A LOWER BOUND ON THE AVERAGE PHYSICAL LENGTH OF EDGES IN THE PHYSICAL REALIZATION OF GRAPHS*

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ABSTRACT

The stereo-realization of a graph is the assignment of positions in Cartesian space to each of its vertices such that vertex density is bounded. A bound is derived on the average edge length in such a realization. It is similar to an earlier reported result, however the new bound can be applied to graphs for which the earlier result is not well suited. A more precise realization definition is also presented. The bound is applied to d-dimensional realizations of de Bruijn graphs, yielding an edge length of $\Omega((1-2^{-d})r^{n/d}/(2n))$, where r is the radix (number of distinct symbols) and n is the number of graph dimensions (number of symbol positions). The bound is also applied to shuffle-exchange graphs; for such graphs with small radix the edge-length bound is $\frac{2}{3}l_\sigma + \frac{1}{3}l_\epsilon \gtrapprox (1-2^{-d})r^{n/d}/(2(n(2-1/r)-1))$, where r is the radix, n is the number of graph dimensions, l_σ is the average length of shuffle edges, and l_ϵ is the average length of exchange edges.

Keywords: Graphs, Realization, Physical limits, Edge length.

1. Introduction

A graph *realization* is an assignment of positional and other information to the elements of a graph; it can be used to bound the performance of an interconnection network based on the graph. In [1] Vitányi describes a graph realization where processors have bounded volume; a lower bound on the average physical edge length is derived for such graph realizations. The bound is based on the graph's diameter, symmetry, and path distribution. It is derived by partitioning the graph's edges into the *orbits* induced by a group of automorphisms on the graph. (Two edges belong to the same orbit iff there is an automorphism in which one edge is the image of the other.) An average edge length is found which applies to all edges in an orbit. The bound is easily applied to many popular graphs such as trees, d-dimensional toroidal meshes, and cube-connected cycles. The bound on average

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edge length using this technique is $\Omega(1)$ for a complete N-vertex binary tree and $\Omega(N^{1/3}/\log N)$ for the hypercube and cube-connected cycles [1].

The bounds are not easy to apply to less-symmetric graphs, in particular the de Bruijn and shuffle-exchange graphs. Any group of automorphisms would divide the edges in these graphs into many orbits, yielding either a trivial or unwieldy bound.

A modified version of Vitányi's bound, easily applied to the de Bruijn and other graphs, will be presented here. Also presented is a more precise realization definition. The remainder of the paper is organized as follows. Preliminaries appear in Section 2; realization is defined in Section 3. The bound is described in Section 4; it is applied to de Bruijn and shuffle-exchange graphs in Section 5. Conclusions appear in Section 6.

2. Preliminaries

For integers a and b, $a \le b$, define $[a,b]$ to be the set $\{a, a+1, \ldots, b-1, b\}$. For integers a and b, $a < b$, define $[a,b)$ to be the set $\{a, a+1, \ldots, b-1\}$. Let r and n be positive integers, $z \in [0, r^n)$, and $a, b \in [-1, n]$. Define

$$z_{(a:b)} = \begin{cases} \lfloor z r^{-b} \rfloor \bmod r^{a-b+1}, & \text{if } a \ge b \ge 0; \\ 0, & \text{otherwise;} \end{cases}$$

and $z_{(a)} = z_{(a:a)}$. Let $x, y \in [0, r^n)$ and $a, b \in [-1, n]$. Then define $xy_{(a:b)} = xr^{a-b+1} + y_{(a:b)}$.

Let $G = (V, E)$ be any graph. A *route*, $P(u,v)$, for $u, v \in V$ is a sequence of edges constituting a path from u to v. (The paths can possibly be cyclic.) Notation $|P(u,v)|$ is the number of edges in the route; $P(u,v,0)$, $P(u,v,1), \ldots,$ $P(u,v,|P(u,v)|-1)$ are the edges in a non-empty route. Define $\sum_{e \in P(u,v)} l(e) = \sum_{i=0}^{|P(u,v)|-1} l(P(u,v,i))$, where l is some mapping of E. A routing, denoted P, for a graph is a collection of routes, exactly one for each vertex pair.

For positive integers r and n, the r,n de Bruijn graph, $G_{\mathrm{dB}} = (V_{\mathrm{dB}}, E_{\mathrm{dB}})$, is defined to be an undirected graph with vertex set $V_{\mathrm{dB}} = [0, r^n)$ and edge set $E_{\mathrm{dB}} = \{\, \langle v,\, v_{(n-2:0)} a_{(0)} \rangle \mid v \in V_{\mathrm{dB}},\ a \in [0, r) \,\}$. See [2] for a description of the work upon which the graph is based, [3] for a discussion of the graph's application to parallel processing, and [4] for an introductory description.

Let $u, v \in V_{\mathrm{dB}}$ be two vertices in an r,n de Bruijn graph. Then the *normal route* between these vertices is defined to be the sequence of edges $P_{\mathrm{dB}}(u,v,i)$ for $i \in [0, n)$ where

$$P_{\mathrm{dB}}(u,v,i) = \langle u_{(n-1-i:0)} v_{(n-1:n-i)},\, u_{(n-2-i:0)} v_{(n-1:n-i-1)} \rangle. \tag{1}$$

For positive integers r and n, the r,n shuffle-exchange graph, denoted $G_{\mathrm{se}} = (V_{\mathrm{se}}, E_{\mathrm{se}})$, is defined to be an undirected graph with vertex set $V_{\mathrm{se}} = [0, r^n)$ and edge set $E_{\mathrm{se}} = E_{\mathrm{se}\sigma} \cup E_{\mathrm{se}\epsilon}$, where $E_{\mathrm{se}\sigma} = \{\, \langle v,\, v_{(n-2:0)} v_{(n-1)} \rangle \mid v \in V_{\mathrm{se}} \,\}$ and $E_{\mathrm{se}\epsilon} = \{\, \langle v,\, v_{(n-1:1)} a_{(0)} \rangle \mid v \in V_{\mathrm{se}},\ a \in ([0, r) - \{v_{(0)}\}) \,\}$. Edges in $E_{\mathrm{se}\sigma}$ are called the *shuffle* edges and edges in $E_{\mathrm{se}\epsilon}$ are called the *exchange* edges. Note that exactly

r shuffle edges form self loops while no exchange edges form self loops. See [5] for an early use of this graph for parallel processing and [4] for an introductory description.

Let $u, v \in V_{\rm se}$ be two vertices in an r, n shuffle-exchange graph. The normal route between these vertices is defined in terms of a *raw normal route*, denoted P'. The raw normal route is divided into $n-1$ *shuffle steps* and n *exchange steps*. Define exchange step i of the raw normal route, $P'_{{\rm se}\epsilon}(u,v,i)$, to be $\langle u_{(n-1-i:1)}v_{(n-1:n-i)}u_{(n-i \bmod n)},\, u_{(n-1-i:1)}v_{(n-1:n-1-i)}\rangle$ if $u_{(n-i \bmod n)} \neq v_{(n-i-1)}$ or ϕ otherwise, for $0 \leq i < n$. Shuffle step i of the raw normal route is defined by $P'_{{\rm se}\sigma}(u,v,i) = \langle u_{(n-1-i:1)}v_{(n-1:n-i-1)},\, u_{(n-2-i:1)}v_{(n-1:n-i-1)}u_{(n-1-i)}\rangle$, for $0 \leq i < n-1$. The normal route is constructed from P' by eliminating all raw-route steps with value ϕ from the sequence $P'_{{\rm se}\epsilon}(u,v,0), P'_{{\rm se}\sigma}(u,v,0), P'_{{\rm se}\epsilon}(u,v,1), P'_{{\rm se}\sigma}(u,v,1), \ldots, P'_{{\rm se}\epsilon}(u,v,n-1)$ and then assigning the element in position i of the resulting sequence to $P_{\rm se}(u,v,i)$ for all remaining elements. Note that the minimum route length using this route is $n-1$ (shuffle edges only) and the maximum length is $2n-1$, the diameter of the graph [4].

3. Realization

Definition 1 *Let $\Re$ be the set of real numbers. A d-*dimensional coordinate *(or* coordinate *when d is understood) is a d-tuple $(x_1, x_2, \ldots, x_d)$ where $x_i \in \Re$ for $0 < i \leq d$. Symbol $\mathbf{C_d}$ will denote the set of all d-dimensional coordinates.*

Definition 2 *Let $X = (x_1, x_2, \ldots, x_d)$ and $Y = (y_1, y_2, \ldots, y_d)$ be two coordinates. Define $L(X, Y)$ to be the physical (Cartesian) distance between X and Y, $(\sum_{i=1}^{d}(x_i - y_i)^2)^{1/2}$.*

The premise upon which the bound described here and in [1] is based is that a limited number of lower-bounded-volume processors can be located in a finite space. That premise is defined rigorously here using the concept of a bounding sphere. As a result, the bound can be applied to processors of any shape. (Note that vertices represent processors.)

It is meaningless to speak of physical distance without simultaneously considering the volume a vertex might occupy, with the volume perhaps being determined by an implementation technology. A realization in which volume is considered will be called a *stereo-realization*.

Definition 3 *Let (V, E) be a graph. A d-*dimensional stereo-realization *of graph (V, E) is a four tuple $(V, E, (T, S))$, where $T : V \to \mathbf{C_d}$ is a one-to-one mapping from vertices to coordinates and $S : \mathbf{C_d} \to V \cup \{\phi\}$ is an onto mapping from coordinates to vertices and ϕ. If $v \in V$, $c \in \mathbf{C_d}$, and $S(c) = v$, then point c is said to be* occupied *by v. Vertex $v \in V$ is said to be* located *at $T(v)$. Let $v \in V$; call set $\sigma(v) = \{\, c \mid c \in \mathbf{C_d},\ S(c) = v \,\}$ vertex v's* space *and let* VOL(v) *denote the volume of v's space.*

Propagation time within processors is as important as propagation time between processors. Therefore the distance between a vertex and any of the points in the vertex's space will be bounded by a constant, ρ. For simplicity, the minimum volume of a vertex's space will be normalized to the volume of a unit sphere.

Definition 4 *A d-dimensional stereo-realization $(V, E, (T, S))$ is said to be* normal *within bound ρ if* $\mathrm{VOL}(v) \geq 2^{\lceil d/2 \rceil}\pi^{\lfloor d/2 \rfloor}/d!!$ *and* $\max\{L(c, T(v)) \mid c \in \sigma(v)\} < \rho$, *for all $v \in V$, where $n!! = n(n-2)(n-4)\cdots\nu$ ($\nu = 1$ or $\nu = 2$) is the double factorial of n. For brevity, the term ρ-normal realization will be used for such a realization.*

Lemma 1 *Let $(V, E, (T, S))$ be a ρ-normal realization. Then a sphere of radius x will be occupied by no more than $(x+\rho)^d$ vertices.*

Proof. A vertex is said to be *contained* in a sphere if all its space falls within the sphere. Consider a sphere of radius x. A vertex can be located within this sphere while some of its space lies outside the sphere. Consider a concentric sphere with radius $x + \rho$. A vertex within the smaller sphere must be contained in the larger sphere. At most $(x+\rho)^d$ vertices can be contained in the larger sphere. $\Box$

4. Lower Bound

Let $G = (V, E, (T, S))$ be ρ-normal realization of a graph and $\langle u, v \rangle \in E$. Define $L(u, v) = L(T(u), T(v))$ and then define $l(\langle u, v \rangle) = L(u, v)$. Let P be a routing for G and $u, v \in V$; define $l(u, v) = \sum_{e \in P(u,v)} l(e)$. (The routing, P, is implicit in $l(u, v)$.)

The following lemma is similar to one described in [1] (they would be identical if the routing from a vertex to all others formed a minimum-spanning tree, if the path from v to v were not considered, and I were eliminated):

Lemma 2 *Edge lengths in ρ-normal realization $(V, E, (T, S))$ under routing P are constrained by*

$$\sum_{v \in V} \sum_{e \in P(u,v)} l(e) \geq |V|(1 - 2^{-d} - I)\frac{|V|^{1/d}}{2}, \tag{2}$$

for all $u \in V$, where $I = \sum_{i=0}^{d-1} |V|^{(i/d)-1} 2^{-i} \rho^{d-i} \binom{d}{i}$.

Proof. By Lemma 1 there are at least $|V|(1 - 2^{-d} - I)$ vertices that are not contained in a radius-($\frac{1}{2}|V|^{1/d}$) sphere centered on u. Suppose v is a vertex not in the sphere. Then $l(u, v) \geq L(u, v) \geq \frac{1}{2}|V|^{1/d}$ and so $\sum_{e \in P(u,v)} l(e) \geq \frac{1}{2}|V|^{1/d}$. If v is a vertex not inside the sphere then $l(u, v) \geq 0$. The lemma is obtained by summing the physical length of the routes from u to all $|V|$ vertices [1]. $\Box$

Those ρ-normal realizations in which I might be considered small will be called *well proportioned.*

Relation (2) bounds the total route length from a vertex, but does not bound the length of individual edges. A bound on the average edge length is obtained from (2) using automorphisms. For example, consider a graph in which an automorphism group can map an edge to any other. Then every edge can appear at any position in every route. This is used to obtain an average edge length by summing (2) over the automorphisms. A bound can also be obtained when the automorphisms partition the edges into more than one orbit; see [1]. Here routing symmetry rather than graph symmetry is exploited.

Let $(V, E, (T, S))$ be a ρ-normal realization with routing P. The total length of routes between all vertex pairs is constrained by

$$\sum_{u,v\in V}\sum_{e\in P(u,v)} l(e) \geq |V|^2(1-2^{-d}-I)\frac{|V|^{1/d}}{2}. \tag{3}$$

If every edge appears in the summation the same number of times then an average edge length can easily be obtained. For some graphs and routings two edges may be summed a different number of times. (*I.e.*, some edges are used more than others in a routing.) This is true for the shuffle-exchange graph under the normal routing. These cases are handled by partitioning the set of edges into a *use partition*.

Let $G=(V,E)$ be a graph, $e\in E$, and P be a routing for the graph. Define $c(e)$ to be the number of times e appears in routing P. That is

$$c(e)=\sum_{u,v\in V}\sum_{e'\in P(u,v)} c(e,e'),$$

where $c(e,e')=1$ if $e=e'$ and $c(e,e')=0$ otherwise.

Definition 5 *Edge sets $E_1,E_2,\ldots,E_t$ are a* use partition *of graph (V,E) under routing P if $E=\cup_{i=1}^{t}E_i$, if $E_1,E_2,\ldots,E_t$ are disjoint, and if for all $i\in[1,t]$ and for all $e_1,e_2\in E_i$, $c(e_1)=c(e_2)$.*

Let E_j be a set in a use partition and $e\in E_j$; define $c(E_j)=c(e)$.

Lemma 3 *If $E_1,E_2,\ldots,E_t$ are a use partition of graph (V,E) under route P with ρ-normal realization $(V,E,(T,S))$ then*

$$\sum_{v,u\in V}\sum_{e\in P(u,v)} l(e)=\sum_{j=1}^{t} c(E_j)\sum_{e\in E_j} l(e).$$

Proof. The right hand side is obtained by rearranging and combining terms appearing on the left hand side using the use-partition definition. □

For graph (V,E) with routing P define the *average distance* to be

$$\overline{D}=|V|^{-2}\sum_{u,v\in V}|P(u,v)|.$$

For a ρ-normal realization of the graph define

$$l_j=\sum_{e\in E_j} l(e)/|E_j| \qquad \text{and} \qquad \delta_j=c(E_j)\,|E_j|\,/(\overline{D}\,|V|^2),$$

where $E_1,E_2,\ldots,E_t$ form a use partition. The quantity l_j is the average length of edges in set E_j and δ_j is called the *usage* of edges in set E_j.

Theorem 1 *The average edge length in ρ-normal realization $(V,E,(T,S))$ with routing P and use-partition $E_1,E_2,\ldots,E_t$ is constrained by*

$$\sum_{j=1}^{t} l_j \geq \sum_{j=1}^{t}\delta_j l_j \geq \frac{(1-2^{-d}-I)|V|^{1/d}}{2\overline{D}}, \tag{4}$$

where $I=\sum_{i=0}^{d-1}|V|^{(i/d)-1}2^{-i}\rho^{d-i}\binom{d}{i}$.

Proof. Rel. (3) is obtained by summing (2) for all vertices. Applying Lemma 3 to (3) yields the relation $\sum_{j=1}^{t} c(E_j) \sum_{e \in E_j} l(e) \geq |V|^2(1 - 2^{-d} - I)\frac{1}{2}|V|^{1/d}$. Divide each side by $\overline{D}\,|V|^2$, multiply the left-hand size by $\frac{|E_j|}{|E_j|}$, and substitute l_j and δ_j to obtain $\sum_{j=1}^{t} \delta_j l_j$. Since $\max_{j=1}^{t} \delta_j \leq 1$, $\sum_{j=1}^{t} l_j \geq \sum_{j=1}^{t} \delta_j l_j$. $\square$

5. Application

Application of the bound requires a use partition for a graph. Let $G_{\mathrm{dB}} = (V_{\mathrm{dB}}, E_{\mathrm{dB}})$ be an r, n de Bruijn graph under the normal routing, P_{dB}.

Lemma 4 *Set E_{dB} is a use partition and $c(E_{\mathrm{dB}}) = nr^{n-1}$.*

Proof. Define $c(i, e) = \{\, (u, v) \mid u, v \in V_{\mathrm{dB}},\ e = P_{\mathrm{dB}}(u, v, i) \,\}$. By enumeration of pairs $u, v \in V_{\mathrm{dB}}$ for which the right-hand side of (1) is equal to $e \in E_{\mathrm{dB}}$ it can be seen that $|c(i, e)| = r^{n-1}$ for all $0 \leq i < n$ and $e \in E_{\mathrm{dB}}$. That is, each edge will appear in position i in the route between r^{n-1} distinct vertex pairs. This is true for all i therefore $c(e) = nr^{n-1}$ for all $e \in E_{\mathrm{dB}}$. Thus, E_{dB} is a use partition of E_{dB}. $\square$

Theorem 2 *The average edge length, l, in any ρ-normal realization of an r, n de Bruijn graph $(V_{\mathrm{dB}}, E_{\mathrm{dB}}, (S, T))$ is constrained by*

$$l \geq (1 - 2^{-d} - I)\frac{r^{n/d}}{2n}.$$

For a well-proportioned 3-dimensional realization, $l \gtrapprox 7r^{n/3}/(16n)$.

Proof. Consider routing P_{dB}. All edges will be placed in a single use partition, E_{dB}. Using $c(E_{\mathrm{dB}})$ from Lemma 4 the usage of E_{dB} is $\delta = \frac{nr^{n-1}\,nr^n}{nr^{2n}} = 1$. Since all routes are of length n, $\overline{D}_{\mathrm{dB}} = n$. Substituting these into (4) yields $l \geq (1 - 2^{-d} - I)\frac{1}{2n}|V|^{1/d}$. Substituting $|V| = r^n$ gives the bound $l \geq (1 - 2^{-d} - I)\frac{1}{2n}r^{n/d}$. For well-proportioned 3-dimensional realizations, $l \gtrapprox \frac{7r^{n/3}}{16n}$. $\square$

Let $G_{\mathrm{se}} = (V_{\mathrm{se}}, E_{\mathrm{se}})$, $E_{\mathrm{se}} = E_{\mathrm{se}\sigma} \cup E_{\mathrm{se}\epsilon}$, be a shuffle-exchange graph.

Lemma 5 *$E_{\mathrm{se}\sigma}$ and $E_{\mathrm{se}\epsilon}$ form a use partition of G_{se} under P_{se}.*

Proof. Consider any shuffle edge $\langle u,\, v\rangle \in E_{\mathrm{se}\sigma}$. The digits in the radix-r representation of v are the same as the digits in the radix-r representation of u, only the order is changed. Consider any exchange edge $\langle u,\, v\rangle \in E_{\mathrm{se}\epsilon}$. The digits in v are not the same as the digits in u, because v is obtained by changing exactly one digit in u. Therefore $E_{\mathrm{se}\sigma} \cap E_{\mathrm{se}\epsilon} = \phi$. By definition, $E_{\mathrm{se}} = E_{\mathrm{se}\sigma} \cup E_{\mathrm{se}\epsilon}$. By enumerating vertex pairs used in the normal route it can be seen that every edge $e_1 \in E_{\mathrm{se}\sigma}$ is used in raw step $i \in [0, n-2]$ of exactly r^n vertex-pair routes. Since every shuffle edge can be used in all the $n-1$ steps that use a shuffle edge, $c(E_{\mathrm{se}\sigma}) = (n-1)r^n$. Also by enumeration, every edge in $e_1 \in E_{\mathrm{se}\epsilon}$ is used in raw step i by exactly r^{n-1} vertex-pair routes. Since every exchange edge can be used in every raw step (although for a given raw step it cannot be used for all vertex pairs), $c(E_{\mathrm{se}\epsilon}) = nr^{n-1}$. All conditions for a use partition are thus satisfied. $\square$

Theorem 3 *The average edge length of a shuffle edge, l_σ, and the average edge length of an exchange edge, l_ϵ, in any ρ-normal realization of an r, n shuffle-exchange graph*

$(V_{\mathrm{se}}, E_{\mathrm{se}}, (S,T))$ *is constrained by*

$$\frac{n-1}{n(2-1/r)-1}l_\sigma + \frac{n(1-1/r)}{n(2-1/r)-1}l_\epsilon \geq \frac{(1-2^{-d}-I)r^{n/d}}{2(n(2-1/r)-1)}.$$

Proof. Consider routing P_{se}. By Lemma 5 the edges are divided into two use partitions, $E_{\mathrm{se}\sigma}$ and $E_{\mathrm{se}\epsilon}$. Quantity $\overline{D}_{\mathrm{se}}$ is computed by enumerating routes by the number of exchange edges used: $\overline{D}_{\mathrm{se}} = \frac{1}{r^{2n}}\sum_{x=0}^{n}(n+x-1)\binom{n}{x}r^{n-x}(r(r-1))^x = n\left(2-\frac{1}{r}\right)-1$. The usages are then $\delta_\sigma = \frac{n-1}{n(2-1/r)-1}$ and $\delta_\epsilon = \frac{n(1-1/r)}{n(2-1/r)-1}$. Substituting these and $|V| = r^n$ into (4) yields the constraint. □

The denominators in Theorem 3 can be cancelled yielding $(n-1)l_\sigma + n(1-1/r)l_\epsilon \geq (1-2^{-d}-I)\frac{1}{2}r^{n/d}$, a form that some might prefer. Perhaps of special interest is the binary shuffle-exchange graph. The edge constraint for a well-proportioned 3-dimensional realization of this graph when n is large is $2l_\sigma + l_\epsilon \gtrapprox \frac{7}{8}2^{n/3}$.

6. Conclusions

The bound presented gives a constraint on average edge length for a graph realization. The bound is similar to Vitányi's but can be applied to less symmetric graphs. The bound could also be applied to symmetric graphs in which the routing induces a manageable use partition. The bound makes use of a more precise realization definition.

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