Risk Assessment Via Monte Carlo Simulation: Tolerances Versus Statistics

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Overview

- Motivation
- The New Monte Carlo Method
- Truncation Principle
- Surprising Results
- Conclusion

Monte Carlo Simulation

- Used Extensively to Assess System Safety
- Uncertain Parameters with Tolerances
- Generate “Thousands” of Sample Realizations
- Determine Range of Outcomes, Averages, Probabilities etc.
- How to Initialize the Random Number Generator
- High Sensitivity to Choice of Distribution
- Unduly Optimistic Risk Assessment

It's Arithmetic Time

- Consider

$$1 + 2 + 3 + 4 + 5 + \cdots + 20 = 210$$

- Data and Parameter Errors

- Classical Error Accumulation Issues

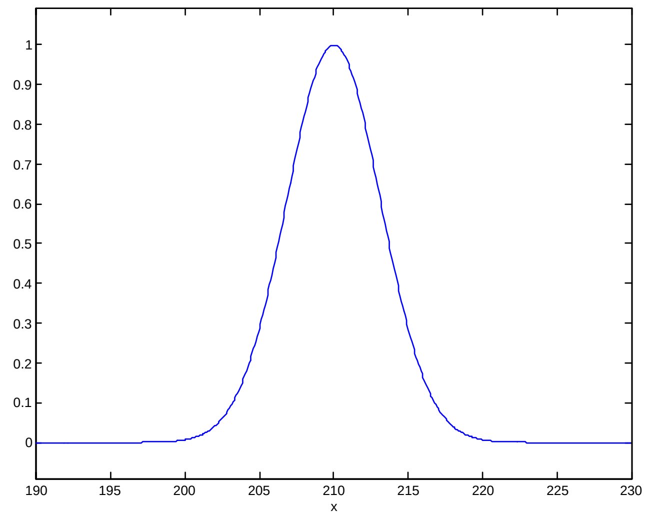
$$(1 \pm 1) + (2 \pm 1) + (3 \pm 1) + \cdots (20 \pm 1) = 210 \pm 20$$

Two Results

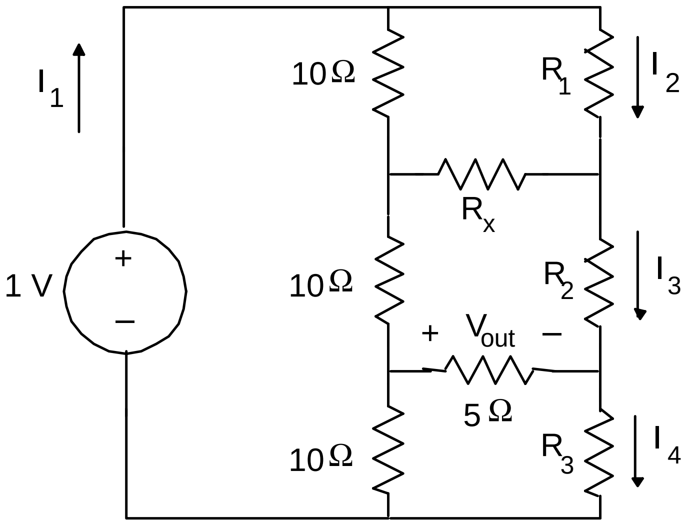
Actual Result

Alternative Result

190 · SUM · 230



Circuit Example



Output Voltage

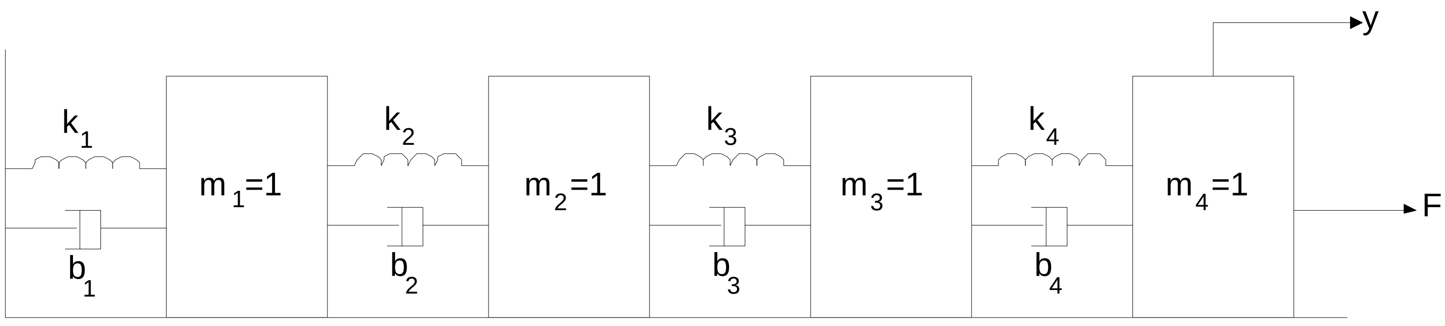
$$V_{out}(R) = \frac{N(R)}{D(R)}$$

$$\begin{aligned} N(R) &= -200(10R_2 + 10R_4 + 10R_4 + R_3R_4) \\ &\quad -350(R_1R_4 + R_2R_4 + R_2R_3 + R_1R_2 + R_1R_3) \\ &\quad -30(R_2R_3R_4 + R_1R_2R_4); \end{aligned}$$

$$\begin{aligned} D(R) &= 50(2R_1R_4 + 10R_4 + R_2R_4 - 10R_3 - R_2R_3) \\ &\quad -50(R_1R_2 - R_1R_3). \end{aligned}$$

- Range of Gain Versus Distribution

More Generally



with tolerances for m_i, b_i, k_i and performance

$$\left|\frac{Y(j\omega)}{F(j\omega)}\right| \leq \overline{g}$$

for all $\omega \geq 0$.

- Desired Simulation

for i = 1:N
RANDOMLY GENERATE m_i, b_i, k_i
EVALUATE PERFORMANCE
end

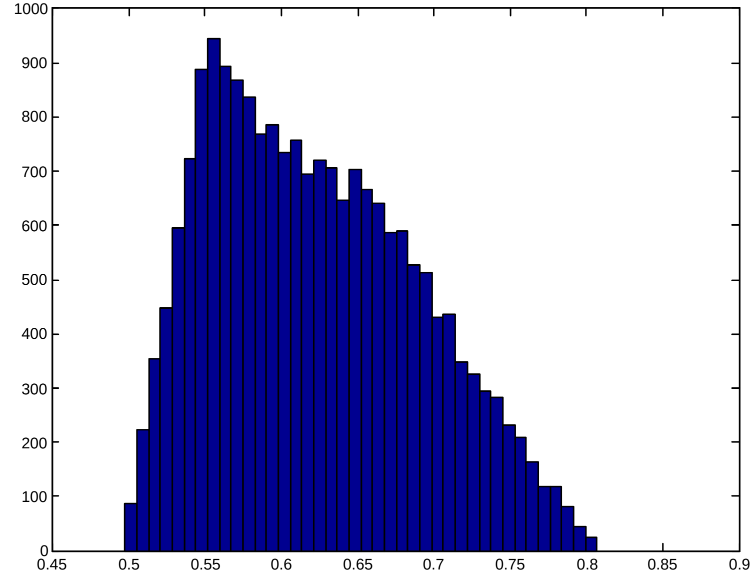
Two Approaches

- Interval of Gain

- Monte Carlo

Generate $k^1, k^2, \ldots, k^N, c^1, c^2, \ldots, c^N$ and

$$\hat{g}(\omega) \doteq \frac{1}{N} \sum_{i=1}^{N} g(\omega, k^i, c^i).$$



Gain Histogram: 20,000 Uniform Samples

- How to Generate Samples?

Key Issue

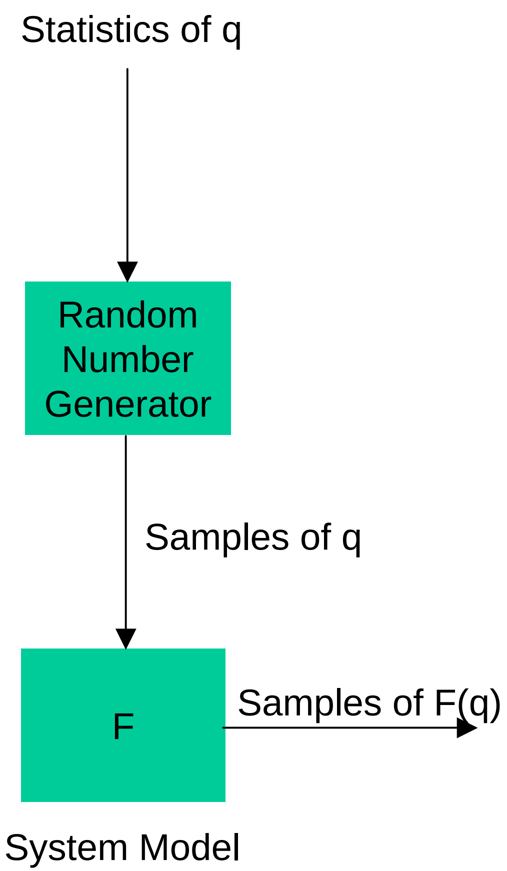
What probability distribution should be used?
Conclusions are often sensitive to choice of distribution.

- Intractability of Trial and Error

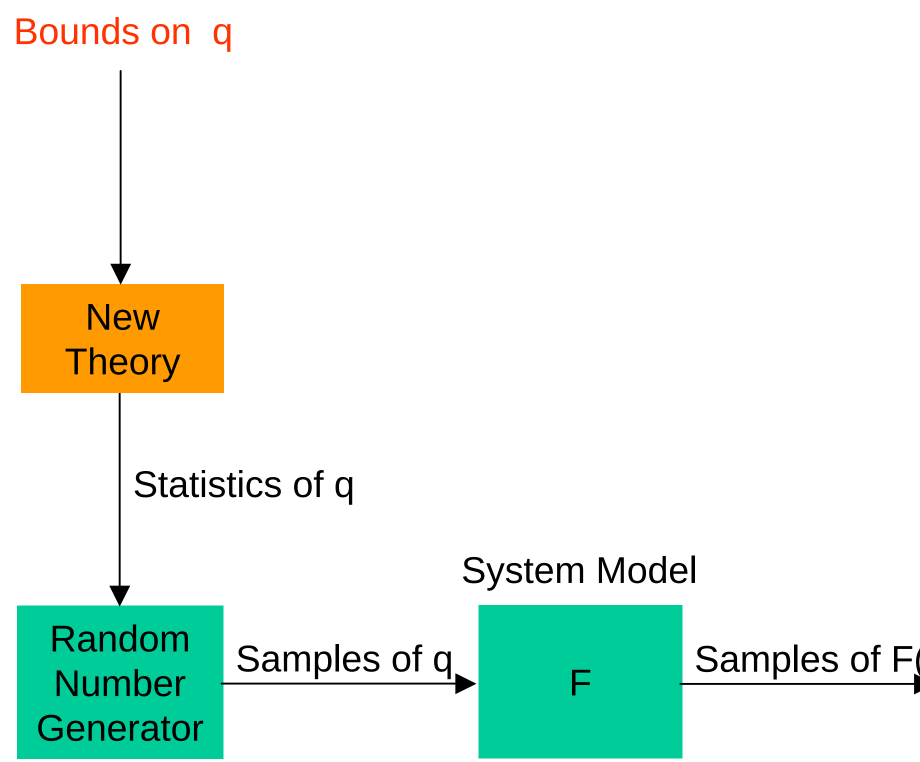
- Combinatoric Explosion Issue

Key Idea in New Research

Classical Monte Carlo



New Monte Carlo



The Central Issue

What probability distribution to use?

Manufacturing Motivation

Uncertain capacitor

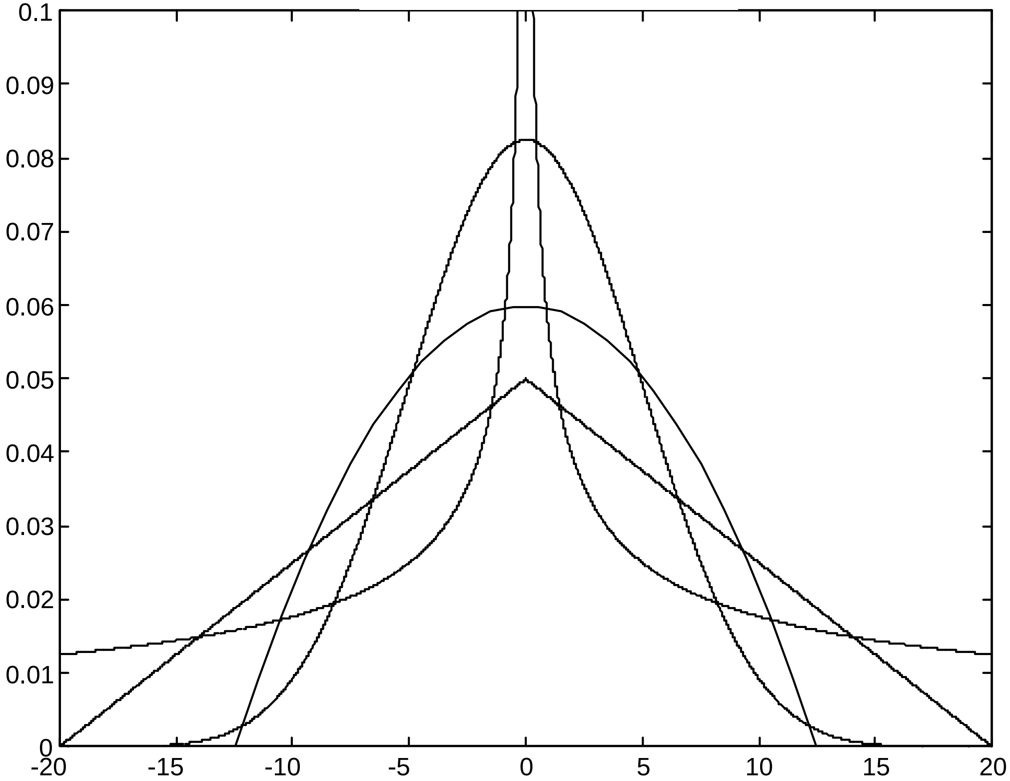
$$30\ \mu\text{fd} \leq C \leq 70\ \mu\text{fd}$$

nominally manufactured with

$$C_0 = 50\mu\text{fd}$$

Positive and negative deviations about C_0 are equally likely. If $|\Delta C_1| < |\Delta C_2|$, ΔC_1 is more likely than ΔC_2.

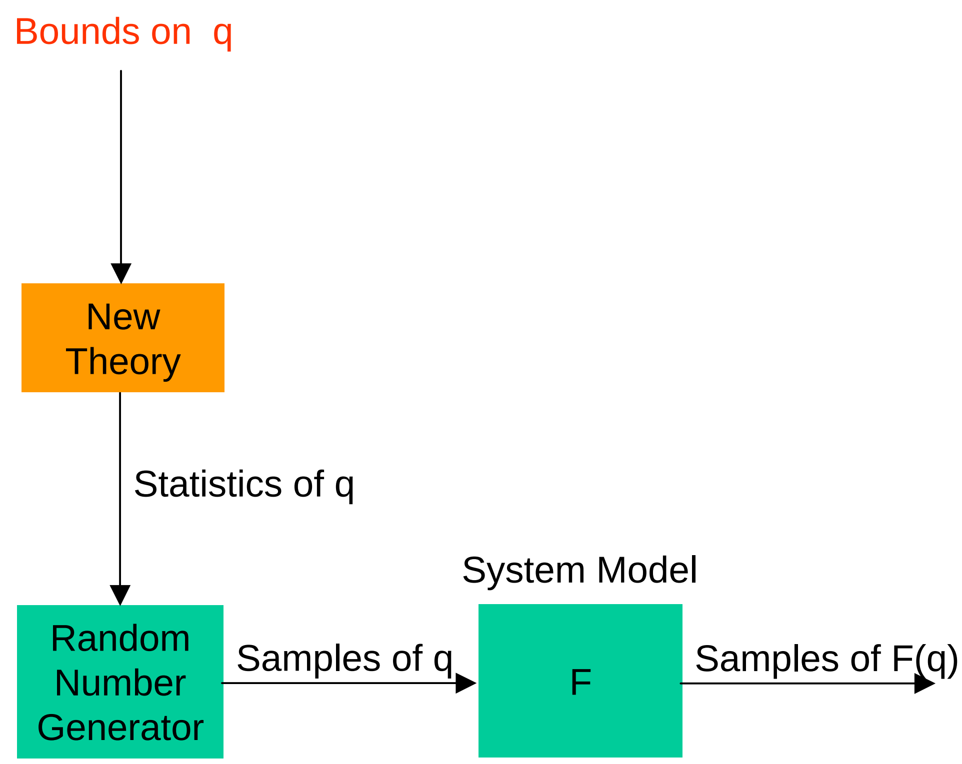
Distributions for Capacitor



Interpretation

- Probabilistic Guarantees
- Robustness With Respect to $f \in \mathcal{F}$
- A Posteriori Versus A Priori

Interpretation (cont.)

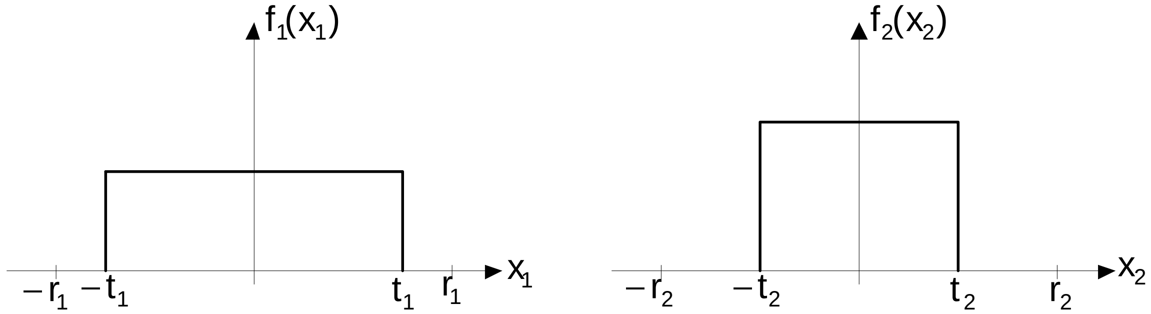


Truncation Principle

Problem is to find $f^* \in \mathcal{F}$ minimizing criterion function, call it $\Phi(f)$.

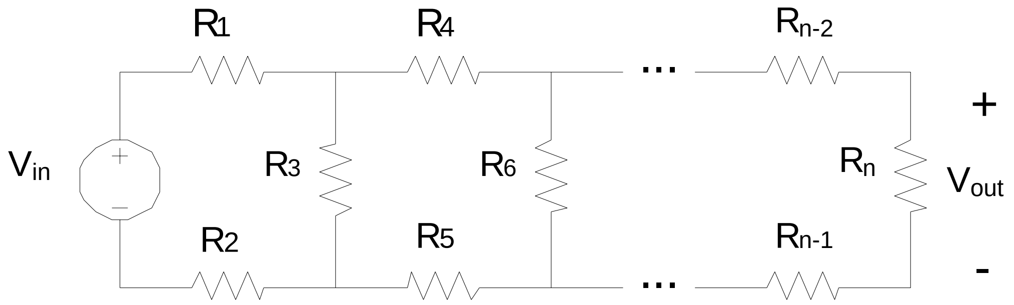
The Truncation Principle indicates that f_i^* is uniform over a sub-interval

$$T_i \doteq [-t_i, t_i] \subseteq [-r_i, r_i].$$



Notation: u^t and $t \in T$.

Example 1: Ladder Network



with density functions $f \in \mathcal{F}$ for each resistor.

- Study Expected Gain

Solution: Set f_i^* to the Dirac Delta function distribution for inter-stage resistors R_i and set f_i^* to the uniform distribution for remaining resistors R_i.

Illustration for Ladder Network

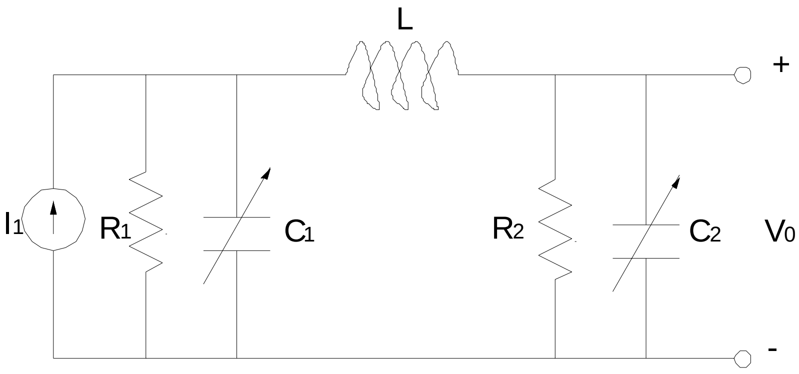
Three stages with with nominal values $R_{1,0} = R_{4,0} = R_{5,0} = R_{7,0} = R_{8,0} = 1$, $R_{2,0} = 2$, $R_{3,0} = 3$, $R_{6,0} = 5$ and $R_{9,0} = 7$, and uncertainty bounds $r_i = 0.8R_{i,0}$ for the inter-stage resistors and $r_i = 0.1R_{i,0}$ for the remaining resistors. Obtain

$$\mathcal{E}(g(q^{f^*})) \approx 0.1864$$

with $n = 100,000$ samples. In contrast, a more traditional Monte Carlo simulation using the uniform distribution for all resistors leads to a 20% difference.

Example 2: RLC Circuit

Consider the RLC circuit



$$R_1 = 1000, R_2 = 100, L = 0.01,$$

$$0.755 \times 10^{-6} \leq C_1 \leq 1.695 \times 10^{-6};$$

$$0.75 \times 10^{-6} \leq C_2 \leq 4.55 \times 10^{-6}.$$

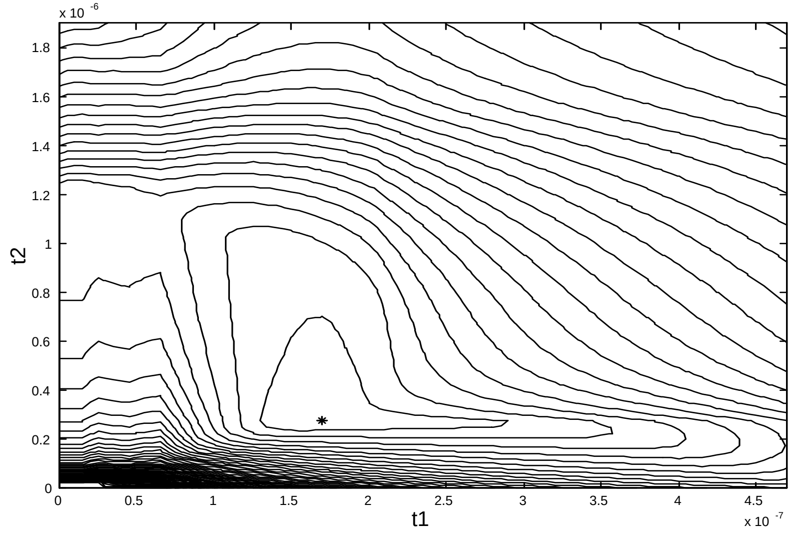
Performance is

$$OS_{max} \leq 96.3$$

Study probability of performance satisfaction with Truncation Principle.

Solution Summary For RLC

Plot contours of equal probability in (t_1, t_2) plane.



Obtain

$$t_1^* \approx 0.17 \times 10^{-6}; \;\; t_2^* \approx 0.275 \times 10^{-6};$$

and compare probability of performance with uniform:

$$\Phi(u'^*) \approx 0.486; \quad \Phi(u) \approx 0.6912.$$

Current and Further Research

- The Optimal Truncation Problem

- Exploitation of Structure

- Correlated Parameters

- New Application Areas; e.g., Cash Flows