Proceedings of the 5th International Conf. Information Fusion
Annapolis, MD, USA, July 2002, pp. 1347–1354

Model-Set Design for Multiple-Model Method Part II: Examples *

X. Rong Li Zhanlue Zhao Peng Zhang Chen He
Department of Electrical Engineering, University of New Orleans
New Orleans, LA 70148, USA
Phone: 504-280-7416, Fax: 504-280-3950, xli@uno.edu

Abstract – *This paper presents several examples that illustrate how the theoretical results presented in Part I [9] and several other places can be applied, along with a demonstration of their effectiveness, in the context of MM estimation for Air Traffic Control surveillance. Model-set design and choice examples are presented. The importance and usefulness of modeling true mode and models as random variables are demonstrated. How such a probabilistic model can be constructed is also demonstrated.*

Keywords: Multiple models, model-set design, variable structure, adaptive estimation, target tracking, air traffic control

1 Introduction

Model-set design is the most important issue in the *application* of multiple-model algorithms. Many publications have appeared reporting various effective application-specific designs. However, very little progress has been made so far in the *theory* of model-set design (see, e.g., [4, 3, 13, 5, 11]). In particular, there is a lack of generally applicable and systematic methodologies for model-set design. Part I [9] presents theoretical results for model-set design in a general setting, including several general design methods. The generality of these results makes them applicable to many practical situations. As usual, however, this generality is achieved at the price of being abstract. This may hamper their application.

Model-set design is closely related with some other issues (e.g., model-set choice, adaptation, and comparison) for which some theoretical results are available, such as those presented in [11, 8, 6, 7].

In this paper, we present several examples that demonstrate how the theoretical results of Part I and previously publication theoretical results can be used in model-set design and choice. A key that underlies these application example is the probabilistic modeling of the true mode and models, as proposed and discussed in Part I.

Most examples presented here deal with the following simple system with position-only measurements

$$x_{k+1} = Fx_k + w_k, \;\; z_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_k + v_k \quad (1)$$

where $x = [x_1, \dot{x}_1, x_2, \dot{x}_2]'$, and process and measurement noises have constant means $\bar{w}$, $\bar{v}$ and covariances Q, R, respectively. Assume that the linear-Gaussian assumption of the Kalman filter is valid. Two generic types of model are considered: nearly constant-velocity (CV) model and coordinated-turn (CT) (with a known turn rate) model, given by [10]

$$F_{\text{CV}} = \text{diag}[F_2, F_2],\; F_2 = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \quad (2)$$

$$F_{\text{CT}}(\omega) = \begin{bmatrix} 1 & \dfrac{\sin\omega T}{\omega} & 0 & -\dfrac{1-\cos\omega T}{\omega} \\ 0 & \cos\omega T & 0 & -\sin\omega T \\ 0 & \dfrac{1-\cos\omega T}{\omega} & 1 & \dfrac{\sin\omega T}{\omega} \\ 0 & \sin\omega T & 0 & \cos\omega T \end{bmatrix} \quad (3)$$

where $T = 5$s is sampling period. Denote by CT(3°/s) a CT model with turn rate $\omega = 3°$/s. Note that CV = CT(0°/s).

All numerical examples presented in this paper are based on the use of the above CT and CV models in one or more sets. These models come primarily from the consideration in maneuvering target tracking for Air Traffic Control (ATC) surveillance [2, 1].

2 Model-Set Design

2.1 The Design Problem

Consider a problem of surveillance for an ATC system. Suppose we decide to use an MM algorithm with a set M of three CT models: $m_1 = \omega_1, m_2 = \omega_2, m_3 = \omega_3$. Clearly, to design this set M optimally in some sense, a probability

* Research supported in part by ONR grant N00014-00-1-0677, NSF grant ECS-9734285, and NASA/LEQSF grant (2001-4)-01.

density function (pdf) $f(s)$ of the true turn rate s is needed. We propose the following Gaussian-mixture model:

$$f(s) = c_0 N_0(s) + cN_1(s) + cN_{-1}(s), \; c_0 + 2c = 1 \quad (4)$$

where

$$N_0(s) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{s^2}{2\sigma_0^2}}, \; N_{\pm 1}(s) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(s \pm \omega_s)^2}{2\sigma^2}} \quad (5)$$

and ω_s is a known standard turn rate. This model can be well justified if most flights will either approach/departure the runway directly without a turn, or with a turn of an extremely slow rate or a rate close to the standard one ($\pm\omega_s$). Note that $f(s)$ may or may not have three peaks, depending on the parameters ω_s, c_0, c, σ_0, and σ.

Consider a specific example with

$$c_0 = c = 1/3, \quad \sigma_0 = \sigma = 1°/\text{s}, \quad \omega_s = 3°/\text{s} \quad (6)$$

Its pdf is plotted in Fig. 1.

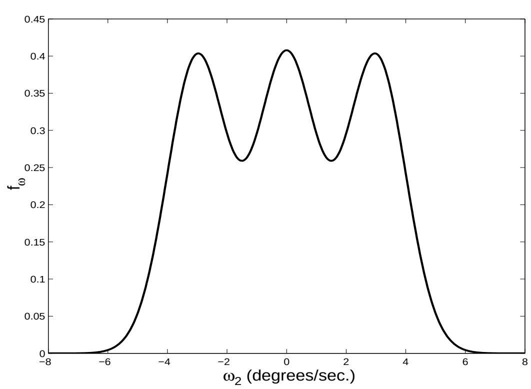


Fig. 1: The probability density function of the true turn rate ω.

2.2 Design by Minimizing Distribution Mismatch

Clearly, the distribution-based approach of Part I can be applied to solve this problem for the specific example with (6). Assume that we want to use only three models. First, plot the cdf $F_s(x)$ of the true mode (its pdf is plotted in Fig. 1) as in Fig. 2. Then, since $F_s(-3°/\text{s}) \approx 1/6$, $F_s(0) = 3/6$, and $F_s(3°/\text{s}) \approx 5/6$, the simple distribution-based design yields the model set $M = \{\omega_1, \omega_2, \omega_3\} = \{0, \pm 3°/\text{s}\}$, with the corresponding initial model probabilities $\{1/3, 1/3, 1/3\}$. Note that it turns out that we have a model at each peak of the pdf $f_s(x)$. This would not be the case if the three peaks are closer or σ is larger. Clearly, this approach is more beneficial when we want to have more models.

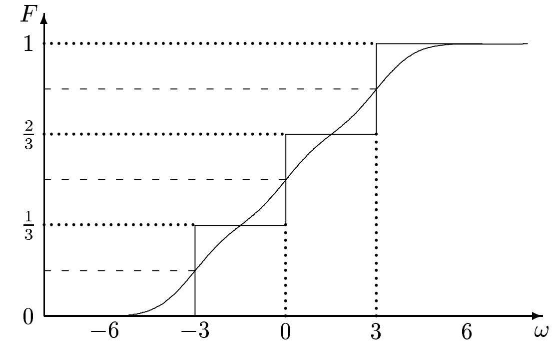


Fig. 2: Model-set design via cdf of turn rate ω.

2.3 Design Procedure for Multivariate Case

The above example is simple because the true mode is one-dimensional. We now describe a design procedure for 2D case, which uses a "minimal" number of models given any tolerance on mismatch between the cdfs of the mode and (random) model. It can be easily extended to higher dimensions.

Consider the cdf of a 2D mode s: $F_s(x, y) = F(x, y)$. As explained in Part I, design of a model set along with the initial model probabilities (i.e., model weights) amounts to constructing a random variable m (i.e., a random model) with a certain cdf $F_m(x, y)$. Our goal is to determine locations of a "minimal" number of models along with probability weights such that the resultant cdf $F_m(x, y)$ satisfies the requirement $\max_{x,y} |F_s(x, y) - F_m(x, y)| \leq \epsilon$.

Let $D(x, y) = F_s(x, y) - F_m(x, y)$ be the difference in cdf. Assume for simplicity that $F(x, y)$ is continuous. In Fig. 1, the origin and the upper right corner stand for $(-\infty, -\infty)$ and (∞, ∞), respectively, at which $F(-\infty, -\infty) = 0$ and $F(\infty, \infty) = 1$. Note that $F(x, y)$ is monotonically increasing.

The procedure consists of three steps, as illustrated in Fig. 3.

First, determine the equal-height lines $l_{1-\epsilon}$, $l_{1-3\epsilon}$, $\ldots$, $l_{1-(2N-1)\epsilon}$, l_ϵ, where N is an integer such that $\epsilon < 1 - (2N-1)\epsilon \leq 3\epsilon$. This means that $F(x_i, y_i) = 1 - i\epsilon$ for any point (x_i, y_i) on the line $l_{1-i\epsilon}$.

Second, determine the points $A_1, A_2, \ldots, A_N$. The location of A_1 is (x_1, y_1). It minimizes $|F(x_1, \infty) - F(\infty, y_1)|$ among all points on $l_{1-\epsilon}$. The point A_1 determines two reference lines for the next point A_2. Its location is (x_2, y_2), which minimizes $|F(x_2, y_1) - F(x_1, y_2)|$ among all points on $l_{1-2\epsilon}$. A_i is determined likewise.

Third, determine the model locations $m_1, m_2, \ldots, m_m$. We place models on the horizontal and vertical lines de-

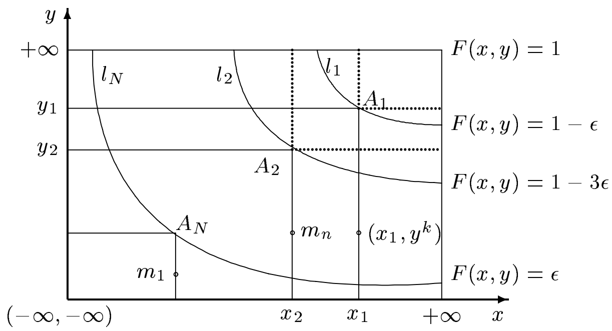


Fig. 3: Illustration of model-set design by approximating cdf.

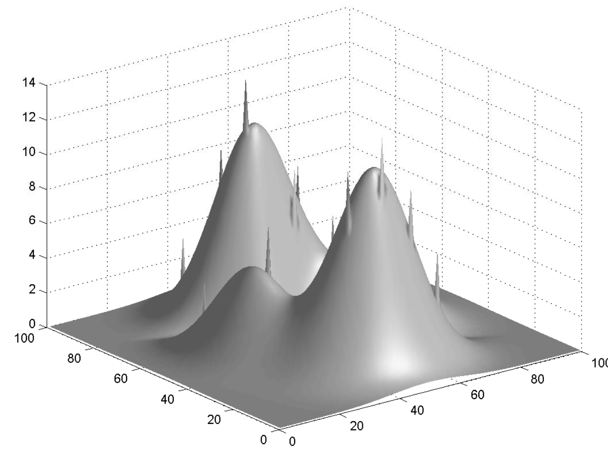


Fig. 4: The true pdf and designed model locations.

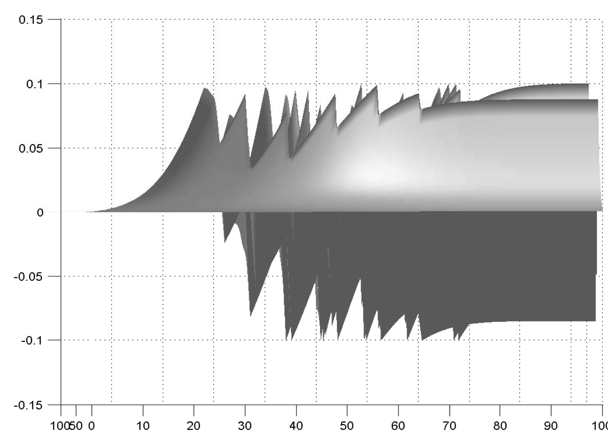


Fig. 5: $D(x, y)$—the difference in cdf.

termined by points $A_1, A_2, \ldots, A_N$. For the line x_2-A_2 it uses the next line x_1-A_1 as a reference. Note that $F(x_1, y)$ is monotonically increasing on the line x_1-A_1. If a point (x_1, y^k) is the lowest point such that $D(x_1, y^k) > \epsilon$, then choose (x_2, y^k) as a model location. This process is done from left to right (i.e., for $x_N, \ldots, x_1$) and from bottom up (i.e., for $y^1, y^2, \ldots$).

The weight of each model is determined at the same time the model location is determined. The weight is "how high" a jump is needed at each model location. The upper bound on the height of a jump of a model at (x_i, y^k) is determined by the difference $D(x, y)$ along the line x_{i-1}-A_{i-1} at or above y^k.

The model locations and weights on the horizontal line y_i-A_i are determined in exactly the same way.

Fig. 4 shows an example of the true pdf and the model locations designed, depicted by the sharp peaks. In the design, the tolerance $\epsilon = 0.1$ was chosen. The resultant model locations concentrate around the major peaks of the true density. Fig. 5 shows the error $D(x, y)$. It is bounded by $\epsilon = 0.1$, as required.

2.4 Design by Minimizing Modal Distance

We now demonstrate how to design M by minimizing an optimality criterion for mode estimation. Consider minimizing the average modal distance squared, dropping $s \in \mathbf{S}$ for simplicity,

$$\begin{aligned} & \|s-m\|^2_{m\in M} = E[(s-\omega)^2\,|m \in M] \\ = \quad & E[E[(s-\omega)^2\,|s, m\in M]|m\in M] \\ = \quad & \int_{-\infty}^{\infty} \sum_i (s-\omega_i)^2\, P\{m=\omega_i|m\in M, s\}\, f(s)ds \end{aligned}$$

By symmetry of $f(s)$, $\omega_1 = 0$, $\omega_2 = \omega$, $\omega_3 = -\omega$ and we need only to determine the optimal ω. Thus, we need only to solve the following optimization problem

$$\begin{aligned} \min_\omega J \quad = \quad & \int [s^2 P\{m=0|s\} + (s-\omega)^2 P\{m=\omega|s\} \\ & +(s+\omega)^2 P\{m=-\omega|s\}] f(s)ds \end{aligned} \tag{7}$$

We emphasize that conditional model probabilities $w_i(s) = P\{m=\omega_i|m\in M, s\}$ given true mode in the above is not to be confused with the unconditional model probabilities $P\{m=\omega_i|m\in M\}$. For example, compare (9) with (12). Thus, the problem becomes

$$\min_\omega J = \int [s^2 w_1(s) + (s-\omega)^2 w_2(s) + (s+\omega)^2 w_3(s)] f(s)ds \tag{8}$$

For simplicity, however, we use the rectangular window, which leads to the following pmf:

$$w_i(s) = P\{m = \omega_i|s\} = \begin{cases} 1[s; (-\omega/2, \omega/2)] & i = 1 \\ 1[s; (\omega/2, \infty)] & i = 2 \\ 1[s; (-\infty, -\omega/2)] & i = 3 \end{cases} \tag{9}$$

where $1[x; R]$ is the indicator function, defined by

$$1[x; R] = \begin{cases} 1 & x \in R \\ 0 & x \notin R \end{cases} \tag{10}$$

(9) indicates that models $m_1 = 0$, $m_2 = \omega$, or $m_3 = -\omega$ is in effect if and only if the true turn rate falls inside the intervals $(-\omega/2, \omega/2)$, $(\omega/2, \infty)$, or $(-\infty, -\omega/2)$, respectively. In other words, these intervals are the effective coverage of the three models. This use of the rectangular window is well justified by Theorem 5.1 of Part I since it satisfies the nearest-neighbor condition (Condition B).

With (9), the cost function is

$$\begin{aligned} J &= \int \sum_{i=1}^{3} (s - \omega_i)^2 P\{m = \omega_i | m \in M, s\} f(s) ds \\ &= \int_{-\omega/2}^{\omega/2} s^2 f(s) ds + \int_{\omega/2}^{\infty} (s - \omega)^2 f(s) ds \\ &\quad + \int_{-\infty}^{-\omega/2} (s + \omega)^2 f(s) ds \\ &= \int_{-\infty}^{\infty} s^2 f(s) ds + 2\omega^2 \int_{\omega/2}^{\infty} f(s) ds - 4\omega \int_{\omega/2}^{\infty} s f(s) ds \\ &= c_0 \sigma_0^2 + 2c(\sigma^2 + \omega_s^2) + 2\omega^2 P_2 - 4\omega J_4 \end{aligned} \tag{11}$$

where $P_2 = \int_{\omega/2}^{\infty} f(s) ds$, given by

$$\begin{aligned} P_i = P\{m = \omega_i | m \in M\} &= \begin{cases} 2\int_0^{\omega/2} f(s) ds, \; i = 1 \\ \int_{\omega/2}^{\infty} f(s) ds, \; i = 2, 3 \end{cases} \\ &= \begin{cases} c_0 \operatorname{erf}\left(\frac{\omega/2}{\sqrt{2}\sigma_0}\right) + c \operatorname{erf}\left(\frac{\omega/2-3}{\sqrt{2}\sigma}\right) \\ \quad + c \operatorname{erf}\left(\frac{\omega/2+3}{\sqrt{2}\sigma}\right), \; i = 1 \\ \frac{1}{2}(1 - P_1^{(2)}), \; i = 2, 3 \end{cases} \end{aligned} \tag{12}$$

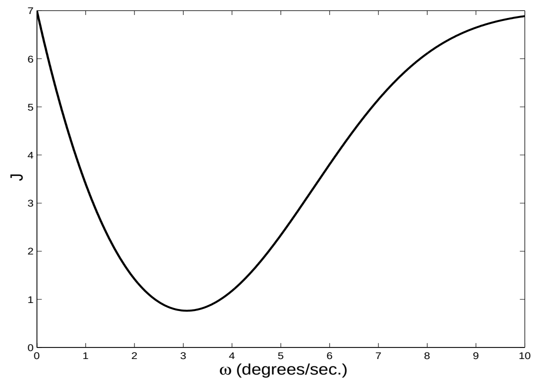
and J_4 can be obtained by direct integration and using the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$ as:

$$J_4 = \int_{\omega/2}^{\infty} s f(s) ds \tag{13}$$

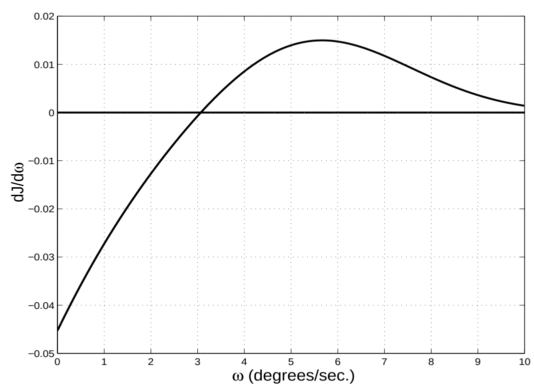
$$\begin{aligned} &= \frac{c_0 \sigma_0}{\sqrt{2\pi}} e^{-\frac{\omega^2}{8\sigma_0^2}} + \frac{c\sigma}{\sqrt{2\pi}} \left[e^{-\frac{(\omega/2+\omega_s)^2}{2\sigma^2}} + e^{-\frac{(\omega/2-\omega_s)^2}{2\sigma^2}} \right] \\ &\quad + \frac{c\omega_s}{2} \left[\operatorname{erf}\left(\frac{\omega/2 + \omega_s}{\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{\omega/2 - \omega_s}{\sqrt{2}\sigma}\right) \right] \end{aligned} \tag{14}$$

For the specific example with (6), the above optimization problem can be solved by plotting the above integrals J vs. ω (Fig. 6) and identifying the minimum within the interval of interest, which is around 3.05. Fig. 6(b) plots the derivative $dJ/d\omega$ obtained from Mathematica, which verifies the minimum. Therefore, the corresponding best model set is approximately $M = \{0, \pm 3°/s\}$ and the (initial) model probabilities are

$$P_1 = P\{m = \omega_1 | m \in M\} = 1/3, \quad P_3 = P_2 = 1/3$$



(a) Modal distance squared



(b) Derivative of modal distance squared

Fig. 6: The average modal distance squared (and its derivative) versus the turn rate ω.

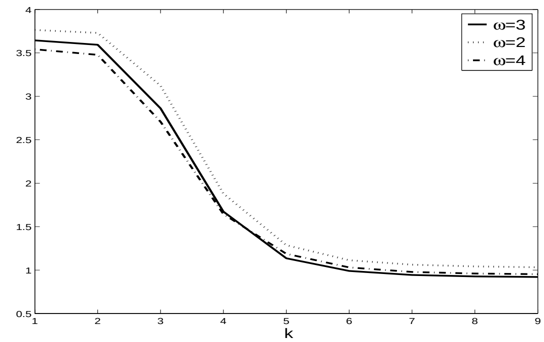
A design based on modal distance is particularly suitable for such applications as fault detection and isolation where the primary objective of hybrid estimation is mode estimation.

Note that the above procedures are still applicable even if a more sophisticated probabilistic model than (4) is used. However, if more than one parameter is to be determined, then the resulting optimization problem is in general multidimensional.

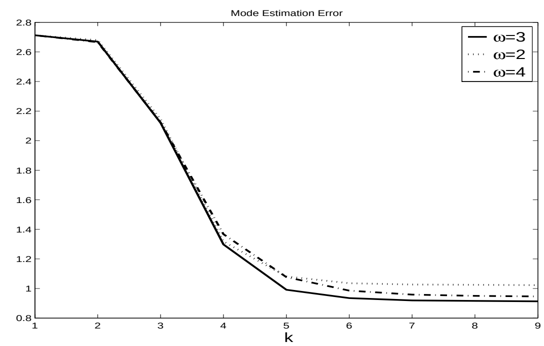
2.5 Effectiveness of Designs

Monte-Carlo simulations were conducted to verify the above model-set designs. In the simulation, 500 true modes were generated randomly with the distribution given by (4). They were unknown to the estimators and not allowed to jump. The initial state of the system was $x_0 = [1000, 100, 200, 120]'$ and for simplicity each estimator used x_0 as the initial state estimate. For each true mode generated, 100 samples of base state trajectories and the corresponding measurement sequences were generated. These measurements were used by an autonomous MM (AMM, also known as static MM) estimator based on a model set—assuming that the true mode belongs to the model set with the corresponding pmf—to estimate the base state and the true mode. Three AMM estimators $\hat{x}_M$, $\hat{x}_{M_1}$, and $\hat{x}_{M_2}$ were obtained, over the time steps $k = 1, 2, \ldots, 10$, based on the model sets $M = \{0, \pm 3°/\text{s}\}$, $M_1 = \{0, \pm 2°/\text{s}\}$, and $M_2 = \{0, \pm 4°/\text{s}\}$, with the initial model probabilities $\{1/3, 1/3, 1/3\}$, $\{0.3786, 0.2427, 0.3786\}$, and $\{0.288, 0.424, 0.288\}$, respectively, calculated by (12).

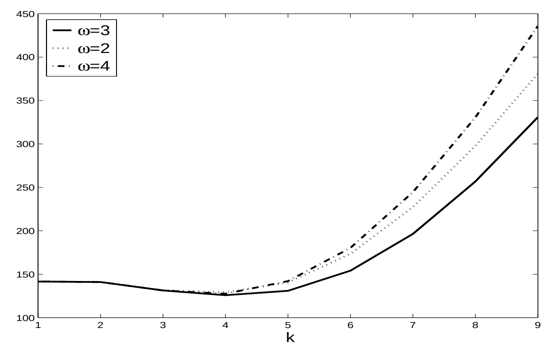
Fig. 7 shows the RMS errors, average modal distances, and mode estimation errors, as defined in [12]. These results support the above designs.



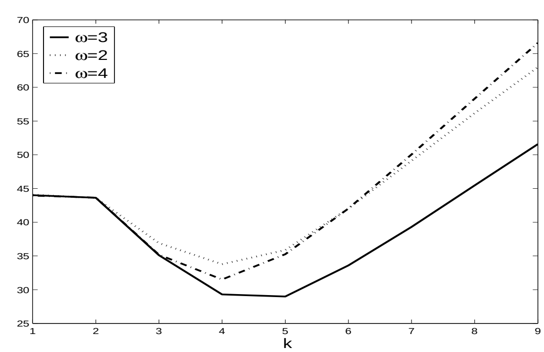
(a) Modal distances



(b) Mode estimation errors



(c) RMS position errors



(d) RMS velocity errors

Fig. 7: RMS errors, modal distances, and mode estimation errors of three MM estimators.

3 Model-Set Design Given Scenarios

Apart from the methods described in Part I and demonstrated above, the hypothesis-testing based approach originally developed for model-set adaptation [8, 7] can also be used for model-set design in a different manner. We now give one such example.

An important question for model-set design is the following: Given a number of interested (or representative) scenarios, how to design a model set with fewer models than the number of scenarios?

Suppose for simplicity that the scenarios of interest are: true turn rates are $0°$/s, $1°$/s, $2°$/s, $3°$/s and $4°$/s, respectively, and the model set to be designed is $\{0, \pm\omega\}$. In other words, the task is to determine ω such that the model set can cover the five possible true turn rates effectively. Varying ω, Fig. 8 shows (over 100 Monte-Carlo runs) the percent of correct decision of model selection for the five true turn rates of interest using multiple model-set sequential probability ratio test (MMS-SPRT) of [8] with CV model as the special model. It seems reasonable from Fig. 8 to choose $\omega = 3°$/s so that the correct decision is still above 80% even in the worst case where the true turn rate is $1°$/s or $2°$/s.

4 Model-Set Choice

It was shown in [11] that for two arbitrary model sets A and B with $A \subset B$, $\|\hat{x}_S - \hat{x}_B\| \leq \|\hat{x}_S - \hat{x}_A\|$ holds if and only if

$$r \leq r_t = \frac{\sqrt{b^2 \cos^2\theta + 1 - b^2} - b\cos\theta}{1 - b} \tag{15}$$

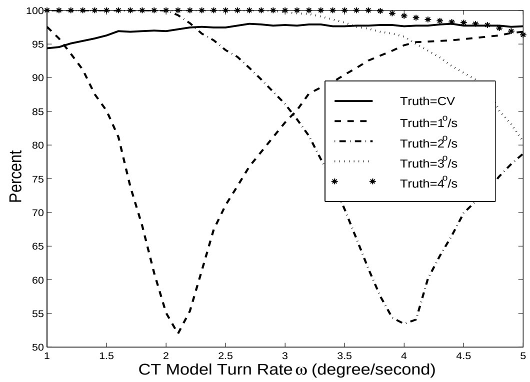


Fig. 8: Percent of correct model selection.

where

$$r = \frac{\|\hat{x}_S - \hat{x}_C\|}{\|\hat{x}_S - \hat{x}_A\|}$$

$$\cos\theta = \frac{(\hat{x}_S - \hat{x}_A)' \, (\hat{x}_S - \hat{x}_C)}{\|\hat{x}_S - \hat{x}_A\| \, \|\hat{x}_S - \hat{x}_C\|}$$

rence $C =$

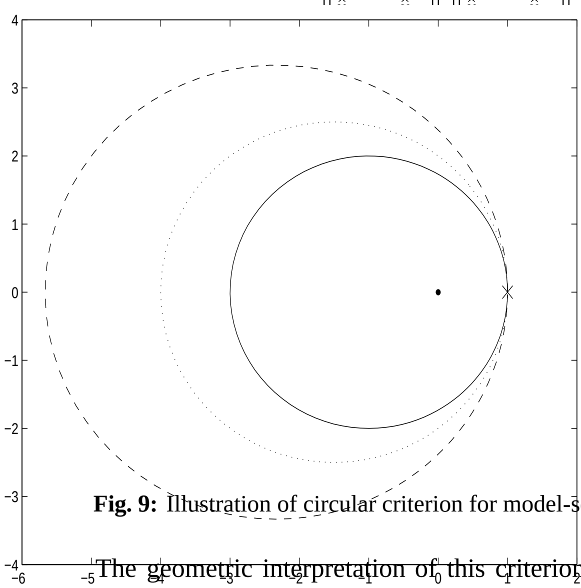


Fig. 9: Illustration of circular criterion for model-set choice.

The geometric interpretation of this criterion is simple and interesting: Refer to Fig. 9. Model set B is better than set A if and only if the estimators based on those models in B but not in A falls inside the corresponding circle (a ball if dimension is higher than two) determined by $b = \frac{P\{s=m_i|s\in B\}}{P\{s=m_i|s\in A\}}$ for some $m_i \in A$; that is, b is the ratio of model probability in set B to the model probability in set A for any identical model.

Note that this result requires the knowledge of the optimal estimator $\hat{x}_S$ using the optimal model set. We demonstrate below how this seemingly unrealistic theoretical result can be used for model-set choice and comparison.

Suppose that the true turn rate s at a given time is a discrete random variable with sample (mode) space $\mathbf{S} = \{0, \pm 1°/\text{s}, \pm 2°/\text{s}, \pm 3°/\text{s}, \pm 4°/\text{s}, 5°/\text{s}, \pm 6°/\text{s}\}$ and its pmf is the following discrete version of (4): s is generated by (4) using the rectangular window and rounded to the nearest one of the above 13 possible turn rates. In other words,

$$P_i^{\text{true}} = P\{s = \omega_i | s \in \mathbf{S}\} = \begin{cases} \int_{5.5}^{\infty} f(s)ds & \omega_i = \pm 6°/\text{s} \\ \int_{\omega_i - 0.5}^{\omega_i + 0.5} f(s)ds & \omega_i \neq \pm 6°/\text{s} \end{cases} \quad (16)$$

Note that at the given time the system is a coordinated-turn system governed by (1) with $F_{\text{CT}}(\omega_i)$ given by (3).

Consider two MM estimators $\hat{x}_A$ and $\hat{x}_B$ based on two model sets $A = \{0, \pm 3°/\text{s}\}$ and $B = \{0, \pm 1°/\text{s}, \pm 3°/\text{s}, \pm 7°/\text{s}\}$, respectively. Consider the following model-set choice problem: decide whether model set A is better than set B using criterion $\|\hat{x}_S - \hat{x}_B\| \leq \|\hat{x}_S - \hat{x}_A\|$ or equivalently (15), for the set of scenarios of interest given above in the form of a pmf (16).

To use criterion (15), we need the estimators $\hat{x}_C$ based on model set $C = B - A = \{\pm 1°/\text{s}, \pm 7°/\text{s}\}$ and the optimal MM estimator $\hat{x}_S$, where its models have the same probability mass as the true modes, given by (16); that is, $P_i^S \triangleq P\{m = \omega_i | m \in \mathbf{S}\} = P_i^{\text{true}}, \forall i$. The model probabilities for the other estimators are defined similarly:

$$P_i^B \triangleq P\{m = \omega_i | m \in B\} = \begin{cases} 2\int_0^{0.5} f(s)ds, \; \omega_i = 0 \\ \int_{0.5}^{2} f(s)ds, \; \omega_i = \pm 1°/\text{s} \\ \int_{2}^{5} f(s)ds, \; \omega_i = \pm 3°/\text{s} \\ \int_{5}^{\infty} f(s)ds, \; \omega_i = \pm 7°/\text{s} \end{cases}$$

P_i^A and P_i^C are induced by P_i^B in that they are obtained from P_i^B by deleting the probabilities of the models in B but not in A, C, respectively, and scaling up the remaining model probabilities such that they sum up to one. For example,

$$P_i^A \triangleq P\{m = \omega_i | m \in A\} = \begin{cases} \frac{2}{c}\int_0^{0.5} f(s)ds & \omega_i = 0 \\ \frac{1}{c}\int_2^{5} f(s)ds & \omega_i = \pm 3°/\text{s} \end{cases} \quad (17)$$

where $c = 2\left[\int_0^{0.5} f(s)ds + \int_2^5 f(s)ds\right]$.

Note that each MM estimator $\hat{x}_A$, $\hat{x}_B$, or $\hat{x}_C$ would be optimal should its model set match exactly the mode space in the sense that there is no approximation in the estimation algorithm used—suboptimality arises only from the fact that none of A, B, and C are equal to $\mathbf{S}$.

For an estimator using model set M, where M could be $\mathbf{S}$, A, B, or C, the average value of the estimate at time T

is given by

$$
\begin{aligned}
\bar{x}_M &= E[\hat{x}_M | m \in M] = E[E(x|z, m \in M)|m \in M] \\
&= E[x|m \in M] \\
&= \sum_{m_i \in M} E[x|m = m_i] P\{m = m_i | m \in M\} \\
&= \sum_{\omega_i \in M} \left[F_{\mathrm{CT}}(\omega_i)\bar{x}_0 + \bar{w}_i\right] P_i^M \qquad (18)
\end{aligned}
$$

where z is the measurement, $x = Fx_0 + w$ follows from (1), and $\bar{x}_0$ is the prior state of the system, assumed be identical for all models. Consider a specific example with (6) and

$$\bar{x} = [1000, 100, 200, 120]', \quad \bar{w}_i = 0, \quad T = 5,$$

Then we have

$$
\begin{aligned}
P_i^S &= \{.1316, .1009, .1009, .1008, .1008, .1296, \\
&\quad .1296, .0807, .0807, .0202, .0202, .0021, .0021\} \\
P_i^A &= \{0.1901, 0.4050, 0.4050\} \\
P_i^B &= \{0.1316, 0.1462, 0.1462, 0.2804, \\
&\quad 0.2804, 0.0076, 0.0076\} \\
P_i^C &= \{0.4753, 0.4753, 0.0247, 0.0247\}
\end{aligned}
$$

and thus

$$
\begin{aligned}
r^2 &= \frac{(\bar{x}_S - \bar{x}_C)'(\bar{x}_S - \bar{x}_C)}{(\bar{x}_S - \bar{x}_A)'(\bar{x}_S - \bar{x}_A)} = 16.7880^2 \\
b &= \frac{P\{m = 0 | m \in B\}}{P\{m = 0 | m \in A\}} = \frac{P\{m = 3^\circ/\mathrm{s} | m \in B\}}{P\{m = 3^\circ/\mathrm{s} | m \in A\}} \\
&= 0.6925 \\
\cos\theta &= \frac{(\bar{x}_S - \bar{x}_A)'(\bar{x}_S - \bar{x}_C)}{\|\bar{x}_S - \bar{x}_A\| \, \|\bar{x}_S - \bar{x}_C\|} = -0.9999
\end{aligned}
$$

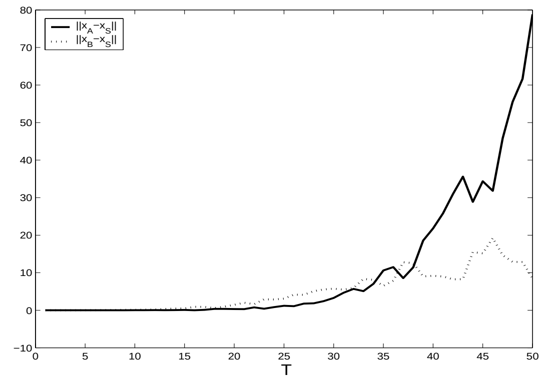
It follows that

$$16.7880 = r > r_t = 5.5041$$

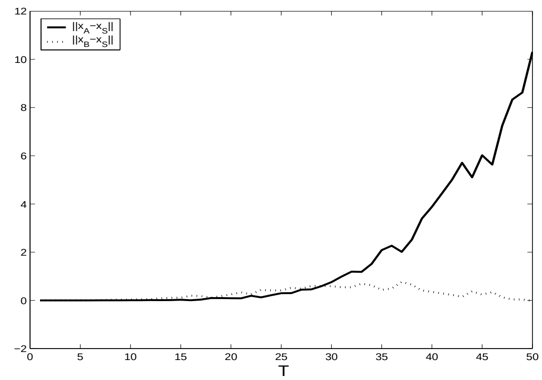
and thus $\|\hat{x}_S - \bar{x}_B\| > \|\hat{x}_S - \bar{x}_A\|$. Consequently, we conclude that model set A is better than set B for this problem. Note that it is hard to say based on our intuition or experience which model set is better. Fig. 9 illustrates the corresponding circular criterion, except that $\bar{x}_C$ should be added at approximately $(16.788, 180^\circ)$.

Fig. 10 shows the RMS position errors of $\hat{x}_S$, $\hat{x}_A$, $\hat{x}_B$ averaged over 500 runs for the time steps $k = 1, 2, \ldots, 10$, which correspond to sampling time $T = 5, 10, \ldots, 50$, respectively. The above numbers correspond to the points of the curves at $T = 5$s. The corresponding curves for the relative merit factor $r - r_t$ of model sets A and B are plotted in Fig. 11 for $T = 5, 10, \ldots, 50$. It is clear that the two figures agree almost perfectly.

In the simulation, the true modes were generated randomly with a distribution given by (16). They were not



(a) RMS position errors



(b) RMS velocity errors

Fig. 10: Difference in prediction of MM estimators: $\|\bar{x}_S - \bar{x}_A\|$ and $\|\bar{x}_S - \bar{x}_B\|$.

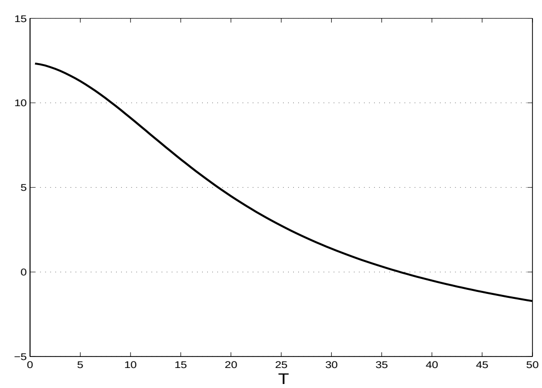


Fig. 11: Relative merit factor $r - r_t$ of model sets A and B versus sampling period T (A is inferior to B if and only if $r - r_t < 0$).

known to the estimators, not allowed to jump (for simplicity), and correspondingly AMM estimators were used based on the model sets S, A, B, respectively, assuming that the true mode has a distribution given by their corresponding pmfs P_i^M, $\forall i \in M$, where $M = \mathbf{S}, A$, or B. The initial state of the system was $x_0 = [1000, 100, 200, 120]'$ and each estimator used x_0 as the initial state estimate.

Clearly, the above procedure still works even if the optimal model set $\mathbf{S}$ is large. In many practical problems, the optimal model set for a given set of scenarios of interest is (approximately) known but may be too large to be used in an MM estimator. This example demonstrates how to choose between two model sets given this optimal model set, without using actual measurements or simulation. Clearly, the introduction of an proper probabilistic model of the scenarios, such as the Gaussian mixture model (4), is a key here. The circular criterion is also applicable for the cases where measurements are involved.

5 Conclusions

The design and choice of a model set is the most important issue in the application of the multiple-model approach. As demonstrated in this paper, the theoretical results obtained in Part I and several other publications are useful for model-set design and choice.

References

[1] Y. Bar-Shalom and X. R. Li, *Multitarget-Multisensor Tracking: Principles and Techniques*. Storrs, CT: YBS Publishing, 1995.

[2] Y. Bar-Shalom, X. R. Li, and T. Kirubarajan, *Estimation with Applications to Tracking and Navigation: Theory, Algorithms, and Software*. New York: Wiley, 2001.

[3] Y. Baram, "A Sufficient Condition for Consistent Discrimination Between Stationary Gaussian Models," *IEEE Trans. Automatic Control*, AC-23(5):958–960, Oct. 1978.

[4] Y. Baram and N. R. Sandell, Jr., "An Information Theoretic Approach to Dynamical Systems Modeling and Identification," *IEEE Trans. Automatic Control*, AC-23(1):61–66, Feb. 1978.

[5] M. J. Caputi, "A Necessary Condition for Effective Performance of the Multiple Model Adaptive Estimator," *IEEE Trans. Aerospace and Electronic Systems*, AES-31(3):1132–1139, July 1995.

[6] X. R. Li, "Hybrid Estimation Techniques," in *Control and Dynamic Systems: Advances in Theory and Applications* (C. T. Leondes, ed.), vol. 76, pp. 213–287, New York: Academic Press, 1996.

[7] X. R. Li, "Engineer's Guide to Variable-Structure Multiple-Model Estimation for Tracking," in *Multitarget-Multisensor Tracking: Applications and Advances* (Y. Bar-Shalom and D. W. Blair, eds.), vol. III, ch. 10, pp. 499–567, Boston, MA: Artech House, 2000.

[8] X. R. Li, "Multiple-Model Estimation with Variable Structure—Part II: Model-Set Adaptation," *IEEE Trans. Automatic Control*, AC-45(11):2047–2060, Nov. 2000.

[9] X. R. Li, "Model-Set Design for Multiple-Model Estimation—Part I," in *Proc. 2002 International Conf. on Information Fusion* (Paper M1A.4), (Annapolis, MD, USA), July 2002.

[10] X. R. Li and Y. Bar-Shalom, "Design of an Interacting Multiple Model Algorithm for Air Traffic Control Tracking," *IEEE Trans. Control Systems Technology*, 1(3):186–194, Sept. 1993. Special issue on Air Traffic Control.

[11] X. R. Li and Y. Bar-Shalom, "Multiple-Model Estimation with Variable Structure," *IEEE Trans. Automatic Control*, AC-41(4):478–493, Apr. 1996.

[12] X. R. Li, Y. M. Zhang, and X. R. Zhi, "Multiple-Model Estimation with Variable Structure—Part IV: Design and Evaluation of Model-Group Switching Algorithm," *IEEE Trans. Aerospace and Electronic Systems*, AES-35(1):242–254, Jan. 1999.

[13] S. N. Sheldon and P. S. Maybeck, "An Optimizing Design Strategy for Multiple Model Adaptive Estimation and Control," *IEEE Trans. Automatic Control*, AC-38(4):651–654, Apr. 1993.