. While it has a limited value in direct application, this EIE algorithm did serve as a basis for further development and refinement for solutions to more realistic tracking problems.

For the case with correlated measurement errors, a practical situation typically arising in radar tracking at high measurement frequency, [82] reported an MTT application of the EIE algorithm, along with some adjustments, after decorrelation using a technique similar to the one described in Sec. 8.4.1 of [10].

5.3.6 Modified Input Estimation Algorithm

The modified input estimation (MIE) algorithm proposed in [25] is a modification and enhancement of the original IE and EIE algorithms.

The time-correlated Singer model of acceleration was adopted in this algorithm as the zero-input model so as to cover the CV motion and maneuvers of a small magnitude since the standard IE algorithm is known to be insensitive to low maneuvers and to require a long detection window for accumulating sufficient maneuver statistics. In such a setting the unknown input stands for a stepwise change in the target acceleration level. A "naive" direct implementation of the idea was found, however, to suffer from the side effect of hampering the maneuver detection since the Singer process noise would tend to suppress the increase in residuals. To improve the maneuver detection sensitivity, it was proposed to estimate the input based on the so-called *pseudoresiduals*, introduced particularly for this purpose, explained below.

The target model (19) is modified as

$$x_{k+1} = \begin{cases} Fx_k + Gu_0 + w_k & k < n \\ F\left[x_k + B\left(u - u_0\right)\right] + Gu + w_k & k = n \\ Fx_k + Gu + w_k & k > n \end{cases} \tag{56}$$

where $x = (\mathsf{x},\,\mathsf{y},\,\mathsf{z},\dot{\mathsf{x}},\,\dot{\mathsf{y}},\,\dot{\mathsf{z}},\ddot{\mathsf{x}},\,\ddot{\mathsf{y}},\,\ddot{\mathsf{z}})'$, F, G and Q are defined in Singer model, w_k is zero-mean and white, $B = [0,\,0,\,I]'$, and u_0 and u are the input levels before and after a change occurs at time n. The use of u_0 and u (rather than $u_0 = 0$ and u) makes it possible to estimate piecewise-constant input levels by a successive application of the input estimator. The measurement matrix is $H = [I,\,0,\,0]$. This model differs from (19) with $G_k u_k = Gu_0$ for $k < n$ and $G_k u_k = Gu$ for $k \geq n$ only in that it assumes the acceleration change occurs suddenly (i.e., as an impulse) at time n and thus does not affect the target position and velocity during the course of change, similarly to the case considered in the implementation of the EIE algorithm [31]. As such, this model represents a maneuver as a specific combination of impulse and step changes.

For any hypothesized maneuver onset time n, the *pseudoresidual* is defined for $\kappa > n$ as

$$\tilde{y}_\kappa(n) = y_\kappa(n) - \hat{H}_\kappa u, \qquad \kappa > n \tag{57}$$

for $\kappa \leq n$ it coincides with the residual $\tilde{z}_\kappa = z_\kappa - H\hat{x}_{\kappa|\kappa-1}$, where

$$\hat{H}_\kappa = H\left(FE\right)^{\kappa-n}B + H\sum_{j=0}^{\kappa-n-1}\left(FE\right)^j G \tag{58}$$

$$y_\kappa(n) = z_\kappa - z_{\kappa-1} - HF\hat{x}_v = z_\kappa - z_{\kappa-1} - HFD\hat{x}_{n|n} \tag{59}$$

with $D = \text{diag}(0, I, 0)$ and $E = \text{diag}(0, I, I)$, which take velocity components and remove position components of the state vector, respectively. Note that $(z_\kappa - z_{\kappa-1})$ is the measured position displacement over the sampling interval $(t_{k-1}, t_k]$ and $HF\hat{x}_v$ is the corresponding predicted nonmaneuvering displacement based on the velocity estimate $\hat{x}_v = D\hat{x}_{n|n}$ at a hypothesized maneuver onset time n. (57) thus provides a measurement model for the unknown input u in the form

$$y_\kappa(n) = \hat{H}_\kappa u + \tilde{y}_\kappa(n), \qquad \kappa = n+1,\,\ldots,k \tag{60}$$

It was claimed in [25] with evidence that the unknown pseudoresidual $\tilde{y}_\kappa(n)$ has zero mean. The covariance of $\tilde{y}_\kappa(n)$ was found approximately

$$S_\kappa^{\tilde{y}} = HFDP_{n|n}DF'H' + R_\kappa + R_{\kappa-1}, \qquad \kappa = n+1,\,\ldots,k \tag{61}$$

Also, it was not shown that $\tilde{y}_\kappa(n)$ is white, which is nonetheless implicitly assumed in the derivation therein.

Based on this measurement model, the MIE algorithm uses the same input estimator and maneuver onset-time estimator as those of the EIE algorithm. Its maneuver detector is given by $[\hat{u}_k(\hat{n}) - u_0]'\hat{\Sigma}_k^{-1}[\hat{u}_k(\hat{n}) - u_0] \geq \lambda$, where [25]

$$\hat{\Sigma}_k = E\{[\hat{u}_k(\hat{n}) - u_0][\hat{u}_k(\hat{n}) - u_0]'\} = \Sigma_k + \Sigma_k \sum_{i=n+1}^{k-1}\left[\sum_{j=i+1}^{k}(V_{i,j} + V_{j,i}) - W_i - W_i'\right]\Sigma_k \tag{62}$$

with $V_{i,j} = \hat{H}_i'(S_i^{\tilde{y}})^{-1}HFDP_{n|n}DF'H'(S_j^{\tilde{y}})^{-1}\hat{H}_j$ and $W_i = \hat{H}_i'(S_i^{\tilde{y}})^{-1}R_i(S_{i+1}^{\tilde{y}})^{-1}\hat{H}_{i+1}$. It is computational more efficient than that of the EIE algorithm. Essentially the same state estimate corrector as those used in the other IE algorithms is employed, given by

$$\hat{x}_{k|k} = \hat{x}_{k|k}^{u_0} + \hat{G}_k[\hat{u}_k(\hat{n}) - u_0], \qquad P_{k|k} \approx P_{k|k}^{u_0} + \hat{G}_k\hat{\Sigma}_k\hat{G}_k'$$

where $\hat{G}_k = (U_kF)(U_{k-1}F)\cdots(U_{n+1}F)B + G_{k,n}G$, and $\{\hat{x}_{k|k}^{u_0}, P_{k|k}^{u_0}\}$ is from the KF assuming input u_0.

The MIE algorithm, by utilizing a more precise target model and more sophisticated measurement model for input estimation, succeeds to improve the EIE algorithm considerably regarding the detection window length and implicitly the computation involved. Additional computational savings also come from the possibility to pre-compute $\hat{H}_\kappa$ off-line and to compute $\hat{G}_k$ for $\hat{n}$ only, rather than for each n in the window, as in the EIE algorithm.

We now suggest an analytic performance comparison between the input estimators in the MIE and EIE algorithms [72]. Given two optimal LS estimators $\hat{u}_1$ and $\hat{u}_2$ of u based on measurement models $z_i = H_i u + v_i$ with zero-mean v_i and $\text{cov}(v_i) = R_i$, $i = 1, 2$, respectively. Let A_1 and A_2 be two nonsingular matrices such that $A_i z_i = Hu + A_i v_i$, $i = 1, 2$. Then, $\hat{u}_1$ has a smaller MSE matrix than $\hat{u}_2$ if and only if $H'[(A_1 R_1 A_1')^{-1} - (A_2 R_2 A_2')^{-1}]H \geq 0$ since $\text{MSE}(\hat{u}_i) = [H'(A_i R_i A_i')^{-1} H]^{-1}$. Treating $\hat{u}_1$ and $\hat{u}_2$ as the input estimators in the MIE and EIE algorithms, respectively, we have $H_1 := \hat{H}_k$, $H_2 := \bar{H}_k$, $R_1 := S_k^{\#}$, and $R_2 := S_k = HFP_{k-1|k-1}F'H' + HQ_{k-1}H' + R_k$. For simplicity, we may consider only the steady state, where $P_{k-1|k-1} = P_{n|n}$. It appears from this analysis that the MIE input estimator is not uniformly superior to the EIE input estimator — their relative merit depends on the problem at hand.

It was claimed in [26] that the MIE algorithm "is better than the IMM method in performance and computation," which is supported by the simulation results presented in [25]. However, our own simulation results indicate that the IMM algorithm performs slightly better than the MIE algorithm in our scenarios, and with a better design than those used in [25], the IMM algorithm performs about the same as the MIE algorithm in their scenarios. Such simulation results depend highly on scenarios, design parameters, and many other factors. They are insufficient for arriving at an *overall* conclusion, which requires a fair performance comparison over an ensemble of realistic tracking scenarios (e.g., the benchmark tracking problems [83, 84] or some random scenarios).

An MIE algorithm using measurements in radar spherical coordinates has been presented in [26] for a more realistic 3D tracking problem. To cope with the problems arising from the use of mixed coordinates, the pseudomeasurement method in the radar LOS coordinate system (CS) was used (see Part III). This enables decoupling of the coordinate axes so that three 1D filters can be used, leading to considerable savings in computation. For this purpose the decoupled KF in LOS CS of [26] was employed to serve as a zero-input KF within the framework of the MIE algorithm. The implementation in the LOS CS was obtained after accounting for the respective coordinate transforms concerning the pseudoresidual and its covariance.

5.3.7 Generalized Input Estimation (GIE) Algorithm

While the EIE and MIE algorithms were developed aimed at overcoming the deficiencies of the original IE algorithm due to ignoring the uncertainty in the maneuver onset time (i.e., Assumption C), the generalized IE (GIE) algorithm, proposed recently in [27], is intended to relax the restrictive assumption concerning the evolution of the input (i.e., Assumption A). Here the unknown input is modeled as a linear combination of known basic time functions, defined over the detection window. This problem setting is more general than the constant-input model of the original IE algorithm. Due to the linearity of this input model, the original IE algorithm works in principle, except that the constant weights used in the linear combination need be determined, which can be easily estimated by essentially identical estimator of a constant input [27].

We now briefly describe some features of the algorithm and its development in a way that is significantly more elegant than the original description in [27].

The unknown input is assumed to have the form

$$\mathsf{u}_\kappa = \sum_{i=1}^{r} a_i \varphi_i(t_\kappa) = \varphi(t_\kappa)' a \tag{63}$$

where $\varphi(t) = [\varphi_1(t), \ldots, \varphi_r(t)]'$ is the vector of *known* (or to be designed) scalar functions of time and $a = [a_1' \ldots a_r']'$ denotes the stacked vector of *unknown constant* vector-valued weights to be determined. Then, we have $\mathbf{u}_k = [(\varphi(t_n)'a)', \ldots, (\varphi(t_{k-1})'a)']' = \Phi_k a$, where $\Phi_k = [\varphi(t_n), \ldots, \varphi(t_{k-1})]'$. It is thus clear that with this input model, estimation of a time-varying input $\mathbf{u}_k$ becomes estimation of the time-invariant weight vector a since Φ_k is assumed known. As such, without Assumption (A) this input model enables us to apply the results that rely on Assumption (A) by recognizing the relationship $\tilde{\mathbf{G}}_k \mathbf{u}_k = \tilde{\mathbf{G}}_k \Phi_k a = \bar{\mathbf{G}}_k a$ and $\mathbf{H}_k \mathbf{u}_k = \mathbf{H}_k \Phi_k a = \bar{\mathbf{H}}_k a$. More specifically, with Assumptions (B) and (C) and the assumption that a maneuver is detected at k, as implicitly made in [27], we have $\hat{a}_k = \Sigma_k e_k$, $\text{MSE}(\hat{a}_k) = \Sigma_k$ and thus $\hat{\mathbf{u}}_k = \Phi_k \hat{a}_k$, $\text{MSE}(\hat{\mathbf{u}}_k) = \Phi_k \Sigma_k \Phi_k'$, where Σ_k and e_k are given by (38)–(39) with the replacement $\bar{H}_i := H_i \mathbf{G}_i \Phi_k = H_i[L_{i,k-s}, L_{i,k-s+1}, \ldots, L_{i,k}]\Phi_k$.

Note, however, that the estimability of $\mathbf{u}_k$ and a differs in general unless $\varphi_i(t_\kappa)$ are chosen such that Φ_k is square and nonsingular. A necessary condition for estimability of a (i.e., invertibility of Σ_k) was given in [27]. In fact, it can be shown using results of [85] that a necessary and sufficient condition is that $\bar{\mathbf{H}}_k$ has full column rank (i.e., $\det(\bar{\mathbf{H}}_k' \bar{\mathbf{H}}_k) \neq 0$) and $\mathbf{S}_k$ as used in (28) is nonsingular.

The same maneuver detector and state estimate corrector as in the original IE algorithm were used in the GIE algorithm of [27] utilizing the relation $\hat{a}'_k\Sigma_k^{-1}\hat{a}_k = e'_k\Sigma_k e_k$. Note, however, that $\hat{\mathrm{u}}'_k \mathrm{MSE}(\hat{\mathbf{u}}_k)^{-1}\hat{\mathrm{u}}_k = \hat{a}'_k\Sigma_k^{-1}\hat{a}_k$ does not hold unconditionally.

Consequently, we have shown that the GIE algorithm for an unknown time-varying input modeled by (63) can be derived from the original IE algorithm for estimating an unknown but constant u in the following system:

$$x_{k+1} = F_k x_k + G_k u + \Gamma_k w_k$$

with a time-varying G_k by replacement $G_k := \Phi_k$ and $u := a$. In this sense, this "generalized" IE algorithm does not really generalize the original IE algorithm. It is in fact the original IE algorithm combined with input model (63). In other words, the real contribution of the so-called generalized IE algorithm is the introduction of the input model (63). The problem setting of input estimation with a time-varying matrix G_k is, however, more general than that of the GIE algorithm since the choice of G_k is not restricted to Φ_k by the input model (63). We emphasize this since in practice this means that the designer is allowed to choose G_k evolving in other manners, depending on the specific tracking application.

5.3.8 Multiple Input-Level Algorithm

In the multiple input-level method, proposed in [86], the set of all possible values of the unknown input u is partitioned into a number of quantization levels and each level i is represented by a single point (value) u^i in it. It then assumes that the unknown input can take on only one of these representative values at any time. As a result, it actually assumes that these representative values form a partition of the set of all possible values of the unknown input. By total expectation theorem, the state estimator is given by $\hat{x}_{k|k} = E[x_k|z^k] = \sum_i \hat{x}^i_{k|k} P\{u_k = u^i|z^k\}$, where $\hat{x}^i_{k|k} = E[x_k|z^k, u_k = u^i]$ is the optimal estimate assuming $u_k = u^i$. Each $\hat{x}^i_{k|k}$ can be found by a KF. Under the linear-Gaussian assumption, it is easy to verify that the state estimate is linear in the control input u:

$$\hat{x}^i_{k|k} = F_{k-1}\hat{x}^i_{k-1|k-1} + K_k(z_k - H_k F_{k-1}\hat{x}^i_{k-1|k-1}) + U_k G_{k-1} u^i$$

where $U_k = I - K_k H_k$. If the same previous estimate is used in each KF (i.e., $\hat{x}^i_{k-1|k-1} = \hat{x}_{k-1|k-1}, \forall i$), then

$$\hat{x}_{k|k} = F_{k-1}\hat{x}_{k-1|k-1} + K_k(z_k - H_k F_{k-1}\hat{x}_{k-1|k-1}) + U_k G_{k-1}\hat{u}_k \tag{64}$$

where $\hat{u}_k = \sum_i u^i P\{u_k = u^i|z^k\}$. As a result, *only a single KF, rather than a bank of KFs, is needed*. As such, the problem of optimal estimation with unknown input reduces to the problem of optimal estimation with known input, along with determining the probabilistic weights $P\{u_k = u^i|z^k\}$. Additional assumption is needed to determine the probabilistic weights. A reasonable one for some applications is that the input sequence $\langle u_k \rangle$ is a semi-Markov process (see Part I), or more specifically, a sojourn-time dependent Markov chain. Under this assumption, an efficient recursion for the calculation of the probabilistic weights can be readily obtained. Superior performance is obtained if it is combined with colored rather than white noise model [86] (see Part I).

Without the assumption that the same previous estimate is used in each KF, a bank of filter would be needed in general and thus this algorithm is in fact a degenerated case of a multiple-model algorithm.

5.3.9 Other IDE Algorithms and Implementations

We now briefly review other references concerning, or closely related to the IE algorithms. [87] proposed a simplified version of the original IE algorithm. [24] further developed the original IE algorithm in an information filter framework and proposed a highly efficient decoupled algorithm for radar target tracking of maneuvering aircraft (see also [88]). [45] (with a subsequent correction) proposed an MTT tracking algorithm which involves recursive estimation of the input and a maneuver detector minimizing detection delay. [28] applied the IE techniques for tracking a reentry vehicle in the terminal phase from radar measurements**.** The overall adaptive tracker, based on a nonlinear reentry vehicle dynamic model [2], consists of an EKF (zero-input based), a recursive version of the IE algorithm, a standard "$k\sigma$"-type maneuver detector, and a standard state estimate corrector. [89] proposed a maneuver detector for the IE algorithm based on a detectable maneuver set. [90] suggested an algorithm as a combination of a multiple input-level estimation and GLR-based maneuver detection. [91, 92] developed input detection and estimation for MTT based on FIR filters.

5.4 Comparison of GLR and IE Algorithms

It has been well known, as made explicit in [93], that there is a duality between maneuver detection (and identification) as well as fault detection and identification (FDI) and a duality between MTT and state estimation of systems subject to faults. As a result, there is a duality between the developments of algorithms for MTT (in particular, IE algorithms) and FDI (in particular, GLR algorithms).

The IE and GLR algorithms were developed largely in the MTT and FDI areas, respectively, which emphasize on estimation and detection, respectively. As a result, the IE and GLR algorithms are slightly more estimation-oriented and detection-oriented, respectively. This difference reflects well in the naming of the algorithms developed in the two areas in view of the fact that generalized likelihood ratio tests were well established in hypothesis testing long before. It is unfortunate that not enough cross connections were given by researchers in these two areas. Results developed originally in one area were often redeveloped in the other area, although some of such developments might have been inspired by results in the other area.

The input-estimation parts of GLR and IE are essentially the same since ML and LS estimators coincide under the linear-Gaussian assumption. They differ mainly in the detection part.

6 Switching-Model Approach

In this approach, there are two classes of models: maneuver (e.g., CA or coordinated turn) models and nonmaneuver (e.g., CV) models; tracking is done by a filter that uses one model at one time; the decision as which model to use is made in real time using measurement information (and prior information if any), hence the name **switching model** approach.

In a broad sense, algorithms in the equivalent-noise or IDE approaches also belong to this approach since following different decisions these algorithms take different actions, which may be construed as filtering based on different models. For instance, an upward noise-level adjustment may be viewed as switching from a model with a lower noise level to one with a higher level; state estimate correction or not using input estimates may also be interpreted as filtering based on different models, although this interpretation may appear farfetched to some people.

The switching-model approach has three aspects: *modeling*, *decision*, and *filtering*.

In principle, all target motion models, including those described in Part I, can be potentially used either as a maneuver or nonmaneuver model. As such, this approach is capable of providing valuable flexibility in actual implementation for vast, diversified applications. In reality, however, only a very small subset of those models described in Part I has been proposed or implemented with this approach. The best explanation here is probably the fact that in history more attractive approaches, such as multiple-model methods, had come into vogue before this approach could take off.

Ideally, different filters may be used for different models so as to take advantage of the specifics of each model. In practice, the variety of filters used for different models is very limited — almost all of them are KFs or EKFs. This stems from two facts: the popularity of the KF, along with its simple extensions such as EKF, and a lack of effective practical nonlinear filters. In this sense, this approach is better not referred to as "switching filter" approach.

We now describe several algorithms in this approach.

6.1 Variable State Dimension Algorithm

The *variable state dimension (*VSD*)* algorithm, proposed in [20], uses a CV model and a CA model for the target dynamics to achieve good tracking performance in both non-maneuvering and maneuvering situations. The CV and CA models have been adequately described in Part I. Switching from the CV model to the CA model is triggered by a detection of maneuver onset, and the CA model is switched back to the CV model when the maneuver is declared over. The maneuver detector proposed in [20] is based on the chi-square test using a fading-memory sum of measurement residuals (see Sec. 3.1). As proposed in [20], once a maneuver is detected at time k, it is assumed that the target had a constant acceleration (see Assumption (C) in Sec. 5.3 starting at $k-s-1$, where s is the effective window length, the CA model based filter is properly initialized at $k-s$, and all measurements after $k-s$ are reprocessed to obtain all components of the state vector. No such retrodiction was proposed in [20, 10] after switching back to the CV model from the CA model. In view of the fact that the maneuver onset and termination times are both uncertain, the main reason appears to be: While use of the CV model when acceleration is present will lead to unacceptable performance, use of the CA model in the absence of acceleration will degrade performance only slightly. This is a fortunate fact, for as explained before the detection of maneuver termination is more difficult than that of maneuver onset.

In hindsight, the VSD algorithm may be referred to as switching acceleration-model algorithm. It can also be viewed as an IDE algorithm where input (acceleration) is modeled as a random process with a (nearly) constant dynamics and estimated as state components.

It was reported in [20, 10] based on limited simulation results that the VSD algorithm outperforms the original IE algorithms for the scenarios considered.

6.2 Enhanced VSD Algorithm

The above original VSD method was modified in [94] to enhance performance by using (a) two-stage decision, (b) measurement concatenation, and (c) an improved initialization of the maneuver filter.

The **two-stage decision**, referred to as double decision logics in [94], consists of two decision logics connected in series. While one and only one filter is running at any particular time, the first decision logic is used to detect any abnormality in the measurement residuals to determine possible maneuver onset or termination by comparing a proper abnormality measure with a threshold. If a possible maneuver onset (or termination) is declared, then the maneuver (or nonmaneuver) filter is initialized and it runs in parallel with the other filter, and the second decision logic is then activated at some time. The second decision logic performs the so-called maximum likelihood test to eliminate the filter with a relatively smaller likelihood (i.e., poorer performance), although it would be better if a sequential probability ratio test were used.

The overall two-stage decision proposed is the same for maneuver onset and termination. The first stage, however, differs in the two situations. While detection of maneuver onset is based on the use of input estimate, a maneuver termination is detected using a fading-memory sum of the residuals from the maneuver filter. This input estimate based detection is chosen for two major reasons: it is more efficient than the fading-memory sum chi-square detector and it provides estimates of the detected acceleration, which serves to initialize the acceleration components of the maneuver filter more precisely than the retrodiction procedure of the original VSD algorithm. (Very similar ideas were followed in the more recent work on the VSD-IE algorithm in [95].) Once maneuver is confirmed, the maneuver filter is initialized using the input estimates by analogy with (65)–(67), discussed later.

In our opinion, the replacement of single-stage decision with two-stage decision is a significant innovation. It may achieve a simultaneous reduction in the rates of false switching and missed switching, at a price of a slightly longer decision time and a possibly higher level of sophistication. Essentially the same idea was introduced and implemented in [96, 97] for variable-structure multiple-model estimation. It appears that multiple-stage decision has an even better potential. In the limit, however, it becomes soft decision, which is hardly distinguishable from estimation, such as what the multiple-model approach does. Fuzzy-set enthusiasts would add that this is also where the merits of a fuzzy set lie relative to a crisp set. This may also inspire some readers to try fuzzy decision based techniques for MTT.

The measurement concatenation proposed is a model whereby fast sampled measurements are stacked while maintaining their proper relationships with the states. It provides a means of improving the tracking performance by increasing the measurement rate without increasing the state estimation update rate. It is the opposite of the sequential processing of vector measurements. Its application may be restricted by the allowable computational resource. It does have some subtleties, such as the presence of correlation between the concatenated measurement and process noise sequences.

6.3 Two-Stage Filtering

A KF based on a CA model can be implemented by a combination of two filters, one based on a CV model and the other for estimating the acceleration alone, even though acceleration is clearly coupled with position and velocity. Such a filter is known as a *two-stage filter* [79]. Compared with the direct implementation, it has reduced computation and memory requirements, among other things.

Since the VSD algorithms run both CV and CA filters, albeit at different times, a natural idea is to implement them as two-stage filters. This idea, suggested in [98], was pursued in [99, 100, 101, 102]. Their tracking filter consists of a primary (CV) filter and secondary (acceleration only) filter. The acceleration filter is turned on and off by a maneuver detector. It operates on the residual data of the CV filter. The output of the secondary (acceleration) filter is used — when maneuver is detected — to correct the output of the CV filter, in a way similar to state estimate correction of the IE algorithm. A principal difference however is modeling the acceleration (input) as a (white or colored) random process [99, 100, 101]. As an implementation of the so-called bias-filter, the acceleration filter estimates the acceleration using the measurement residuals of the bias-free (i.e., CV) filter. The overall estimate is based on the estimates of the CV and acceleration filters. The two-stage algorithm of [103] was proven therein to be equivalent to the CA filter under a restrictive algebraic sufficient condition, which would be rarely satisfied for practical systems [104]. Another two-stage estimator was proposed in [104] as an extension of the original estimator of [79]. It is optimal (equivalent to the augmented optimal estimator) in a general setting. Further, optimal two-stage KFs were obtained in [105]. These optimal algorithms can facilitate development of two-stage trackers because two-stage tracking has certain advantages over the VSD algorithms in terms of reduced computational load and faster response to manuever ending. For example, it may be integrated with the IDE approach: detect maneuver onset by a bias estimate based test, and use the optimal bias filter and optimal correction upon detection in an optimal manner. A number of publications exist for separate estimation of the bias (unknown input) for nonlinear dynamic systems [106, 107, 108]. They can also be applied to input detection and estimation.

An implementation of a two-stage tracker was proposed in [102] within the α-β filter framework. This implementation was also discussed in [7].

6.4 An Integrated VSD-IE Algorithm

The EIE algorithm of Sec. 5.3.5 uses a CA model for nonmaneuvering motion, which is known to cause performance degradation when the target moves at nearly constant velocity. To overcome this drawback, a variation was proposed in [95] that integrates three major techniques: (a) the input estimator (in a recursive form) of the IE algorithm, as practically given by (50)–(52), to estimate the input and detect maneuver using the CV filter, (b) the ML estimator of the maneuver onset time, together with the respective maneuver detector and state estimate corrector, and (c) the VSD technique of running a CA filter initialized by the estimated input with the estimated onset time. Simply put, this algorithm applies a recursive form of the EIE algorithm to obtain maneuver onset time estimate $\hat{n}$ and input estimate $\hat{u}_k(\hat{n})$ and to detect maneuver using the CV filter $\{\hat{x}^*, P^*\}$, and then if a maneuver is declared at time k, it switches to a CA filter $\{\hat{x}, P\}$, initialized by[3]

$$\begin{bmatrix} \hat{x}_{k|k-1} \\ \hat{\dot{x}}_{k|k-1} \end{bmatrix} = \begin{bmatrix} \hat{x}^*_{k|k-1} \\ \hat{\dot{x}}^*_{k|k-1} \end{bmatrix} + \hat{G}_{k|k-1}\hat{u}_k(\hat{n}) \tag{65}$$

$$\hat{\ddot{x}}_{k|k-1} = \hat{u}_k(\hat{n}) \tag{66}$$

$$P_{k|k-1} \approx \begin{bmatrix} P^*_{k|k-1} + \hat{G}'_k\hat{\Sigma}_k\hat{G}_k & \hat{G}_k\hat{\Sigma}_k \\ \hat{\Sigma}_k\hat{G}'_k & \hat{\Sigma}_k \end{bmatrix} \tag{67}$$

where $\hat{\Sigma}_k = \text{MSE}[\hat{u}_k(\hat{n})]$ and $\hat{G}_k = G_{k,n}|_{n=\hat{n}}$, given by (36). The unknown target acceleration is to be further estimated by the CA filter until the maneuver is declared over.

This integrated VSD-IE algorithm gives a better solution to the MTT problem than both the EIE algorithm and the VSD algorithm. Since an accurately initialized CA filter is used, its performance is superior to that of the IE algorithm during maneuver. On the other hand, its maneuver detection and onset time estimation appears to be more accurate than what is done in the VSD algorithm. Additionally, the simple yet accurate initialization of the CA filter eliminate the need for the awkward retrodictive filtering in the VSD algorithm that causes computational discontinuities.

6.5 Other Switching-Model Algorithms and Implementations

[109] proposed a switching-model scheme between nonmaneuvering and maneuvering models by a likelihood ratio detector. In [30], a switching-model algorithm, referred to as a decision directed tracker therein, was developed for tracking 2D maneuvering aircraft. It uses a nearly CV model for nonmaneuvering motion and the Singer model for maneuvers. The maneuver detection is based on a GLR test for detecting a maneuver-induced bias in the CV filter's residual sequence. In [110, 111] switching to a (nonlinear) turn model is triggered by a maneuver detection. Many other earlier implementations that belong to the switching-model approach can be found in [21] and references therein.

6.6 Model Switching vs. Input Detection and Estimation

While the input is treated as a deterministic process in the IDE approach, it is usually modeled as a random process in the switching model approach. Since no good prior dynamic model of the input u_k as a random process is available, it is customarily modeled as nearly constant: $u_{k+1} = u_k + w_k$. When the input responsible for the maneuver is acceleration, this becomes the popular CA model. In this case, all differences between the VSD algorithm and the IE algorithm stems from the fact that the unknown input (acceleration) is estimated as part of the state vector in the former, but as a separate nonrandom parameter in the latter.

[3] We point out that this initialization of the CA filter by the state estimate corrector in the input estimator actually fits well into the two-stage filter.

Maneuver onset time estimation is well integrated with input estimation in some IDE algorithms, which has not been found in the switching model algorithms. On the hand, almost all IDE algorithms developed so far are limited to the linear Gaussian case. Switching model approach does not have this limitation in general.

In view of the pros and cons of each approach, the following integration is worth exploring, as suggested in [72]. Maneuver detection is based on two-stage decision, where the first stage is based on a sophisticated change-point detection algorithm and the second stage is based on sequential probability ratio test between the two filters, one based on the model currently in effect and the other based on the newly activated model. The input and its onset time estimates are obtained as in the IDE approach using results from the nonmaneuver filter. These estimates are used only to initialize the two-stage maneuver filter in the same spirit as in the integrated VSD-IE algorithm in that the state estimate correction of the IDE approach is used.

7 Concluding Remarks

Many decision-based algorithms and techniques have been developed for maneuvering target tracking (MTT). They can be classified into three categories: equivalent noise, input detection and estimation, and switching model.

The equivalent-noise approach assumes that target maneuver can be covered by an equivalent noise and then MTT becomes state estimation in the presence of this unknown nonstationary noise. It appears that the only major attractive feature of this approach is its simplicity. While many techniques have been developed in the area of adaptive Kalman filtering, few of them have a good applicability in MTT mainly due to the highly nonstationary nature of the equivalent noise.

The input detection and estimation approach converts the MTT problem into that of state estimation with unknown input, generally considered as another area of adaptive Kalman filtering. It explicitly estimates the unknown input that is responsible for target maneuver. This approach is attractive in that it is basically free of maneuver models: It attempts to estimate the unknown target maneuver *directly* from the available measurement information without the need to have a maneuver model. A fundament dilemma here is: Potentially time-varying unknown input is better modeled as nonrandom, but little prior knowledge about its dynamics is available.

In the switching-model approach, the model in effect switches between a maneuver model and a nonmaneuver model. It is most general and flexible of all three approaches. In a broad sense, it includes the other two approaches as special cases. Its distinctive power for adaptation to the environment lies in its flexibility in choosing an appropriate model for the situation encountered. If needed, each model-based filter can be adaptive. However, it relies on good maneuver detection more crucially than the other two approaches.

Although the basic ideas of the equivalent-noise and input detection and estimation approaches work for nonlinear tracking problems, their algorithms developed so far are mostly restricted to linear systems and their simple nonlinear extensions. In contrast, the switching-model approach applies to nonlinear problems almost as easily as to linear problems.

Within the framework of decision-based adaptation for MTT, integration of different approaches appears to be most promising.

Advances in maneuver detection have been surprisingly limited, although it is the basis of all decision-based approaches. Essentially only two types of maneuver detectors have been developed: those based on the simple chi-square test (using either residuals or input estimates) and those based on the generalized likelihood ratio test. Considerably more effort should be directed towards searching and adapting effective techniques in the rich resources of statistics, in particular, change-point detection.

References

[1] X. R. Li and V. P. Jilkov. A Survey of Maneuvering Target Tracking: Dynamic Models. In *Proc. 2000 SPIE Conf. on Signal and Data Processing of Small Targets*, vol. 4048, pages 212–236, Orlando, Florida, USA, April 2000.

[2] X. R. Li and V. P. Jilkov. A Survey of Maneuvering Target Tracking—Part II: Ballistic Target Models. In *Proc. 2001 SPIE Conf. on Signal and Data Processing of Small Targets,* vol. 4473, pages 559–581, San Diego, CA, USA, 2001.

[3] X. R. Li and V. P. Jilkov. A Survey of Maneuvering Target Tracking—Part III: Measurement Models. In *Proc. 2001 SPIE Conf. on Signal and Data Processing of Small Targets,* vol. 4473, pages 423–446, San Diego, CA, USA, 2001.

[4] A. Farina and F. A. Studer. *Radar Data Processing, vol. I: Introduction and Tracking, vol. II: Advanced Topics and Applications*. Research Studies Press, Letchworth, Hertfordshire, England, 1985.

[5] S. S. Blackman. *Multiple Target Tracking with Radar Applications*. Artech House, Norwood, MA, 1986.

[6] Y. Bar-Shalom and T. E. Fortmann. *Tracking and Data Association*. Academic Press, New York, 1988.

[7] S. S. Blackman and R. F. Popoli. *Design and Analysis of Modern Tracking Systems*. Artech House, Norwood, MA, 1999.

[8] P. L. Bogler. *Radar Principles with Applications to Tracking Systems*. Wiley, 1990.

[9] Y. Bar-Shalom and X. R. Li. *Estimation and Tracking: Principles, Techniques, and Software*. Artech House, Boston, MA, 1993. (Reprinted by YBS Publishing, 1998).

[10] Y. Bar-Shalom, X. R. Li, and T. Kirubarajan. *Estimation with Applications to Tracking and Navigation: Theory, Algorithms, and Software*. Wiley, New York, 2001.

[11] A. N. Shiryaev. *Optimal Stopping Rules*. Springer-Verlag, New York, 1977.

[12] N. Kligene and L. Telksnys. The Methods for Detecting Instants of Change in Random Process Properties. *Automation and Remote Control*, pages 1241–1316, 1983. Translation from Russian.

[13] M. Basseville and A. Benveniste. Design and Comparative Study of Some Sequential Jump Detection Algorithms for Digital Signals. *IEEE Trans. on ASSP*, 31(3):521 – 535, June 1983.

[14] M. Basseville and A. Benveniste, editors. *Detection of Abrupt Changes in Signals and Dynamical Systems*. Springer-Verlag, 1986.

[15] M. Basseville. Detecting Changes in Signals and Systems. *Automatica*, 24(3):309–326, May 1988.

[16] L. Telksnys, editor. *Detection of Changes in Random Processes*. Optimization Software, Inc., New York, 1986.

[17] M. Basseville and I. Nikiforov. *Detection of Abrupt Changes: Theory and Application*. Prentice Hall, Englewood Cliffs, NJ, 1993.

[18] T. L. Lai. Sequential Changepoint Detection in Quality Control and Dynamical Systems. *J. R. Statist. Soc.*, 57(4):613–658, 1995.

[19] F. Gustafsson. *Adaptive Filtering and Change Detection*. Wiley, 2001.

[20] Y. Bar-Shalom and K. Birmiwal. Variable Dimension Filter for Maneuvering Target Tracking. *IEEE Trans. Aerospace and Electronic Systems*, AES-18(5):621–629, Sept. 1982.

[21] M. S. Woolfson. An Evaluation of Manoeuvre Detector Algorithms. *GEC J. of Research (Chelmsford, England)*, 3(3):181–190, 1985.

[22] J. R. Cloutier, C. F. Lin, and C. Yang. Maneuvering Target Tracking via Smoothing and Filtering Through Measurement Concatenationn. *AIAA Journal of Guidance, Control, and Dynamics*, 16(2):377–384, March-Apr. 1993.

[23] Y. T. Chan, A. G. C. Hu, and J. B. Plant. A Kalman Filter Based Tracking Scheme with Input Estimation. *IEEE Trans. Aerospace and Electronic Systems*, AES-15(2):237–244, Mar. 1979.

[24] M. Farooq and S. Bruder. Information Type Filters for Tracking a Maneuvering Target. *IEEE Trans. Aerospace and Electronic Systems*, 26(3):441–454, 1990.

[25] I. H. Whang, J. Lee, and T. Sung. Modified Input Estimation Technique Using Pseudoresiduals. *IEEE Trans. Aerospace and Electronic Systems*, 30(1):220–228, 1994. Also published in No. 2, pp. 591-598.

[26] T. K. Sung and J. G. Lee. A Decoupled Adaptive Tracking Filter for Real Applications. *IEEE Trans. Aerospace and Electronic Systems*, 33(3):1025–1030, 1997.

[27] H. Lee and M.-J. Tahk. Generalized input-estimation technique for tracking maneuvering targets. *IEEE Transactions on Aerospace and Electronic Systems*, 35(4):1388–1402, 1999.

[28] S.-C. Lee and C.-Y. Liu. Trajectory Estimation of Reentry Vehicle by Use of on-Line Input Estimator. *Journal of Guidance, Control, and Dynamics*, 22(6):808–, 1999.

[29] A. S. Willsky. A Survey of Design Methods for Failure Detection in Dynamic Systems. *Automatica*, 12(6):601–611, Nov. 1976.

[30] R. J. McAulay and E. J. Denlinger. A Decision-Directed Adaptive Tracker. *IEEE Trans. Aerospace and Electronic Systems*, AES-9(2):229–236, Mar. 1973.

[31] P. L. Bogler. Tracking a Maneuvering Target Using Input Estimation. *IEEE Trans. Aerospace and Electronic Systems*, AES-23(3):298–310, May 1987.

[32] T. H. Kerr. Decentralized Filtering and Redundancy Management in Multisensor Navigation. *IEEE Trans. Aerospace and Electronic Systems*, AES-23:83–119, 1987.

[33] F. Gustafsson. The Marginalized Likelihood Test for Detecting Abrupt Changes. *IEEE Trans. Automatic Control*, AC-41:66–78, Jan. 1996.

[34] E. Bekir. Adaptive Kalman Filter for Tracking Maneuvering Targets. *AIAA Journal of Guidance*, 6(5):414–416, Sept.-Oct. 1983.

[35] E. S. Page. Continuous Inspection Schemes. *Biometrika*, 41:100–115, 1954.

[36] D. V. Hinkley. Inference about the Change-Point in a Sequence of Random Variables. *Biometrika*, 57:1–17, 1970.

[37] Y. T. Chan and F. Couture. Manoeuvre Detection and Track Correction by Input Estimation. *IEE Proc.-F*, 140(1):21–28, Feb. 1993.

[38] A. Wald. *Sequential Analysis*. Wiley, New York, 1947.

[39] J. O. Berger. *Statistical Decision Theory and Bayesian Analysis*. Springer-Verlag, Mew York, 1985.

[40] P. M. Newbold and H. C. Yu. Detection of Changes in the Characteristics of a Gauss-Markov Process. *IEEE Trans. Aerospace and Electronic Systems*, AES-4(5):707–718, 1968.

[41] D. Bertzekas. *Dynamic Programing and Stochastic Control*. Mathematics in Science and Engineering. Academic Press, New York, 1976.

[42] J. L. Speyer and J. E. White. Shiryaev Sequential Probability Ratio Test for Redundancy Management. *AIAA Journal of Guidance, Control, and Dynamics*, 7(5):588–595, 1984.

[43] I. V. Nikiforov. A Generalized Change Detection Problem. *IEEE. Trans. Information Theory*, IT-41(1):171–187, 1995.

[44] D. P. Malladi and J. L. Speyer. A Generalized Shiryaev Sequential Probability Ratio Test for Change Detection and Isolation. *IEEE Trans. Automatic Control*, AC-44(8):1522–1534, 1999.

[45] T. C. Wang and P. K. Varshney. A Tracking Algorithm for Maneuvering Targets. *IEEE Trans. Aerospace and Electronic Systems*, AES-29(3):910–924, July 1993.

[46] A. H. Jazwinski. *Stochastic Processes and Filtering Theory*. Academic Press, New York, 1970.

[47] R. K. Mehra. On the Identification of Variances and Adaptive Kalman Filtering. *IEEE Trans. Automatic Control*, AC-15(2):175–184, Apr. 1970.

[48] R. K. Mehra. Approaches to Adaptive Filtering. *IEEE Trans. Automatic Control*, AC-17(5):693–698, Oct. 1972.

[49] R. L. T. Hampton and J. R. Cooke. Unsupervised Tracking of Maneuvering Vehicles. *IEEE Trans. Aerospace and Electronic Systems*, AES-9:197–207, Mar. 1973.

[50] G. Chen and C. K. Chui. A Modified Adaptive Kalman Filter for Real-Time Applications. *IEEE Trans. Aerospace and Electronic Systems*, AES-27:149–154, Jan. 1991.

[51] L. Chin. Advances in Adaptive Filtering. In C. T. Leondes, editor, *Control and Dynamic Systems: Advances in Theory and Applications*, vol. 15. Academic Press, New York, 1979.

[52] A. Moghaddamjoo. Approaches to Adaptive Kalman Filtering. *Control-Theory and Advanced Technology*, 5(1):1–18, Mar. 1989.

[53] A. P. Sage and G. W. Husa. Algorithms for Sequential Adaptive Estimation of Prior Statistics. In *Proc. 1969 IEEE Symp. Adaptive Processes*, University Park, Nov. 1969. Penn. State Univ.

[54] P. L. Smith. Estimation of the Covariance Parameters of Nonstationary Time-Discrete Linear Systems. In *Proc. 2nd Symp. Nonlinear Estimation Theory and Its Applications*, pages 325–328, San Diego, CA, Sept. 1971.

[55] P. R. Belanger. Estimation of Noise Covariance Matrices for a Linear Time-Varying Stochastic Process. *Automatica*, 10:267–275, 1974.

[56] A. H. Jazwinski. Adaptive Filtering. *Automatica*, 5(4):475–485, July 1969.

[57] X. R. Li and Y. Bar-Shalom. A Recursive Hybrid System Approach to Noise Identification. In *Proc. 1st IEEE Conf. Control Applications*, pages 847–852, Dayton, OH, Sept. 1992.

[58] X. R. Li and Y. Bar-Shalom. A Recursive Multiple Model Approach to Noise Identification. *IEEE Trans. Aerospace and Electronic Systems*, AES-30(3):671–684, July 1994.

[59] C. B. Chang and J. Tabaczynski. Application of State Estimation to Target Tracking. *IEEE Trans. Automatic Control*, AC-29(2):98–109, Feb. 1984.

[60] R. P. Wishner, R. E. Larson, and M. Athans. Status of Radar Tracking Algorithms. In *Proc. Symp. Nonlinear Estimation*, San Diego, CA, Sept. 1970.

[61] C. B. Chang, R. H. Whiting, and M. Athans. On the State and Parameter Estimation for Maneuvering Reentry Vehicles. *IEEE Trans. Automatic Control*, AC-22(2):99–105, Feb. 1977.

[62] K. Spingarn and H. L. Weidemann. Linear Regression Filtering and Prediction for Tracking Maneuvering Aircraft Targets. *IEEE Trans. Aerospace and Electronic Systems*, AES-8:800–810, Nov. 1972.

[63] F. R. Castella. An Adaptive Two-Dimensional Kalman Tracking Filter. *IEEE Trans. Aerospace and Electronic Systems*, AES-16:822–829, Nov. 1980.

[64] K. A. Myers and B. D. Tapley. Adaptive Sequential Estimation with Unknown Noise Statistics. *IEEE Trans. Automatic Control*, AC-21:520–523, Aug. 1976.

[65] D. G. Hull, J. L. Speyer, and W. M. Greenwell. Adaptive Noise Estimation for Homing Missiles. *AIAA J. Guidance, Control, and Dynamics*, 7(3):322–328, May-June 1984.

[66] P. O. Gutman and M. Vegler. Tracking Targets Using Adaptive Kalman Filtering. *IEEE Trans. Aerospace and Electronic Systems*, AES-26:691–699, Sept. 1990.

[67] M. E . Hough. Improved Performance of Recursive Tracking Filters Using Batch Initialization and Process Noise Adaptation. *AIAA Journal of Guidance, Control, and Dynamics*, 22(5):675–681, 1999.

[68] R. L. Moose, M. K. Sistanizadeh, and G. Skagejord. Adaptive Estimation for a System with Unknown Measurement Bias. *IEEE Trans. Aerospace and Electronic Systems*, AES-22(6):732–739, Nov. 1986.

[69] M. Niedzwiecki. Identification of Nonstationary Stochastic Systems Using Parallel Estimation Schemes. In *Proc. 27th Conf. Decision and Control*, pages 258–263, Austin, TX, Dec. 1988.

[70] M. Niedzwiecki. Multiple-Model Approach to Finite Memory Adaptive Filtering. In *Preprints of 11th IFAC World Congress*, vol. 3, pages 154–159, Tallinn, USSR, Aug. 1990.

[71] G. P. Cardillo, A. V. Mrstik, and T. Plambeck. A Track Filter for Reentry Objects with Uncertain Drag. *IEEE Trans. Aerospace and Electronic Systems*, AES-35(2):395–409, Apr. 1999.

[72] X. R. Li, V. P. Jilkov, and P. Zhang. Input Detection and Estimation for Maneuvering Target Tracking. Submitted for publication, 2002.

[73] X. R. Li, Y. M. Zhu, J. Wang, and C. Z. Han. Optimal Linear Estimation Fusion—Part I: Unified Fusion Rules. *Submitted for journal publication*, May 2002.

[74] A. S. Willsky and H. L. Jones. A Generalized Likelihood Ratio Approach to the Estimation in Linear Systems Subject to Abrupt Changes. In *Proc. 1974 IEEE Conf. On Decision and Control*, Nov. 1974.

[75] A. S. Willsky and H. L. Jones. A Generalized Likelihood Ratio Approach to the Detection and Estimation of Jumps in Linear Systems. *IEEE Trans. Automatic Control*, AC-21:108–112, Feb. 1976.

[76] J. Korn, S. W. Gully, and A. S. Willsky. Application of the Generalized Likelihood Ratio Algorithm to Maneuver Detection and Estimation. In *Proc. 1982 American Control Conf.*, Arlington, VA, June 1982.

[77] K. Schnepper. A Comparison of GLR and Multiple Model Filters for a Target Tracking Problem. In *Proceedings of the 25 Conference on Decision and Control*, pages 666–670, Athens, Greece, December 1986.

[78] C. B. Chang and K.P. Dunn. On The GLR Detection and Estimation of Unexpected Inputs in Linear Systems. *IEEE Trans. Automatic Control*, AC-24(3):499–501, 1979.

[79] B. Friedland. Treatment of Bias in Recursive Filtering. *IEEE Trans. Automatic Control*, AC-14:359–367, Aug. 1969.

[80] Y. Bar-Shalom, K. C. Chang, and H. A. P. Blom. Tracking a Maneuvering Target Using Input Estimation Versus the Interacting Multiple Model Algorithm. *IEEE Trans. Aerospace and Electronic Systems*, AES-25(2):296–300, Apr. 1989.

[81] M. Farooq and S. Bruder. Comments on ”Tracking a Maneuvering Target Using Input Estimation. *IEEE Trans. Aerospace and Electronic Systems*, 25(2):300–302, 1989.

[82] J.-A. Guu and C.-H. Wei. Tracking a Maneuvering Target Using Input Estimation at High Measurement Frequency. *International Journal of System Science*, pages 871–883, 1992.

[83] W. D. Blair, G. A. Watson, and S. A. Hoffman. Benchmark Problem for Beam Pointing Control of Phased Array Radar Against Maneuvering Target. In *Proc. 1994 American Control Conf.*, pages 2071–2075, Baltimore, MD, June 1994.

[84] W. D. Blair, G. A. Watson, T. Kirubarajan, and Y. Bar-Shalom. Benchmark for Radar Resource Allocation and Tracking Targets in the Presence of ECM. *IEEE Trans. Aerospace and Electronic Systems*, AES-34(4):1097–1114, Oct. 1998.

[85] X. R. Li, K. S. Zhang, J. Zhao, and Y. M. Zhu. Optimal Linear Estimation Fusion—Part V: Relationships. In *Proc. 2002 International Conf. on Information Fusion*, Annapolis, MD, USA, Aug. 2001.

[86] R. L. Moose. An Adaptive State Estimator Solution to the Maneuvering Target Problem. *IEEE Trans. Automatic Control*, AC-20(3):359–362, June 1975.

[87] Y. T. Chan, J. B. Plant, and J. Bottomley. A Kalman tracker with a simple input estimator. *IEEE Transactions on Aerospace and Electronic Systems*, AES-18(2):235–241, 1982.

[88] M. Farooq, S. Bruder, T. Quach, and S.S. Lim. Adaptive Filtering Techniques for Maneuvering Targets. In *Proceedings of the 34th Midwest Symposium on Circuits and Systems*, pages 31–34, Monterey, CA, USA, 1992.

[89] H. Lee and M. Tahk. A Detection Scheme of Input Estimation Filter. In *Proc. KACC'95*, Seoul, Korea, Oct. 1995.

[90] S. J. Shin and T. L. Song. Input Estimation with Multiple Model for Maneuvering Target Tracking. Submitted.

[91] O.-K. Kwon and K.-S. Yoo. Manoeuvre Detection and Target Tracking in Uncertain Systems Using FIR Filters. In *Proc. 31st IEEE Conf. on Decision and Control*, pages 64–65, Tucson AZ, Dec. 1992.

[92] S. H. Park, W. H. Kwon, O.-K. Kwon, and P. S. Kim. Maneuver Detection and Target Tracking Using State-Space Optimal FIR Filters. In *Proc. 1999 American Control Conf.*, page 4253, San Diego, CA, June 1999. Also Tech.Rep. No. SNU-EE-TR-1999-8, Seoul National University. Available online: http//haedong.snu.ac.kr.

[93] T. H. Kerr. Duality Between Failure Detection and Radar/Optical Maneuver Detection. *IEEE Trans. Aerospace and Electronic Systems*, AES-25:520–528, July 1989.

[94] J. R. Cloutier, C. F. Lin, and C. Yang. Enhanced Variable Dimension Filter for Maneuvering Target Tracking. *IEEE Trans. Aerospace and Electronic Systems*, 29(3):786–797, July 1993.

[95] Y. H. Park, , J. H. Seo, and J. G. Lee. Tracking Using Variable-Dimension Filter With Input Estimation. *IEEE Trans. Aerospace and Electronic Systems*, AES-31(1):399–408, Jan. 1995.

[96] X. R. Li, X. R. Zhi, and Y. M. Zhang. Multiple-Model Estimation with Variable Structure—Part III: Model-Group Switching Algorithm. *IEEE Trans. Aerospace and Electronic Systems*, AES-35(1):225–241, Jan. 1999.

[97] X. R. Li, Y. M. Zhang, and X. R. Zhi. Multiple-Model Estimation with Variable Structure—Part IV: Design and Evaluation of Model-Group Switching Algorithm. *IEEE Trans. Aerospace and Electronic Systems*, AES-35(1):242–254, Jan. 1999.

[98] T.E. Bullock and S. Sangsuk-Iam. Maneuver Detection and Tracking with a Nonlinear Target Model. In *Proc. 23 IEEE Conf. Decision and Control*, Las Vegas, NV, Dec. 1984.

[99] A. T. Alouani, P. Xia, T. R. Rice, and W. D. Blair. A Two-Stage Kalman Estimator for State Estimation in the Presence of Random Bias and for Tracking Maneuvering Targets. In *Proc. 30th IEEE Conf. on Decision and Control*, pages 2059–2062, Brighton, England, Dec. 1991.

[100] A. T. Alouani, P. Xia, T. R. Rice, and W. D. Blair. A Two-Stage Kalman Estimator for Tracking Maneuvering Targets. In *Proc. of IEEE 1991 International Conference on Systems, Man, and Cybernetics*, pages 761–766, Sharlottsville, VA, Oct. 1991.

[101] A. T. Alouani, T. R. Rice, and W. D. Blair. Two-Stage Filter for Estimation in the Presense of Stochastical Bias. In *Proc. 31th IEEE Conf. on Decision and Control*, Chicago, IL, June 1992.

[102] W. D. Blair. Fixed-Gain Two-Stage Estimators for Tracking Maneuvering Targets. *IEEE Trans. Aerospace and Electronic Systems*, AES-29(3):1004–1014, July 1993.

[103] A. T. Alouani and W. D. Blair. On the Optimality of Two-Stage State Estimation in the Presence of Random Bias. *IEEE Trans. Automatic Control*, AC-38(8):1279–1282, Aug. 1993.

[104] C.-S. Hsieh and F.-C. Chen. Optimal Solution of the Two-Stage Kalman Estimator. *IEEE Trans. Automatic Control*, AC-44(1):195–199, Jan. 1999.

[105] C.-S. Hsieh and F.-C. Chen. General Two-Sage Kalman Filters. *IEEE Trans. Automatic Control*, AC-45(4):819–824, Jan. 2000.

[106] J. M. Mendel. Extension of Friedland's Bias Filter Technique to a Class of Nonlinear Systems. *IEEE Trans. Automatic Control*, AC-21:296–299, Apr. 1976.

[107] A. K. Caglayan and R. E. Lancraft. A Separated Bias Identification and State Estimation Algorithm for Nonlinear Systems. *Automatic*, 19(5), 1983.

[108] D. H. Zhou, Y. X. Sun, Y. X. Xi, and Z. J. Zhang. Extension of Friedland's Separate- Bias Estimation to Randomly-Varying Bias of Nonlinear Systems. *IEEE Trans. Automatic Control*, AC-38(8):1270–1273, Aug. 1993.

[109] J. S. Thorp. Optimal Tracking of Maneuvering Targets. *IEEE Trans. Aerospace and Electronic Systems*, AES-9:512–519, July 1973.

[110] J. A. Roecker and C. D. McGillem. Target Tracking in Maneuver Centered Coordinates. *IEEE Trans. Aerospace and Electronic Systems*, AES-25:836–843, Nov. 1989.

[111] T. Kawase, H. Tsurunosono, N. Ehara, and I. Sasase. Two-Stage Kalman Estimator Using an Advanced Circular Prediction for Tracking Highly Maneuvering Targets. In *Proceedings of ICASSP98*, 1998. Submitted to IEEE Trans. Aerospace and Electronic Systems.