Algorithms for Robustness Analysis

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Outline

- Motivation
- Classical Deterministic Techniques
- Deterministic Robustness Margin
- Probabilistic Robustness Margin
- Robustness Degradation Function
- Randomized Algorithms
- Examples

Summary

- Robustness analysis is the first line to ensure system safety.
  - Controller design is based on inexact model.
  - Robustness analysis predicts how well a system can tolerate uncertainty (e.g. model inaccuracy).
  - Robust analysis provide guidance for improving system's tolerance to accidental faults.
- Classical Deterministic Techniques
  - The central problem of robustness analysis is computing the *deterministic robustness margin*, which is an indicator of the system's ability to tolerate uncertainty.
  - Computing the deterministic robustness margin is in general intractable.
  - Deterministic robustness margin can be an extreme conservative of system robustness.
  - NP hard (Non polynomial complexity)
- Probabilistic Robustness Analysis
  - *Probabilistic robustness margin* is a new concept which overcomes the conservatism of the classic deterministic robustness margin.
    - Existing methods of computing the probabilistic robustness margin depend on the feasibility of computing the deterministic robustness margin. Therefore, the application of existing techniques is limited

to some specific problem. Many important problems, for example robustness problem with time specifications, can not be solved by existing techniques.

  * We develop efficient randomized algorithms with universal applicability.
    · Explicit formula for binomial confidence interval
    · Probabilistic comparison
    · Probabilistic bisection

– *Robustness degradation function* provides more insight for the robustness of the system.

  * Existing methods of computing the robustness degradation function depend on the feasibility of computing the deterministic robustness margin. Hence, many important problems, for example robustness problem with time specifications, can not be solved by existing techniques.
  * We develop efficient randomized algorithms with universal applicability.
    · Probabilistic robustness margin can be efficiently computed and can be taken as a starting point.
    · Sample reuse algorithm
    · Backward iteration

Recent Research Works

1) J. Aravena, X. Chen and F. Chowdhury, "Controllers for Highly Safe System", accepted for *Safeprocess'03 Symposium*, Washington D.C, June 2003.

2) X. Chen, K. Zhou and J. Aravena, "Fast Universal Algorithms for Robustness Analysis", submitted to *IEEE Conferences on Decision and Control* (invited paper), Maui, Hawaii, December 2003.

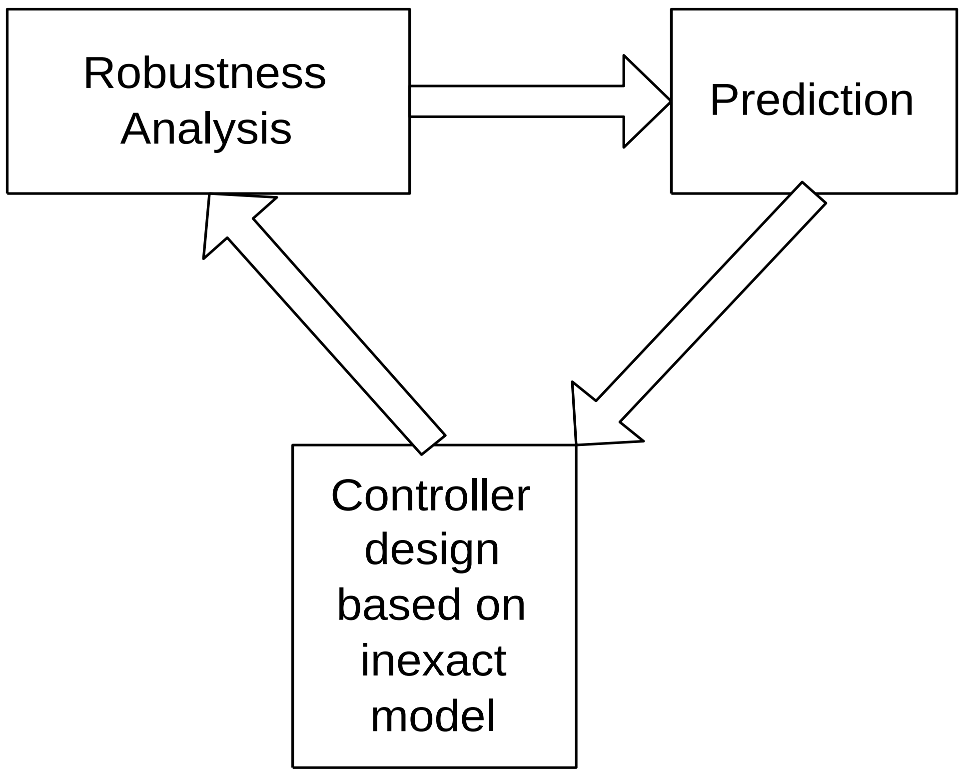
3) X. Chen, K. Zhou and J. Aravena, "Fast construction of robustness degradation function", *IEEE Conferences on Decision and Control*, Las Vegas, Nevada, December 2002, also submitted to *SIAM Journal on Control and Optimization*, June 2002 (revised for further review).

4) X. Chen, K. Zhou and J. Aravena, "On the exact binomial confidence interval and probabilistic robust control", submitted to *Automatica*, July 2002 (revised for further review).

5) X. Chen, K. Zhou and J. Aravena, "Explicit formula for constructing binomial confidence interval with guaranteed coverage probability", submitted to *SIAM Journal on Probability and Its Applications*, July 2002.

Robustness Analysis is the First Line to Ensure System Safety

- ***Controller design is based on inexact model***



Robustness Requirements

Robustness requirements —

- Stability or $\mathcal{D}$-stability;
- Time specifications such as overshoot, rise time, settling time and steady state error.
- H_∞ norm of the closed loop transfer function;

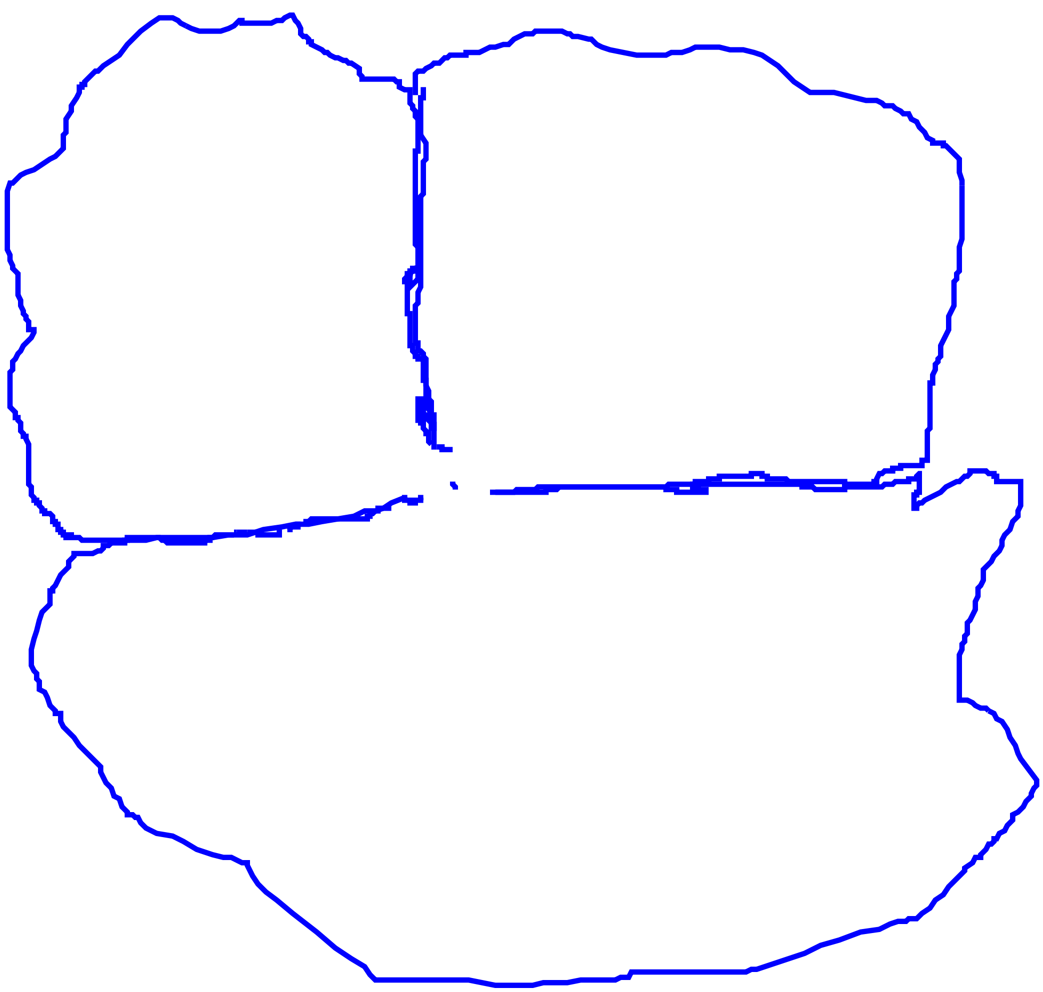
Uncertainty Bounding Sets

Let $\mathcal{B}(r)$ denote the set of uncertainties with size smaller than r.

- l_p ball
- Spectral norm ball
- Homogeneous star-shaped bounding set

Deterministic Robustness Margin

- The **maximal** uncertainty radius r such that the robustness requirement is guaranteed for **every** value of the uncertainty set with radius r



Limitations of Classical Deterministic Techniques

- Computing deterministic robustness margin is in general intractable
- Conservatism
- NP Hardness — Non-polynomial Complexity

Probabilistic Robustness Margin

- The *maximal* uncertainty radius r such that the robustness requirement is guaranteed with probability *at least* $1-\epsilon$ for the uncertainty set with radius r

Robustness Degradation Function

- ***Proportion*** of systems guaranteeing robustness requirements —— ***Radius*** of uncertainty set

Probabilistic Perspective

Truncation Principle [Barmish, Lagoa, Tempo (1997)]

Suppose that uncertainty is "peak" around its nominal value. Then the worst-case probability is attained by uniform distribution

Separable Assumption

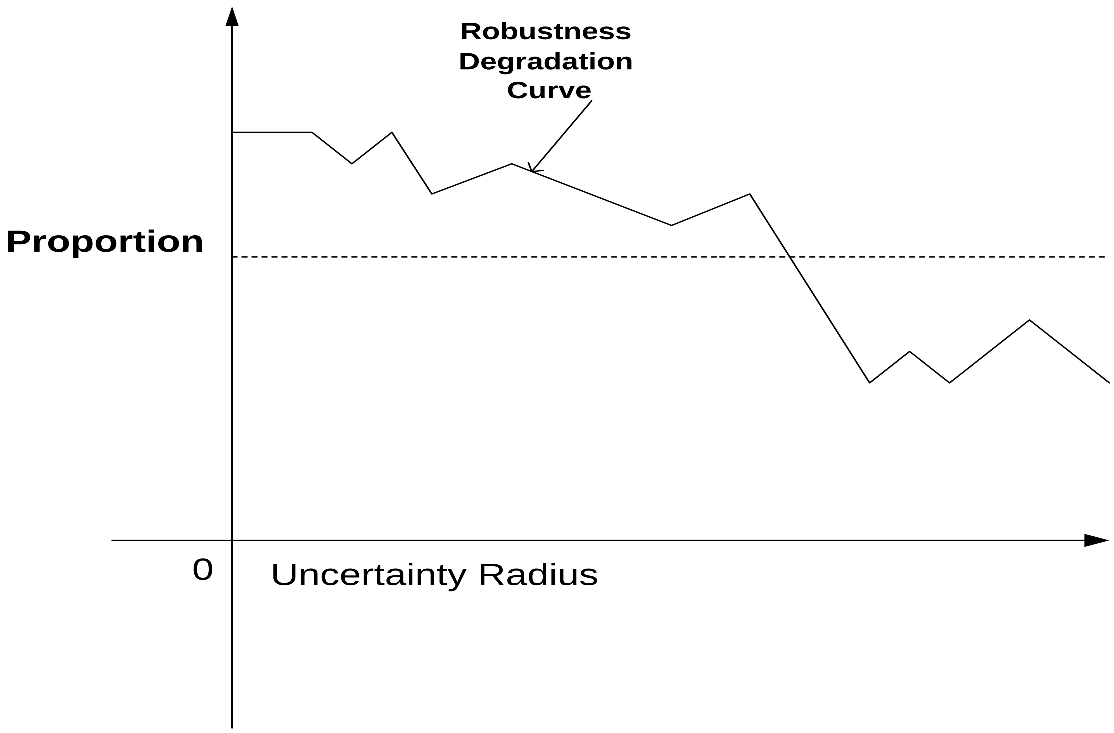


Figure 1: Illustration of Separable Assumption

Estimating Probabilistic Robustness Margin

- **Probabilistic Comparison**
- **Computing Initial Interval**
- **Probabilistic Bisection**

Probabilistic Comparison

- **Confidence Interval**

  Let

  $$\mathcal{L}(k) = \frac{k}{N} + \frac{3}{4}\,\frac{1 - \frac{2k}{N} - \sqrt{1 + 4\theta\; k(1 - \frac{k}{N})}}{1 + \theta N}$$

  and

  $$\mathcal{U}(k) = \frac{k}{N} + \frac{3}{4}\,\frac{1 - \frac{2k}{N} + \sqrt{1 + 4\theta\; k(1 - \frac{k}{N})}}{1 + \theta N}$$

  with $\theta = \frac{9}{8 \ln \frac{2}{\delta}}$. Then

  $$\Pr\left\{\mathcal{L}(K) < \mathbb{P}_X < \mathcal{U}(K)\right\} > 1 - \delta.$$

Complexity of Probabilistic Comparison

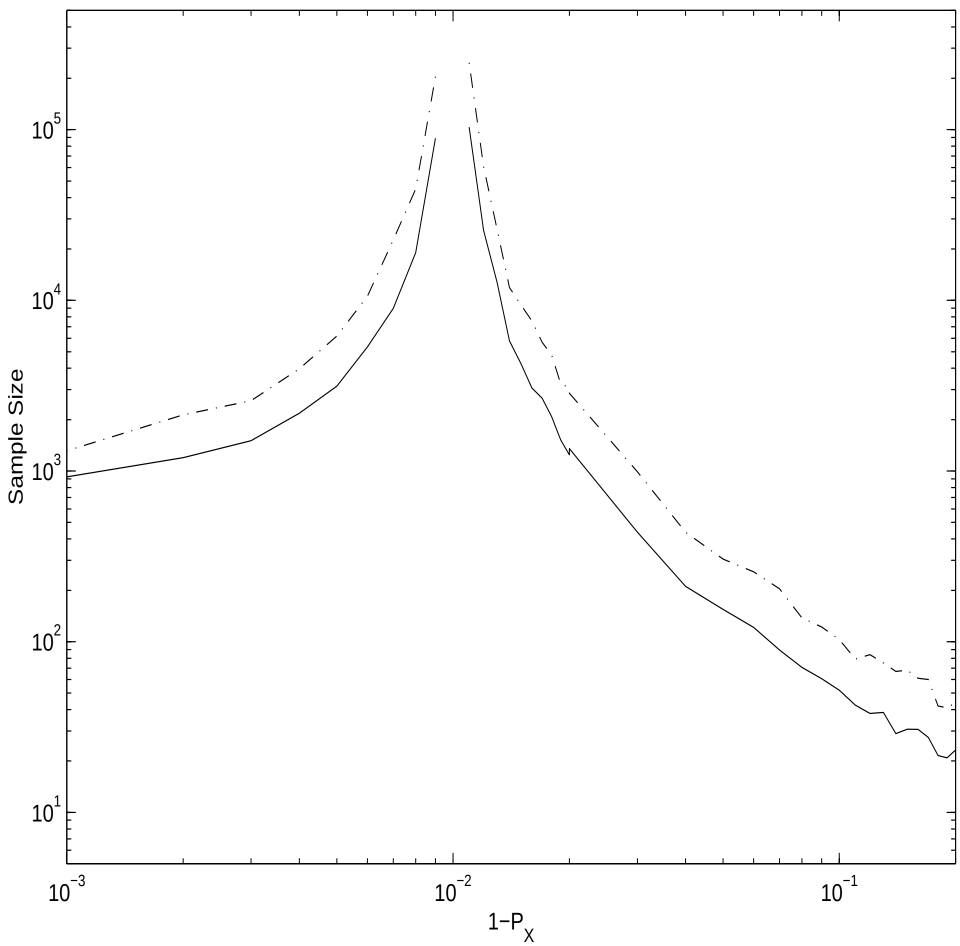


Figure 2: Complexity of Probabilistic Comparison

Constructing Robustness Degradation Curve

- Step 1. Compute an estimate R for the probabilistic robustness margin.

- Step 2. Successively apply *Sample Reuse Algorithm* as a subroutine to construct robustness degradation curve for uncertainty radius interval $[\frac{R}{2^{n+1}}, \frac{R}{2^n}]$ for $n = 0, 1, \cdots$.

Mechanisms of Sample Reuse Algorithm

- Backward Iteration
- Reuse Sample
- Reuse Performance Evaluation

Sample Reuse Algorithm

Sample Reuse Factor

Let $\mathbf{n}_i$ be the number of simulations required at r_i. Define *sample reuse factor*

$$\mathcal{F}_{reuse} := \frac{Nl}{\mathcal{E}[\sum_{i=1}^{l} \mathbf{n}_i]}.$$

Then

$$\mathcal{F}_{reuse} = \frac{l}{l - \sum_{i=2}^{l} \left(\frac{r_i}{r_{i-1}}\right)^d}$$

Evaluation of Sample Reuse Factor

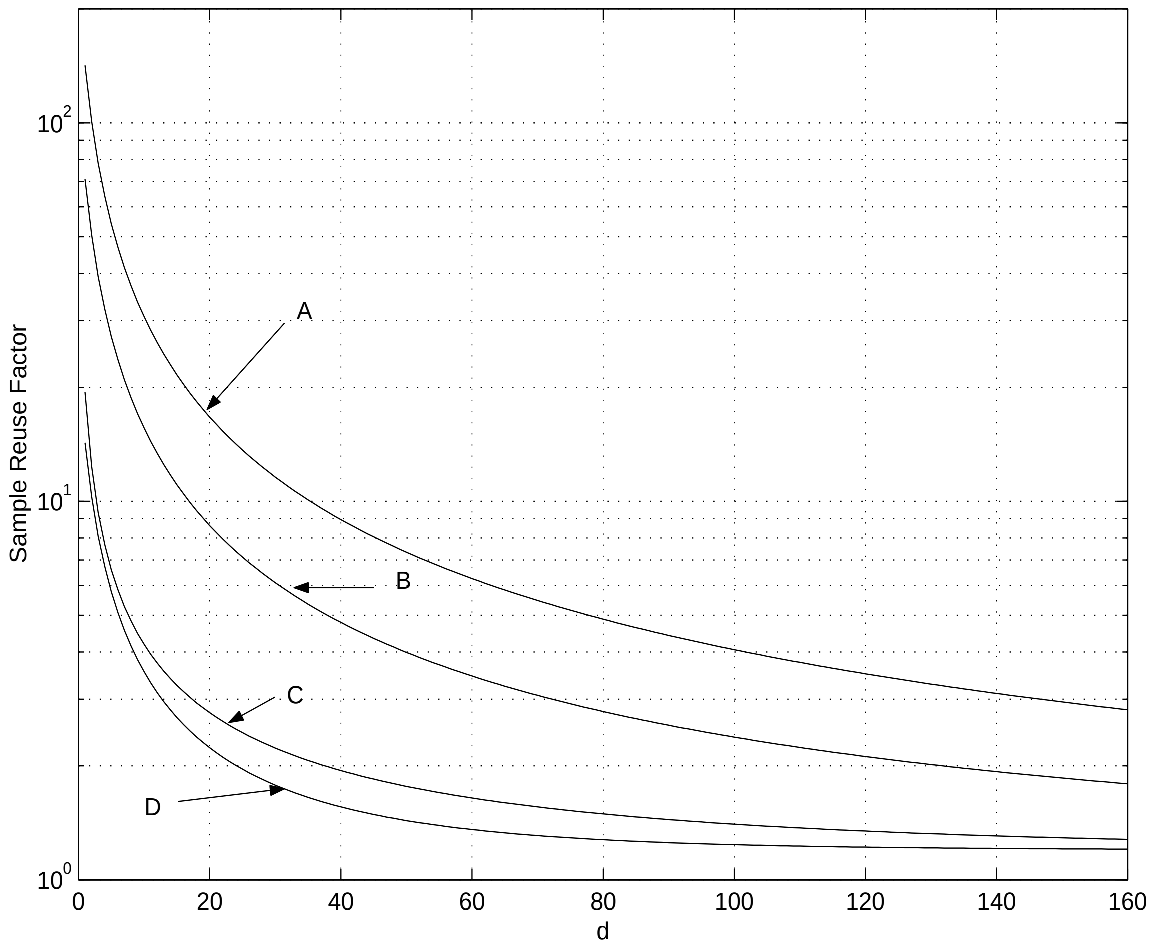


Figure 3: Performance Improvement ($\mathrm{A}:\ l = 200,\ b = 2a;\ \ \mathrm{B}:\ l = 100,\ b = 2a;\ \ \mathrm{C}:\ l = 100,\ a = 0;\ \ \mathrm{D}:\ l = 20,\ b = 2a$)

Evaluation of Sample Reuse Factor

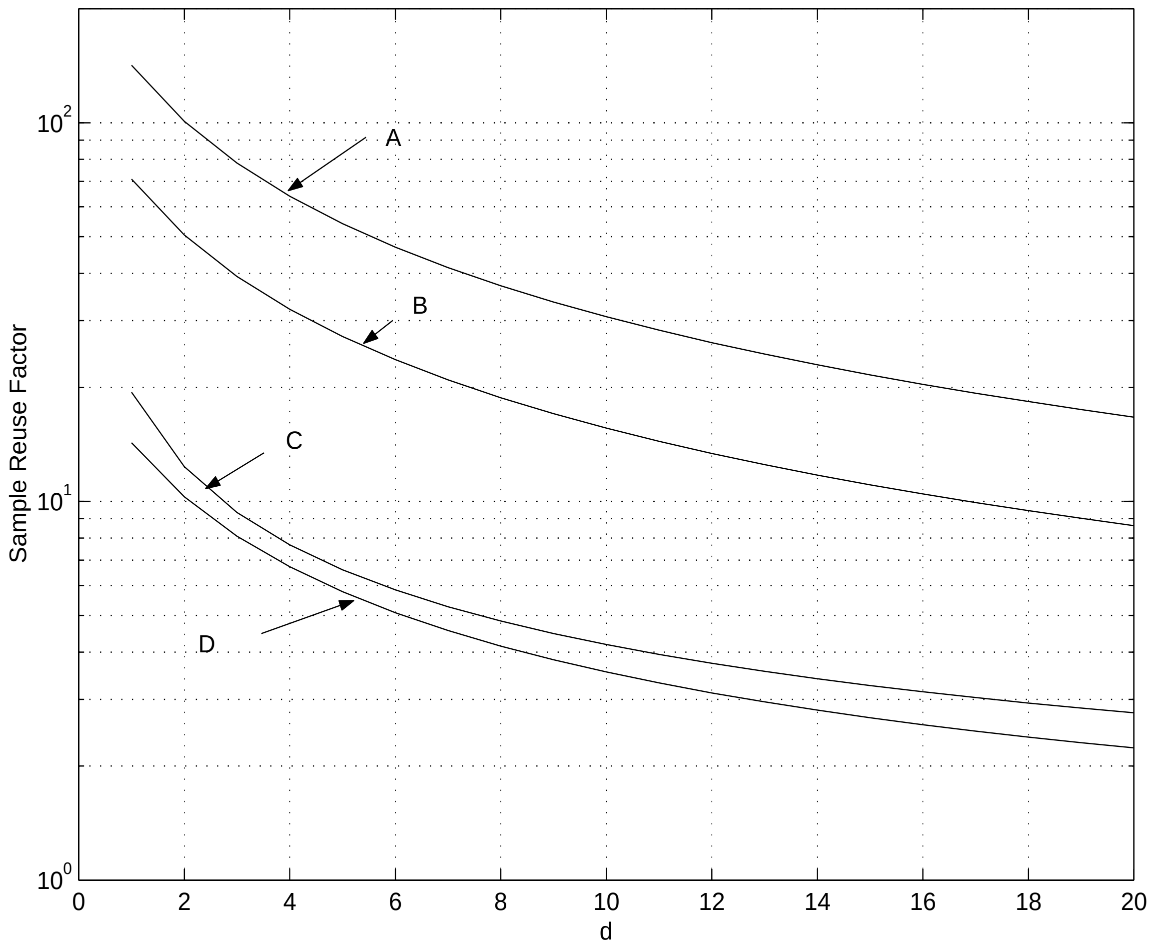


Figure 4: Performance Improvement (A : $l = 200,\ b = 2a$; B : $l = 100,\ b = 2a$; C : $l = 100,\ a = 0$; D : $l = 20,\ b = 2a$)

Illustrative Examples

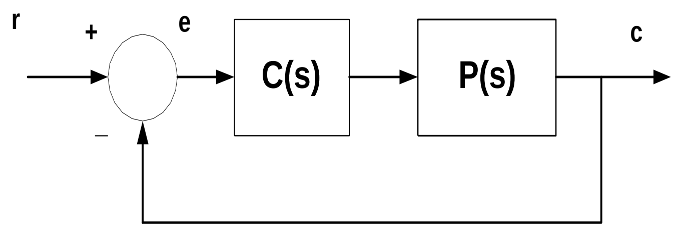


Figure 5: Uncertain System

$$C(s) = \frac{s+2}{s+10}$$

$$P(s) = \frac{800(1+0.1\delta_1)}{s(s+4+0.2\delta_2)(s+6+0.3\delta_3)}$$

$$\Delta = [\delta_1, \delta_2, \delta_3]^{\mathrm{T}}$$

Robustness Problem with Time-Domain Specifications

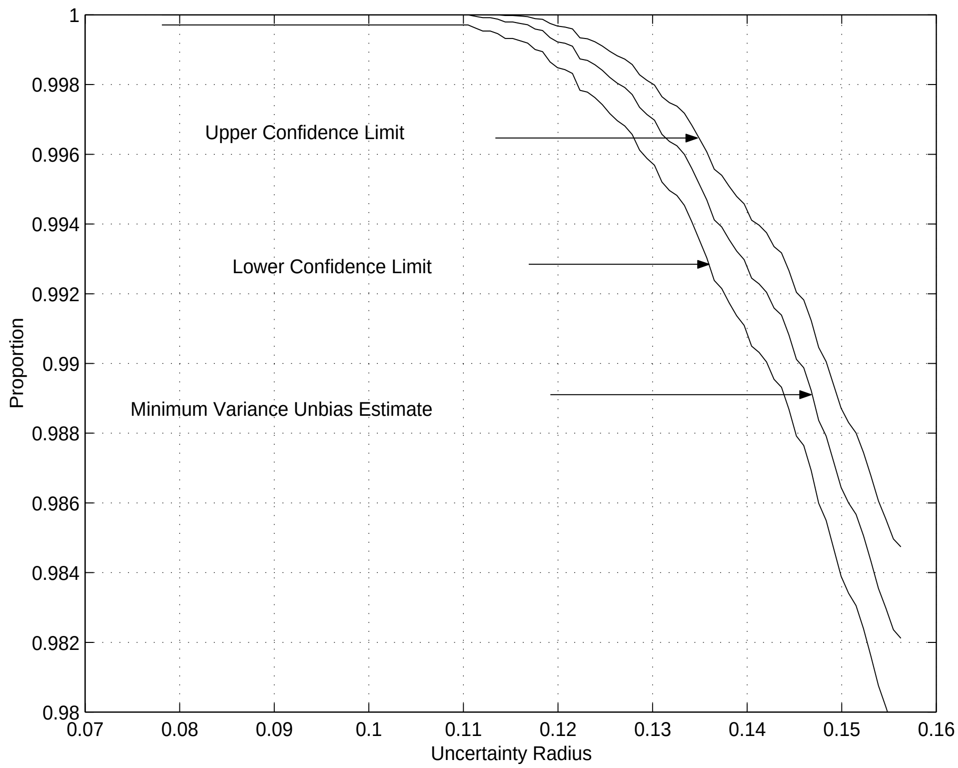


Figure 6: Robustness Degradation Curve

$$\left[\frac{1}{2}, 1\right] \longrightarrow \left[\frac{1}{4}, \frac{1}{2}\right] \longrightarrow \left[\frac{1}{8}, \frac{1}{4}\right] \longrightarrow \left[\frac{1}{8}, \frac{3}{16}\right] \longrightarrow \left[\frac{1}{8}, \frac{5}{32}\right].$$

Robust $\mathcal{D}$-stability over Polytope

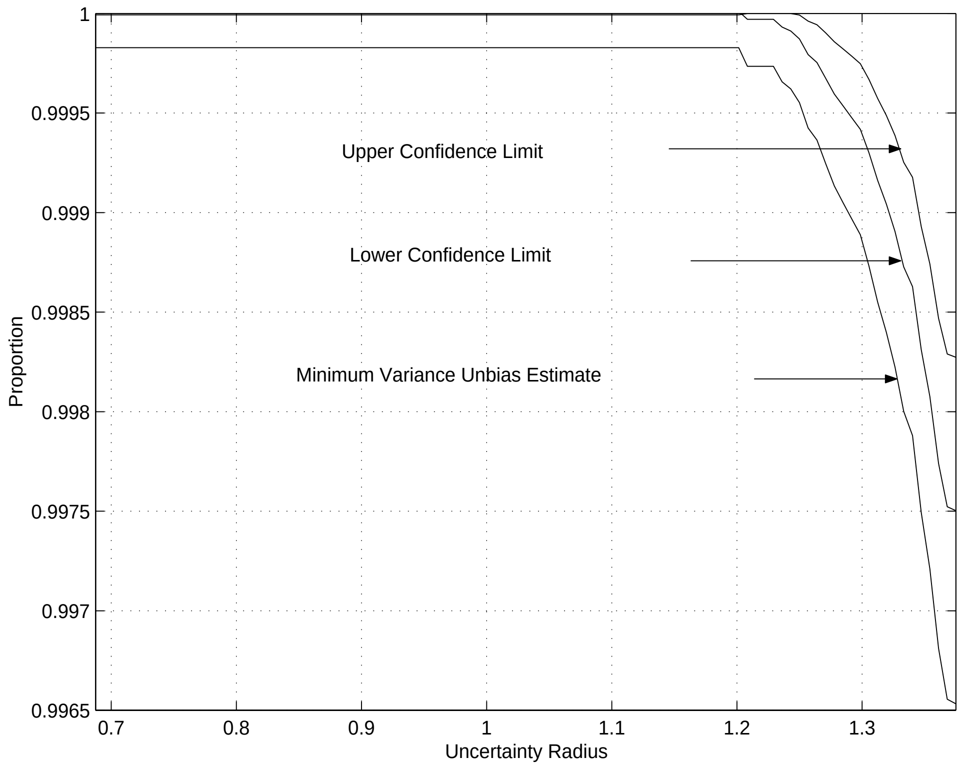


Figure 7: Robustness Degradation Curve

$$[1,2] \longrightarrow \left[1, \frac{3}{2}\right] \longrightarrow \left[\frac{5}{4}, \frac{3}{2}\right] \longrightarrow \left[\frac{5}{4}, \frac{11}{8}\right].$$