Mathematics for 3D Graphics

Topics

Points, Vectors, Vertices, Coordinates
Dot Products, Cross Products
Lines, Planes, Intercepts

References

Many texts cover the linear algebra used for 3D graphics . . .
. . . the texts below are good references, Akenine-Möller is more relevant to the class.


Points and Vectors

**Point:**
Indivisible location in space.

\[ P_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \]

**Vector:**
Difference between two points.

\[ V = P_2 - P_1 = P_1 - P_2 = \begin{bmatrix} 4 - 1 \\ 5 - 2 \\ 6 - 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}. \]

Equivalently: \( P_2 = P_1 + V \).

Don’t confuse points and vectors!
Point-Related Terminology

Will define several terms related to points.

At times they may be used interchangeably.

**Point:**
A location in space.

**Coordinate:**
A representation of location.

**Vertex:**
Term may mean point, coordinate, or part of graphical object.

As used in class, vertex is a less formal term.

It might refer to a point, its coordinate, and other info like color.
Coordinate:
A representation of where a point is located.

Familiar representations:

3D Cartesian $P = (x, y, z)$.

2D Polar $P = (r, \theta)$.

In class we will use 3D homogeneous coordinates.
Homogeneous Coordinates

**Homogeneous Coordinate:**
A coordinate representation for points in 3D space consisting of four components... 
... each component is a real number... 
... and the last component is non-zero.

\[
P = \begin{bmatrix}
x \\
y \\
z \\
w
\end{bmatrix}, \text{ where } w \neq 0.
\]

\(P\) refers to same point as Cartesian coordinate \((x/w, y/w, z/w)\).

To save paper sometimes written as \((x, y, z, w)\).
Homogeneous Coordinates

Each point can be described by many homogeneous coordinates ... 

\[
\begin{bmatrix} 10 \\ 20 \\ 30 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 20 \\ 40 \\ 60 \\ 2 \end{bmatrix} = \begin{bmatrix} 10w \\ 20w \\ 30w \\ w \end{bmatrix} = \ldots
\]

... these are all equivalent so long as \( w \neq 0 \).

Column matrix \[
\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}
\]

could not be a homogeneous coordinate ... 

... but it could be a vector.
Homogeneous Coordinates

Why not just Cartesian coordinates like \((x, y, z)\)?

The \(w\) simplifies certain computations.

Confused?

Then for a little while pretend that
\[
\begin{bmatrix}
 x \\
 y \\
 z \\
 1
\end{bmatrix}
\]
is just \((x, y, z)\).
Homogenized Homogeneous Coordinate

A homogeneous coordinate is *homogenized* by dividing each element by the last.

For example, the homogenization of

\[
\begin{bmatrix}
  x \\
  y \\
  z \\
  w
\end{bmatrix}
\]

is

\[
\begin{bmatrix}
  x/w \\
  y/w \\
  z/w \\
  1
\end{bmatrix}
\]

Homogenization is also known as *perspective divide.*
Points just sit there, it’s vectors that do all the work.

In other words, most operations defined on vectors.

Point/Vector Sum

The result of adding a point to a vector is a point.

Consider point with homogenized coordinate \( P = (x, y, z, 1) \) and vector \( V = (i, j, k) \).

The sum \( P + V \) is the point with coordinate

\[
\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix}
+ \begin{bmatrix}
  i \\
  j \\
  k
\end{bmatrix}
= \begin{bmatrix}
  x + i \\
  y + j \\
  z + k \\
  1
\end{bmatrix}
\]

This follows directly from the vector definition.
Scalar/Vector Multiplication

The result of multiplying scalar $a$ with a vector is a vector...
... that is $a$ times longer but points in the same or opposite direction...
... if $a \neq 0$.

Let $a$ denote a scalar real number and $V$ a vector.

The scalar vector product is $aV = a \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax \\ ay \\ az \end{bmatrix}$. 
Vector/Vector Addition

The result of adding two vectors is another vector.

Let \( V_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \) and \( V_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \) denote two vectors.

The vector sum, denoted \( U + V \), is \( \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \)

Vector subtraction could be defined similarly...
... but doesn’t need to be because we can use scalar/vector multiplication: 
\( V_1 - V_2 = V_1 + (-1 \times V_2) \).
Vector Addition Properties

Vector addition is associative:

\[ U + (V + W) = (U + V) + W. \]

Vector addition is commutative:

\[ U + V = V + U. \]
Vector Magnitude, Normalization

Vector Magnitude

The magnitude of a vector is its length, a scalar.

The magnitude of $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ denoted $\|V\|$, is $\sqrt{x^2 + y^2 + z^2}$.

The magnitude is also called the length and the norm.

Vector $V$ is called a unit vector if $\|V\| = 1$.

A vector is normalized by dividing each of its components by its length.

The notation $\hat{V}$ indicates $V/\|V\|$, the normalized version of $V$. 
Dot Product

The Vector Dot Product

The dot product of two vectors is a scalar.

Roughly, it indicates how much they point in the same direction.

Consider vectors \( \mathbf{V}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \) and \( \mathbf{V}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \).

The dot product of \( \mathbf{V}_1 \) and \( \mathbf{V}_2 \), denoted \( \mathbf{V}_1 \cdot \mathbf{V}_2 \), is \( x_1 x_2 + y_1 y_2 + z_1 z_2 \).
Dot Product Properties

Let $U$, $V$, and $W$ be vectors.

Let $a$ be a scalar.

Miscellaneous Dot Product Properties

$$(U + V) \cdot W = U \cdot W + V \cdot W$$

$$(aU) \cdot V = a(U \cdot V)$$

$$U \cdot V = V \cdot U$$
Dot Product Properties

Orthogonality

The more casual term is perpendicular.

Vectors $U$ and $V$ are called orthogonal iff $U \cdot V = 0$.

This is an important property for finding intercepts.
Dot Product Properties

Angle

Let $U$ and $V$ be two vectors.

Then $U \cdot V = \|U\|\|V\| \cos \phi$...

... where $\phi$ is the smallest angle between the two vectors.
Cross Product

The cross product of two vectors results in a vector orthogonal to both.

The cross product of vectors $V_1$ and $V_2$, denoted $V_1 \times V_2$, is

$$V_1 \times V_2 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}.$$
Cross Product Properties

Let $U$ and $V$ be two vectors and let $W = U \times V$.

Then both $U$ and $V$ are orthogonal to $W$.

$$\|U \times V\| = \|U\|\|V\| \sin \phi.$$  

$U \times V = -V \times U.$

$(aU + bV) \times W = a(U \times W) + b(V \times W).$
Lines, Planes, Intercepts

Lines, planes, and intercepts covered on the blackboard.
Transforms

**Transformation:**
A mapping (conversion) from one coordinate set to another (e.g., from feet to meters) or to a new location in an existing coordinate set.

Particular Transformations to be Covered

- **Translation:** Moving things around.
- **Scale:** Change size.
- **Rotation:** Rotate around some axis.
- **Projection:** Moving to a surface.

Transform by multiplying $4 \times 4$ matrix with coordinate.

$$P_{\text{new}} = M_{\text{transform}} P_{\text{old}}.$$
Transforms

**Scale Transform**

\[ S(s, t, u) = \begin{pmatrix}
    s & 0 & 0 & 0 \\
    0 & t & 0 & 0 \\
    0 & 0 & u & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix}. \]

\( S(s, t, u) \) stretches an object \( s \) times along the \( x \)-axis, \( t \) times along the \( y \)-axis, and \( u \) times along the \( z \)-axis.

Scaling centered on the origin.
Transforms

Rotation Transformations

$R_x(\theta)$ rotates around $x$ axis by $\theta$; likewise for $R_y$ and $R_z$.

\[
R_x(\theta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

\[
R_y(\theta) = \begin{pmatrix}
\cos \theta & 0 & \sin \theta & 0 \\
0 & 1 & 0 & 0 \\
-\sin \theta & 0 & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

\[
R_z(\theta) = \begin{pmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
Transforms

Translation Transform

\[ T(s, t, u) = \begin{pmatrix} 1 & 0 & 0 & s \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

Moves point \( s \) units along \( x \) axis, etc.
Miscellaneous Matrix Multiplication Math

Let $M$ and $N$ denote arbitrary $4 \times 4$ matrices.

**Identity Matrix**

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

$$IM = MI = M.$$
Transforms and Matrix Arithmetic

**Matrix Inverse**

Matrix $A$ is an inverse of $M$ iff $AM = MA = I$.

Will use $M^{-1}$ to denote inverse.

Not every matrix has an inverse.

Computing inverse of an arbitrary matrix expensive . . .

. . . but inverse of some matrices are easy to compute . . .

. . . for example, $T(x, y, z)^{-1} = T(-x, -y, -z)$.

**Matrix Multiplication Rules**

Is associative: $(LM)N = L(MN)$.

Is **not** commutative: $MN \neq NM$ for arbitrary $M$ and $N$.

$(MN)^{-1} = N^{-1}M^{-1}$. (Note change in order.)
Projection Transformations

Projection Transform:
A transform that maps a coordinate to a space with fewer dimensions.

A projection transform will map a 3D coordinate from our physical or graphical model . . .
. . . to a 2D location on our monitor (or a window).

Projection Types

Vague definitions on this page.

Perspective Projection

Points appear to be in “correct” location, . . .
. . . as though monitor were just a window into the simulated world.

This projection used when realism is important.

Orthographic Projection

A projection without perspective foreshortening.

This projection used when a real ruler will be used to measure distances.
Perspective Projection Derivation

Given

A graphical (or physical) model.

Let's put user and user's monitor in graphical model:

Location of user's eye: $E$.

A point on the user's monitor: $Q$.

Normal to user's monitor pointing away from user: $n$.

Goal:

Find $S$, point where line from $E$ to $P$ intercepts monitor (plane $Q, n$).

Line from $E$ to $P$ called the projector.

The user's monitor is in the projection plane.

The point $S$ is called the projection of point $P$ on the projection plane.
Solution:

Projector equation: \( S = E + t\vec{EP} \).

Projection plane equation: \( \vec{QS} \cdot n = 0 \).

Find point \( S \) that’s on projector and projection plane:

\[
\overrightarrow{Q(E + t\vec{EP})} \cdot n = 0 \\
(E + t\vec{EP} - Q) \cdot n = 0 \\
\overrightarrow{QE} \cdot n + t\vec{EP} \cdot n = 0
\]

\[ t = \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n} \]

\[ S = E + \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n} \overrightarrow{EP} \]

Note: \( \overrightarrow{EQ} \cdot n \) is distance from user to plane in direction \( n \) . . .

. . . and \( \overrightarrow{EP} \cdot n \) is distance from user to point in direction \( n \).
Perspective Projection Derivation

To simplify projection:

Fix $E = (0, 0, 0)$: Put user at origin.

Fix $n = (0, 0, -1)$: Make “monitor” parallel to $xy$ plane.

Result:

$$S = E + \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n} \overrightarrow{EP}$$

$$S = \frac{q_z}{p_z} P$$

The key operation in perspective projection is dividing out by $z$ (given our geometry).