Mathematics for 3D Graphics

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Topics

Points, Vectors, Vertices, Coordinates

Dot Products, Cross Products

Lines, Planes, Intercepts

References

Many texts cover the linear algebra used for 3D graphics . . .
. . . the texts below are good references, Akenine-Möller is more relevant to the class.

Appendix A in T. Akenine-Möller, E. Haines, N. Hoffman, “Real-Time Rendering,” Third Edition, A. K. Peters Ltd.

Appendix A in Foley, van Dam, Feiner, Huges, “Computer Graphics: Principles and Practice,” Second Edition, Addison Wesley.

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Points and Vectors

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Point:

Indivisible location in space.

$$E.g.,\ P_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\ P_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Vector:

Difference between two points.

$$E.g.,\ V = P_2 - P_1 = \overrightarrow{P_1P_2} = \begin{bmatrix} 4-1 \\ 5-2 \\ 6-3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

Equivalently: $P_2 = P_1 + V$.

Don't confuse points and vectors!

Point-Related Terminology

Will define several terms related to points.

At times they may be used interchangeably.

Point:
A location in space.

Coordinate:
A representation of location.

Vertex:
Term may mean point, coordinate, or part of graphical object.

As used in class, vertex is a less formal term.

It might refer to a point, its coordinate, and other info like color.

Coordinate:
A representation of where a point is located.

Familiar representations:

3D Cartesian $P = (x, y, z)$.

2D Polar $P = (r, \theta)$.

In class we will use 3D *homogeneous coordinates*.

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Homogeneous Coordinates

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Homogeneous Coordinate:

A coordinate representation for points in 3D space consisting of four components...

... each component is a real number...

... and the last component is non-zero.

Representation: $P = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$, where $w \neq 0$.

P refers to same point as Cartesian coordinate $(x/w, y/w, z/w)$.

To save paper sometimes written as (x, y, z, w).

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Homogeneous Coordinates

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Each point can be described by many homogeneous coordinates ...

... for example, $(10,20,30) = \begin{bmatrix} 10 \\ 20 \\ 30 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 20 \\ 40 \\ 60 \\ 2 \end{bmatrix} = \begin{bmatrix} 10w \\ 20w \\ 30w \\ w \end{bmatrix} = \ldots$

... these are all equivalent so long as $w \neq 0$.

Column matrix $\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$ could not be a homogeneous coordinate ...

... but it could be a vector.

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Homogeneous Coordinates

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Why not just Cartesian coordinates like (x, y, z)?

The w simplifies certain computations.

Confused?

Then for a little while pretend that $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$ is just (x, y, z).

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Homogenized Homogeneous Coordinate

A homogeneous coordinate is *homogenized* by dividing each element by the last.

For example, the homogenization of $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is $\begin{bmatrix} x/w \\ y/w \\ z/w \\ 1 \end{bmatrix}$

Homogenization is also known as *perspective divide*.

Vector Arithmetic

Points just sit there, it's vectors that do all the work.

In other words, most operations defined on vectors.

Point/Vector Sum

The result of adding a point to a vector is a point.

Consider point with homogenized coordinate $P = (x, y, z, 1)$ and vector $V = (i, j, k)$.

The sum $P + V$ is the point with coordinate $$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} + \begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} x+i \\ y+j \\ z+k \\ 1 \end{bmatrix}$$

This follows directly from the vector definition.

Scalar/Vector Multiplication

The result of multiplying scalar a with a vector is a vector...
... that is a times longer but points in the same or opposite direction...
... if $a \neq 0$.

Let a denote a scalar real number and V a vector.

The *scalar vector product* is $aV = a\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax \\ ay \\ az \end{bmatrix}$.

Vector/Vector Addition

The result of adding two vectors is another vector.

Let $V_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ denote two vectors.

The *vector sum*, denoted $U + V$, is $\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$

Vector subtraction could be defined similarly...

... but doesn't need to be because we can use scalar/vector multiplication: $V_1 - V_2 = V_1 + (-1 \times V_2)$.

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Vector Addition Properties

Vector addition is associative:

$$U + (V + W) = (U + V) + W.$$

Vector addition is commutative:

$$U + V = V + U.$$

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Vector Magnitude, Normalization

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Vector Magnitude

The magnitude of a vector is its length, a scalar.

The *magnitude* of $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ denoted $\|V\|$, is $\sqrt{x^2 + y^2 + z^2}$.

The magnitude is also called the *length* and the *norm*.

Vector V is called a *unit vector* if $\|V\| = 1$.

A vector is *normalized* by dividing each of its components by its length.

The notation $\hat{V}$ indicates $V/\|V\|$, the normalized version of V.

Dot Product

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The Vector Dot Product

The dot product of two vectors is a scalar.

Roughly, it indicates how much they point in the same direction.

Consider vectors $V_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$.

The *dot product* of V_1 and V_2, denoted $V_1 \cdot V_2$, is $x_1x_2 + y_1y_2 + z_1z_2$.

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Dot Product Properties

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Let U, V, and W be vectors.

Let a be a scalar.

Miscellaneous Dot Product Properties

$$(U+V)\cdot W = U\cdot W + V\cdot W$$

$$(aU)\cdot V = a(U\cdot V)$$

$$U\cdot V = V\cdot U$$

$$\mathrm{abs}(U\cdot U) = \|U\|^2$$

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Dot Product Properties

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Orthogonality

The more casual term is perpendicular.

Vectors U and V are called *orthogonal* iff $U \cdot V = 0$.

This is an important property for finding intercepts.

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Dot Product Properties

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Angle

Let U and V be two vectors.

Then $U \cdot V = \|U\| \|V\| \cos \phi \ldots$
$\ldots$ where ϕ is the smallest angle between the two vectors.

Cross Product

The cross product of two vectors results in a vector orthogonal to both.

The *cross product* of vectors V_1 and V_2, denoted $V_1 \times V_2$, is

$$V_1 \times V_2 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}.$$

Cross Product Properties

Let U and V be two vectors and let $W = U \times V$.

Then both U and V are orthogonal to W.

$\|U \times V\| = \|U\|\|V\| \sin \phi$.

$U \times V = -V \times U$.

$(aU + bV) \times W = a(U \times W) + b(V \times W)$.

If U and V define a parallelogram, its area is $\|U \times V\|$...
... if they define a triangle its area is $\frac{1}{2}\|U \times V\|$.

Line Definition

A line will be defined in terms of a point and a non-zero vector.

Line:
A set of points generated from a given point, L, and vector, v: $\{S \mid L + tv, \forall t \in \Re\}$.

One can imagine "drawing" the line by varying the parameter t.

Plane Definition

Point P and vector $\overrightarrow{n}$ define a *plane* in which a point S is on the plane iff $\overrightarrow{PS} \cdot \overrightarrow{n} = 0$.

The vector $\overrightarrow{n}$ if referred to as a *normal*.

Plane/Line Intercept

Consider the plain defined by point P and vector $\overrightarrow{n}$, and the line defined by point L and vector $\overrightarrow{v}$; let S denote the point at which the line intercepts the plane (if any).

Since S is on the line, $S = L + t\overrightarrow{v}$.

Since S is on the plane, $\overrightarrow{SP} \cdot \overrightarrow{n} = 0$

Substituting for S and solving for t:

$$\begin{aligned} \overrightarrow{(L + t\overrightarrow{v})P} \cdot \overrightarrow{n} &= 0 \\ (P - L - t\overrightarrow{v}) \cdot \overrightarrow{n} &= 0 \\ (\overrightarrow{LP} - t\overrightarrow{v}) \cdot \overrightarrow{n} &= 0 \\ t &= \frac{\overrightarrow{LP} \cdot \overrightarrow{n}}{\overrightarrow{v} \cdot \overrightarrow{n}} \end{aligned}$$

Use this expression for t to find S

$$S = L + \frac{\overrightarrow{LP} \cdot \overrightarrow{n}}{\overrightarrow{v} \cdot \overrightarrow{n}} \overrightarrow{v}$$

Sample Problem

Problem: *A light model specifies that in a scene with a light of brightness b (scalar) at location L (coordinate), and a point P on a surface with normal $\hat{n}$, the lighted color, c, of P (a scalar) will be the dot product of the surface normal with the direction to the light divided by the distance to the light.*

Restate this as a formula.

Estimate the number of floating point operations in a streamlined computation.

Solution:

Formula: $c = b\widehat{PL} \cdot \hat{n} \frac{1}{\|\overrightarrow{PL}\|}$.

Transformation:

A mapping (conversion) from one coordinate set to another (*e.g.*, from feet to meters) or to a new location in an existing coordinate set.

Particular Transformations to be Covered

Translation: Moving things around.

Scale: Change size.

Rotation: Rotate around some axis.

Projection: Moving to a surface.

Transform by multiplying 4×4 matrix with coordinate.

$$P_{\text{new}} = M_{\text{transform}} P_{\text{old}}.$$

Scale Transform

$$S(s)=\begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$S(s,t,u)=\begin{pmatrix} s & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$S(s)$ stretches an object s times along each axis.

$S(s,t,u)$ stretches an object s times along the x-axis, t times along the y-axis, and u times along the z-axis.

Scaling centered on the origin.

Rotation Transformations

$R_x(\theta)$ rotates around x axis by θ; likewise for R_y and R_z.

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Translation Transform

$$T(s,t,u) = \begin{pmatrix} 1 & 0 & 0 & s \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moves point s units along x axis, etc.

Miscellaneous Matrix Multiplication Math

Let M and N denote arbitrary 4×4 matrices.

Identity Matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$IM = MI = M.$$

Matrix Inverse

Matrix A is an inverse of M iff $AM = MA = I$.

Will use M^{-1} to denote inverse.

Not every matrix has an inverse.

Computing inverse of an arbitrary matrix expensive ...
... but inverse of some matrices are easy to compute ...
... for example, $T(x, y, z)^{-1} = T(-x, -y, -z)$.

Matrix Multiplication Rules

Is associative: $(LM)N = L(MN)$.

Is not commutative: $MN \neq NM$ for arbitrary M and N.

$(MN)^{-1} = N^{-1}M^{-1}$. (Note change in order.)

Projection Transformations

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Projection Transform:

A transform that maps a coordinate to a space with fewer dimensions.

A projection transform will map a 3D coordinate from our physical or graphical model . . .

. . . to a 2D location on our monitor (or a window).

Projection Types

Vague definitions on this page.

Perspective Projection

Points appear to be in "correct" location,. . .

. . . as though monitor were just a window into the simulated world.

This projection used when realism is important.

Orthographic Projection

A projection without perspective foreshortening.

This projection used when a real ruler will be used to measure distances.

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Perspective Projection Derivation

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Lets put user and user's monitor in world coordinate space:

Location of user's eye: E.

A point on the user's monitor: Q.

Normal to user's monitor pointing away from user: n.

Goal:

Find S, point where line from E to P intercepts monitor (plane Q, n).

Line from E to P called the *projector*.

The user's monitor is in the *projection plane*.

The point S is called the *projection* of point P on the projection plane.

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Solution:

Projector equation: $S = E + t\overrightarrow{EP}$.

Projection plane equation: $\overrightarrow{QS} \cdot n = 0$.

Find point S that's on projector and projection plane:

$$\overrightarrow{Q(E + t\overrightarrow{EP})} \cdot n = 0$$

$$(E + t\overrightarrow{EP} - Q) \cdot n = 0$$

$$\overrightarrow{QE} \cdot n + t\overrightarrow{EP} \cdot n = 0$$

$$t = \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n}$$

$$S = E + \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n}\overrightarrow{EP}$$

Note: $\overrightarrow{EQ} \cdot n$ is distance from user to plane in direction n ...

... and $\overrightarrow{EP} \cdot n$ is distance from user to point in direction n.

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Perspective Projection Derivation

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To simplify projection:

Fix $E = (0,0,0)$: Put user at origin.

Fix $n = (0,0,-1)$: Make "monitor" parallel to xy plane.

Result:

$$S = E + \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n} \overrightarrow{EP}$$

$$S = \frac{q_z}{p_z} P,$$

where q_z is the z component of Q, and p_z defined similarly.

The key operation in perspective projection is dividing out by z (given our geometry).

Simple Perspective Projection Transformation

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Simple Projection Transform

Eye at origin, projection surface at $(x, y, -q_z)$, normal is $(0, 0, -1)$.

$$F_{q_z} = \begin{pmatrix} q_z & 0 & 0 & 0 \\ 0 & q_z & 0 & 0 \\ 0 & 0 & 0 & q_z \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Note: $F_{q_z} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} q_z x \\ q_z y \\ q_z \\ -z \end{bmatrix} = \begin{bmatrix} \frac{q_z}{-z} x \\ \frac{q_z}{-z} y \\ \frac{q_z}{-z} \\ 1 \end{bmatrix}$

This maps z coordinates to q_z/z, which will be useful.

View Volume, Frustum

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View-Volume Related Definitions

View Volume:

Parts of the scene which should be visible to the user.

Frustum:

A shape constructed by slicing off the top of a square-base pyramid with a plane parallel to the base.

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Frustum View Volume Motivation

Consider the simple projection transformation:

Shape of view volume consists of two pyramids ...
... one pyramid in front, the other in back, ...
... and both points on eye.

Some points are behind the user...
... and we don't want these to be visible (because they would be unnatural).

Some points in view volume are so far from the user...
... that they would be invisible.

For example, points might form a triangle that covers 1% of a pixel.

These points waste computing power.

Frustum View Volume

View volume in shape of frustum with smaller square on projection plane.

The smaller square of frustum defines a *near plane*.

The larger square defines a *far plane*.

Variables describing a frustum view volume:

n: Distance from eye to near plane.

f: Distance from eye to far plane.

Coordinates of lower-left corner of $(l, b, -n)$.

Coordinates of upper-right corner of $(r, t, -n)$.

Frustum Perspective Transform

Given six values: l, r, t, b, n, f (left, right, top, bottom, near, far).

Eye at origin, projection surface at (x, y, n), normal is $(0, 0, -1)$.

Viewer screen is rectangle from $(l, b, -n)$ to $(r, t, -n)$.

Points with $z > -t$ and $z < -f$ are not of interest.

$$F_{l,r,t,b,n,f} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -2\frac{fn}{f-n} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$