Mathematics for 3D Graphics

Topics

Points, Vectors, Vertices, Coordinates

Dot Products, Cross Products

Lines, Planes, Intercepts

References

Many texts cover the linear algebra used for 3D graphics . . .
. . . the texts below are good references, Akenine-Möller is more relevant to the class.

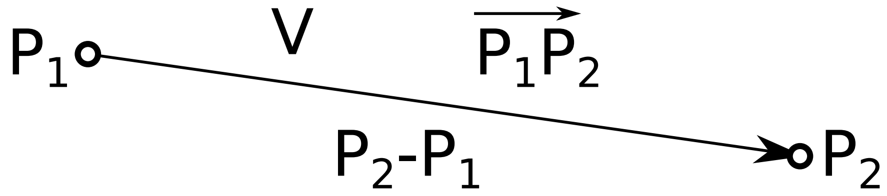
Appendix A in T. Akenine-Möller, E. Haines, N. Hoffman, "Real-Time Rendering," Third Edition, A. K. Peters Ltd.

Appendix A in Foley, van Dam, Feiner, Huges, "Computer Graphics: Principles and Practice," Second Edition, Addison Wesley.

Points and Vectors

Point:

Indivisible location in space.



$$E.g., P_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, P_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Vector:

Difference between two points.

$$E.g., V = P_2 - P_1 = \overrightarrow{P_1P_2} = \begin{bmatrix} 4-1 \\ 5-2 \\ 6-3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

Equivalently: $P_2 = P_1 + V$.

Don't confuse points and vectors!

Point-Related Terminology

Will define several terms related to points.

At times they may be used interchangeably.

Point:
A location in space.

Coordinate:
A representation of location.

Vertex:
Term may mean point, coordinate, or part of graphical object.

As used in class, vertex is a less formal term.

It might refer to a point, its coordinate, and other info like color.

Coordinate:
A representation of where a point is located.

Familiar representations:

3D Cartesian $P = (x, y, z)$.

2D Polar $P = (r, \theta)$.

In class we will use 3D *homogeneous coordinates*.

Homogeneous Coordinates

math-5

Homogeneous Coordinate:

A coordinate representation for points in 3D space consisting of four components...

... each component is a real number...

... and the last component is non-zero.

Representation: $P = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$, where $w \neq 0$.

P refers to same point as Cartesian coordinate $(x/w, y/w, z/w)$.

To save paper sometimes written as (x, y, z, w).

Homogeneous Coordinates

math-6

Each point can be described by many homogeneous coordinates ...

$$\ldots \text{ for example, } (10,20,30) = \begin{bmatrix}10\\20\\30\\1\end{bmatrix} = \begin{bmatrix}5\\10\\15\\0.5\end{bmatrix} = \begin{bmatrix}20\\40\\60\\2\end{bmatrix} = \begin{bmatrix}10w\\20w\\30w\\w\end{bmatrix} = \ldots$$

... these are all equivalent so long as $w \neq 0$.

Column matrix $\begin{bmatrix}x\\y\\z\\0\end{bmatrix}$ could not be a homogeneous coordinate ...

... but it could be a vector.

Homogeneous Coordinates

math-7

Why not just Cartesian coordinates like (x, y, z)?

The w simplifies certain computations.

Confused?

Then for a little while pretend that $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$ is just (x, y, z).

Homogenized Homogeneous Coordinate

A homogeneous coordinate is *homogenized* by dividing each element by the last.

For example, the homogenization of $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is $\begin{bmatrix} x/w \\ y/w \\ z/w \\ 1 \end{bmatrix}$

Homogenization is also known as *perspective divide*.

Vector Arithmetic

Points just sit there, it's vectors that do all the work.

In other words, most operations defined on vectors.

Point/Vector Sum

The result of adding a point to a vector is a point.

Consider point with homogenized coordinate $P = (x, y, z, 1)$ and vector $V = (i, j, k)$.

The sum $P + V$ is the point with coordinate $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} + \begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} x+i \\ y+j \\ z+k \\ 1 \end{bmatrix}$

This follows directly from the vector definition.

Scalar/Vector Multiplication

The result of multiplying scalar a with a vector is a vector...
... that is a times longer but points in the same or opposite direction...
... if $a \neq 0$.

Let a denote a scalar real number and V a vector.

The *scalar vector product* is $$aV = a \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax \\ ay \\ az \end{bmatrix}.$$

Vector/Vector Addition

The result of adding two vectors is another vector.

Let $V_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ denote two vectors.

The *vector sum*, denoted $U + V$, is $\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$

Vector subtraction could be defined similarly…

… but doesn't need to be because we can use scalar/vector multiplication: $V_1 - V_2 = V_1 + (-1 \times V_2)$.

Vector Addition Properties

Vector addition is associative:

$$U + (V + W) = (U + V) + W.$$

Vector addition is commutative:

$$U + V = V + U.$$

Vector Magnitude, Normalization

math-13

Vector Magnitude

The magnitude of a vector is its length, a scalar.

The *magnitude* of $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ denoted $\|V\|$, is $\sqrt{x^2 + y^2 + z^2}$.

The magnitude is also called the *length* and the *norm*.

Vector V is called a *unit vector* if $\|V\| = 1$.

A vector is *normalized* by dividing each of its components by its length.

The notation $\hat{V}$ indicates $V/\|V\|$, the normalized version of V.

math-13

Dot Product

math-14

The Vector Dot Product

The dot product of two vectors is a scalar.

Roughly, it indicates how much they point in the same direction.

Consider vectors $V_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$.

The *dot product* of V_1 and V_2, denoted $V_1 \cdot V_2$, is $x_1 x_2 + y_1 y_2 + z_1 z_2$.

What a Dot Product Does

Let

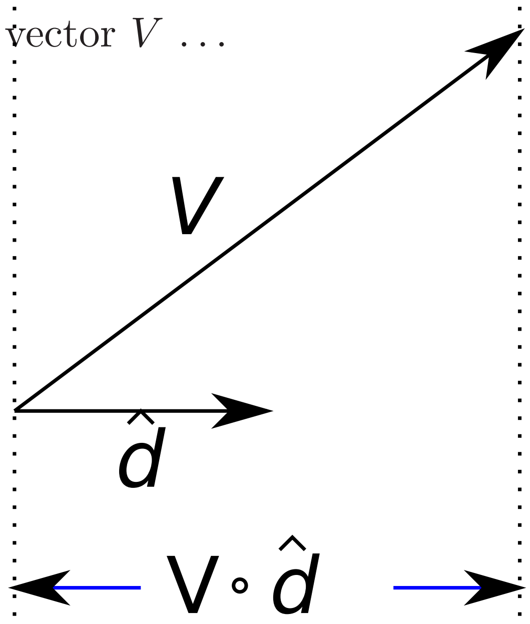
V be some arbitrary vector and

$\hat{d}$ be a unit vector.

Then $V \cdot \hat{d}\ldots$

… measures the length of the vector V …

… in the direction of $\hat{d}$.



Dot Product Properties

math-16

Let U, V, and W be vectors.

Let a be a scalar.

Miscellaneous Dot Product Properties

$$(U+V)\cdot W = U\cdot W + V\cdot W$$

$$(aU)\cdot V = a(U\cdot V)$$

$$U\cdot V = V\cdot U$$

$$\mathrm{abs}(U\cdot U) = \|U\|^2$$

math-16

Dot Product Properties

math-17

Orthogonality

The more casual term is perpendicular.

Vectors U and V are called *orthogonal* iff $U \cdot V = 0$.

This is an important property for finding intercepts.

math-17

Dot Product Properties

math-18

Angle

Let U and V be two vectors.

Then $U \cdot V = \|U\| \|V\| \cos \phi \ldots$
$\ldots$ where ϕ is the smallest angle between the two vectors.

math-18

Cross Product

The cross product of two vectors results in a vector orthogonal to both.

The *cross product* of vectors V_1 and V_2, denoted $V_1 \times V_2$, is

$$V_1 \times V_2 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}.$$

Cross Product Properties

Let U and V be two vectors and let $W = U \times V$.

Then both U and V are orthogonal to W.

$\|U \times V\| = \|U\|\|V\| \sin \phi$.

$U \times V = -V \times U$.

$(aU + bV) \times W = a(U \times W) + b(V \times W)$.

If U and V define a parallelogram, its area is $\|U \times V\|$…
… if they define a triangle its area is $\frac{1}{2}\|U \times V\|$.

Line Definition

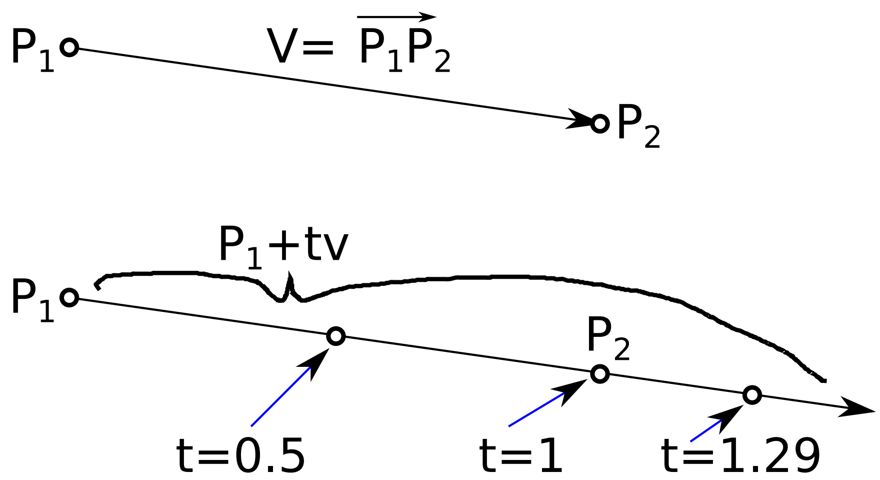
A line will be defined in terms of a point and a non-zero vector.

Line:

A set of points generated from a given point, P_1, and vector, v: $\{S \mid P_1 + tv,\ \forall t \in \Re\}$.

One can imagine "drawing" the line by varying the parameter t.

Illustration of defining a line in terms of two points:



Plane Definition

Point P and vector $\overrightarrow{n}$ define a *plane* in which a point S is on the plane iff $\overrightarrow{PS} \cdot \overrightarrow{n} = 0$.

The vector $\overrightarrow{n}$ if referred to as a *normal*.

Plane/Line Intercept

Consider the plain defined by point P and vector $\overrightarrow{n}$, and the line defined by point L and vector $\overrightarrow{v}$; let S denote the point at which the line intercepts the plane (if any).

Since S is on the line, $S = L + t\overrightarrow{v}$.

Since S is on the plane, $\overrightarrow{SP} \cdot \overrightarrow{n} = 0$

Substituting for S and solving for t:

$$\begin{aligned}
\overrightarrow{(L + t\overrightarrow{v})P} \cdot \overrightarrow{n} &= 0 \\
(P - L - t\overrightarrow{v}) \cdot \overrightarrow{n} &= 0 \\
(\overrightarrow{LP} - t\overrightarrow{v}) \cdot \overrightarrow{n} &= 0 \\
t &= \frac{\overrightarrow{LP} \cdot \overrightarrow{n}}{\overrightarrow{v} \cdot \overrightarrow{n}}
\end{aligned}$$

Use this expression for t to find S

$$S = L + \frac{\overrightarrow{LP} \cdot \overrightarrow{n}}{\overrightarrow{v} \cdot \overrightarrow{n}}\overrightarrow{v}$$

Problem: *A light model specifies that in a scene with a light of brightness b (scalar) at location L (coordinate), and a point P on a surface with normal $\hat{n}$, the* lighted color*, c, of P (a scalar) will be the dot product of the surface normal with the direction to the light divided by the distance to the light.*

Restate this as a formula.

Estimate the number of floating point operations in a streamlined computation.

Solution:

Formula: $c = b\widehat{PL} \cdot \hat{n} \frac{1}{\|\overrightarrow{PL}\|}$.

Transforms

Transformation:
A mapping (conversion) from one coordinate set to another (*e.g.*, from feet to meters) or to a new location in an existing coordinate set.

Particular Transformations to be Covered

Translation: Moving things around.

Scale: Change size.

Rotation: Rotate around some axis.

Projection: Moving to a surface.

Transform by multiplying 4×4 matrix with coordinate.

$P_{\text{new}} = M_{\text{transform}} P_{\text{old}}$.

Scale Transform

$$S(s) = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$S(s,t,u) = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$S(s)$ stretches an object s times along each axis.

$S(s,t,u)$ stretches an object s times along the x-axis, t times along the y-axis, and u times along the z-axis.

Scaling centered on the origin.

Rotation Transformations

$R_x(\theta)$ rotates around x axis by θ; likewise for R_y and R_z.

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Translation Transform

$$T(s,t,u) = \begin{pmatrix} 1 & 0 & 0 & s \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moves point s units along x axis, etc.

Miscellaneous Matrix Multiplication Math

Let M and N denote arbitrary 4×4 matrices.

Identity Matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$IM = MI = M.$$

Transforms and Matrix Arithmetic

math-30

Matrix Inverse

Matrix A is an inverse of M iff $AM = MA = I$.

Will use M^{-1} to denote inverse.

Not every matrix has an inverse.

Computing inverse of an arbitrary matrix expensive ...
... but inverse of some matrices are easy to compute ...
... for example, $T(x, y, z)^{-1} = T(-x, -y, -z)$.

Matrix Multiplication Rules

Is associative: $(LM)N = L(MN)$.

Is not commutative: $MN \neq NM$ for arbitrary M and N.

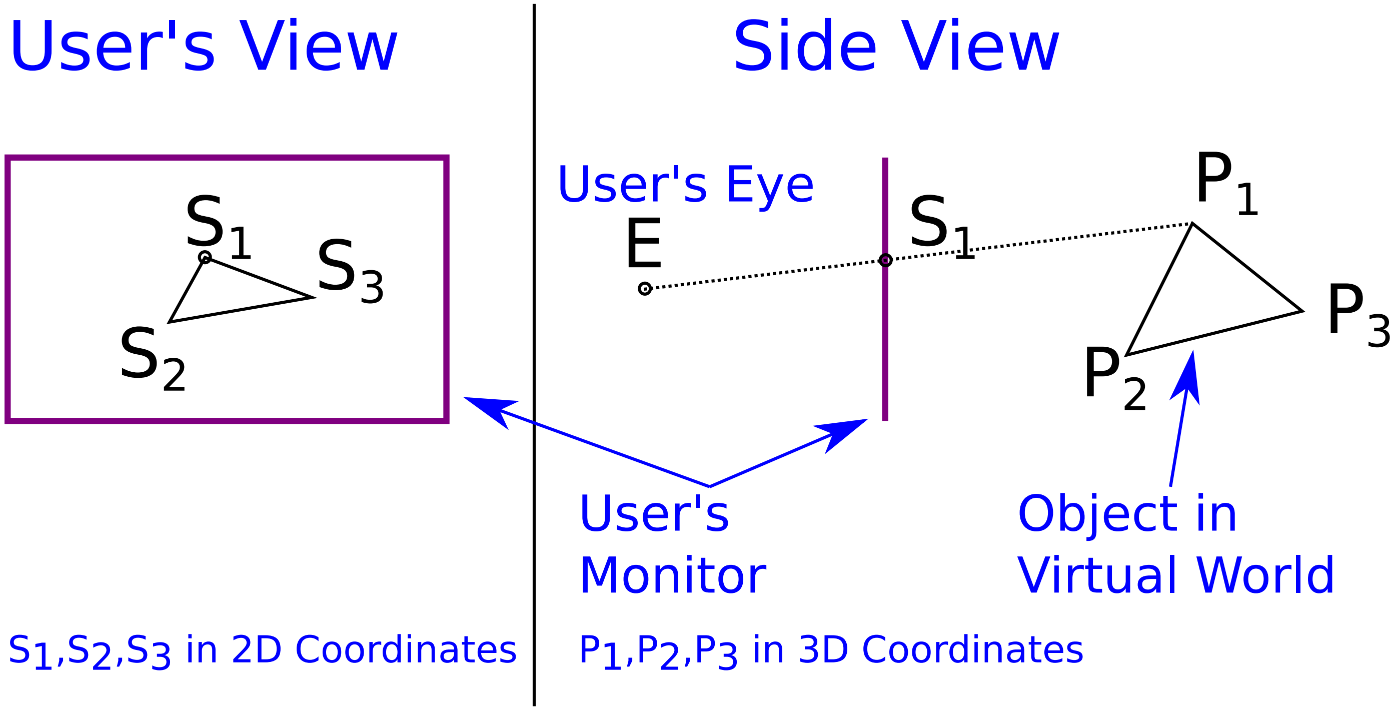
$(MN)^{-1} = N^{-1}M^{-1}$. (Note change in order.)

Projection Transform:

A transform that maps a coordinate to a space with fewer dimensions.

A projection transform maps a 3D coord. from our virtual world (such as P_1) ...

... to a 2D location on our monitor (such as S_1).



$$S_1 = T_{\text{projection}} P_1$$

Projection Types

Vague definitions on this page.

Perspective Projection

Points appear to be in "correct" location,...
... as though monitor were just a window into the simulated world.

This projection used when realism is important.

Orthographic Projection

A projection without perspective foreshortening.

This projection used when a real ruler will be used to measure distances.

Perspective Projection Derivation

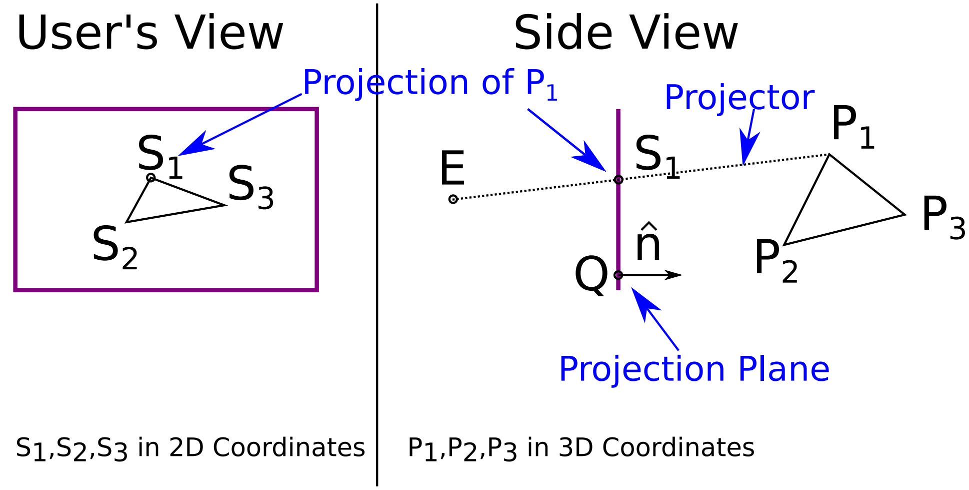
math-33

Lets put user and user's monitor in world coordinate space:

Location of user's eye: E.

A point on the user's monitor: Q.

Normal to user's monitor pointing away from user: $\hat{n}$.



Goal:

Find S_1, point where line from E to P_1 intercepts monitor (plane $Q, \hat{n}$).

Line from E to P called the *projector*.

The user's monitor is in the *projection plane*.

The point S is called the *projection* of point P on the projection plane.

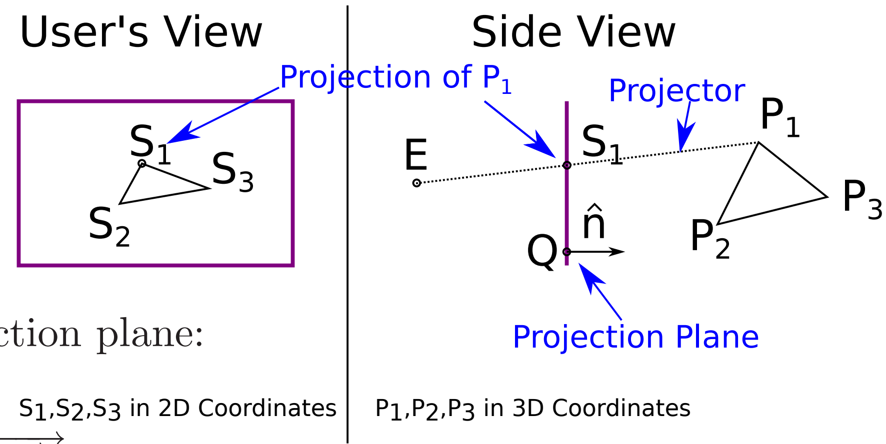
math-33

Solution:

Projector equation: $S = E + t\overrightarrow{EP}$.

Projection plane equation: $\overrightarrow{QS} \cdot n = 0$.

Find point S that's on projector and projection plane:



$$\overrightarrow{Q(E + t\overrightarrow{EP})} \cdot n = 0$$

$$(E + t\overrightarrow{EP} - Q) \cdot n = 0$$

$$\overrightarrow{QE} \cdot n + t\overrightarrow{EP} \cdot n = 0$$

$$t = \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n}$$

$$S = E + \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n}\overrightarrow{EP}$$

Note: $\overrightarrow{EQ} \cdot n$ is distance from user to plane in direction n ...

... and $\overrightarrow{EP} \cdot n$ is distance from user to point in direction n.

Perspective Projection Derivation

math-35

To simplify projection:

Fix $E = (0,0,0)$: Put user at origin.

Fix $n = (0,0,1)$: Make "monitor" parallel to xy plane.

Before: $$S = E + \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n} \overrightarrow{EP}$$

After: $$S = \frac{q_z}{p_z} P,$$

where q_z is the z component of Q, and p_z defined similarly.

The key operation in perspective projection is dividing out by z (given our geometry).

math-35

Simple Perspective Projection Transformation

math-36

Simple Projection Transform 1

Eye at origin, projection surface at (x, y, q_z), normal is $(0, 0, 1)$.

$$F_{q_z} = \begin{pmatrix} q_z & 0 & 0 & 0 \\ 0 & q_z & 0 & 0 \\ 0 & 0 & q_z & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Applying the projection to coordinate $(x, y, z, 1)$:

$$F_{q_z} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} q_z x \\ q_z y \\ q_z z \\ z \end{bmatrix} = \begin{bmatrix} \frac{q_z}{z} x \\ \frac{q_z}{z} y \\ \frac{q_z z}{z} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{q_z}{z} x \\ \frac{q_z}{z} y \\ q_z \\ 1 \end{bmatrix}$$

This maps the z coordinate to the constant q_z . . .
. . . *meaning that the position along the z axis has been lost.*

But we'll need the z position to determine visibility of overlapping objects.

math-36

Simple Perspective Projection Transformation

math-37

Simple Projection Transform, *Preserving* z

Eye at origin, projection surface at (x, y, q_z), normal is $(0, 0, 1)$.

$$F_{q_z} = \begin{pmatrix} q_z & 0 & 0 & 0 \\ 0 & q_z & 0 & 0 \\ 0 & 0 & 0 & q_z \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Applying the projection to coordinate $(x, y, z, 1)$:

$$F_{q_z} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} q_z x \\ q_z y \\ q_z \\ z \end{bmatrix} = \begin{bmatrix} \frac{q_z}{z} x \\ \frac{q_z}{z} y \\ \frac{q_z}{z} \\ 1 \end{bmatrix}$$

This maps z coordinate to q_z/z, ...
... which though a reciprocal, will still be useful.

  math-37

View Volume, Frustum

math-38

View-Volume Related Definitions

View Volume:

Parts of the scene which should be visible to the user.

Frustum:

A shape constructed by slicing off the top of a square-base pyramid with a plane parallel to the base.

Frustum View Volume Motivation

Consider the simple projection transformation:

Shape of view volume consists of two pyramids . . .
. . . one pyramid in front, the other in back, . . .
. . . and both points on eye.

Some points are behind the user. . .
. . . and we don't want these to be visible (because they would be unnatural).

Some points in view volume are so far from the user. . .
. . . that they would be invisible.

For example, points might form a triangle that covers 1% of a pixel.

These points waste computing power.

Frustum View Volume

View volume in shape of frustum with smaller square on projection plane.

The smaller square of frustum defines a *near plane*.

The larger square defines a *far plane*.

Variables describing a frustum view volume:

n: Distance from eye to near plane.

f: Distance from eye to far plane.

Coordinates of lower-left corner of $(l, b, -n)$.

Coordinates of upper-right corner of $(r, t, -n)$.

Frustum Perspective Transform

Given six values: l, r, t, b, n, f (left, right, top, bottom, near, far).

Eye at origin, projection surface at (x, y, n), normal is $(0, 0, -1)$.

Viewer screen is rectangle from $(l, b, -n)$ to $(r, t, -n)$.

Points with $z > -t$ and $z < -f$ are not of interest.

$$F_{l,r,t,b,n,f} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -2\frac{fn}{f-n} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$