Mathematics for 3D Graphics

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Topics

Points, Vectors, Vertices, Coordinates

Dot Products, Cross Products

Lines, Planes, Intercepts

References

Many texts cover the linear algebra used for 3D graphics . . .

. . . the texts below are good references, Akenine-Möller is more relevant to the class.

Appendix A in T. Akenine-Möller, E. Haines, N. Hoffman, "Real-Time Rendering," Third Edition, A. K. Peters Ltd.

Appendix A in Foley, van Dam, Feiner, Huges, "Computer Graphics: Principles and Practice," Second Edition, Addison Wesley.

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Points and Vectors

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Point:

Indivisible location in space.

$$E.g., P_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, P_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Vector:

Difference between two points.

$$E.g., V = P_2 - P_1 = P_1P_2 = \begin{bmatrix} 4-1 \\ 5-2 \\ 6-3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

Equivalently: $P_2 = P_1 + V$.

Don't confuse points and vectors!

Point-Related Terminology

Will define several terms related to points.

At times they may be used interchangeably.

Point:
A location in space.

Coordinate:
A representation of location.

Vertex:
Term may mean point, coordinate, or part of graphical object.

As used in class, vertex is a less formal term.

It might refer to a point, its coordinate, and other info like color.

Coordinate:
A representation of where a point is located.

Familiar representations:

3D Cartesian $P = (x, y, z)$.

2D Polar $P = (r, \theta)$.

In class we will use 3D *homogeneous coordinates*.

Homogeneous Coordinates

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Homogeneous Coordinate:

A coordinate representation for points in 3D space consisting of four components...

... each component is a real number...

... and the last component is non-zero.

Representation: $P = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$, where $w \neq 0$.

P refers to same point as Cartesian coordinate $(x/w, y/w, z/w)$.

To save paper sometimes written as (x, y, z, w).

Homogeneous Coordinates

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Each point can be described by many homogeneous coordinates ...

... for example, $(10, 20, 30) = \begin{bmatrix} 10 \\ 20 \\ 30 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 20 \\ 40 \\ 60 \\ 2 \end{bmatrix} = \begin{bmatrix} 10w \\ 20w \\ 30w \\ w \end{bmatrix} = \ldots$

... these are all equivalent so long as $w \neq 0$.

Column matrix $\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$ could not be a homogeneous coordinate ...

... but it could be a vector.

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Homogeneous Coordinates

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Why not just Cartesian coordinates like (x, y, z)?

The w simplifies certain computations.

Confused?

Then for a little while pretend that $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$ is just (x, y, z).

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Homogenized Homogeneous Coordinate

A homogeneous coordinate is *homogenized* by dividing each element by the last.

For example, the homogenization of $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is $\begin{bmatrix} x/w \\ y/w \\ z/w \\ 1 \end{bmatrix}$

Homogenization is also known as *perspective divide*.

Vector Arithmetic

Points just sit there, it's vectors that do all the work.

In other words, most operations defined on vectors.

Point/Vector Sum

The result of adding a point to a vector is a point.

Consider point with homogenized coordinate $P = (x, y, z, 1)$ and vector $V = (i, j, k)$.

The sum $P + V$ is the point with coordinate $$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} + \begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} x+i \\ y+j \\ z+k \\ 1 \end{bmatrix}$$

This follows directly from the vector definition.

Scalar/Vector Multiplication

The result of multiplying scalar a with a vector is a vector...

... that is a times longer but points in the same or opposite direction...

... if $a \neq 0$.

Let a denote a scalar real number and V a vector.

The *scalar vector product* is $aV = a \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax \\ ay \\ az \end{bmatrix}$.

Vector/Vector Addition

The result of adding two vectors is another vector.

Let $V_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ denote two vectors.

The *vector sum*, denoted $U + V$, is $\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$

Vector subtraction could be defined similarly...

... but doesn't need to be because we can use scalar/vector multiplication: $V_1 - V_2 = V_1 + (-1 \times V_2)$.

Vector Addition Properties

Vector addition is associative:

$$U + (V + W) = (U + V) + W.$$

Vector addition is commutative:

$$U + V = V + U.$$

Vector Magnitude, Normalization

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Vector Magnitude

The magnitude of a vector is its length, a scalar.

The *magnitude* of $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ denoted $\|V\|$, is $\sqrt{x^2 + y^2 + z^2}$.

The magnitude is also called the *length* and the *norm*.

Vector V is called a *unit vector* if $\|V\| = 1$.

A vector is *normalized* by dividing each of its components by its length.

The notation $\hat{V}$ indicates $V/\|V\|$, the normalized version of V.

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The Vector Dot Product

The dot product of two vectors is a scalar.

Roughly, it indicates how much they point in the same direction.

Consider vectors $V_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$.

The *dot product* of V_1 and V_2, denoted $V_1 \cdot V_2$, is $x_1 x_2 + y_1 y_2 + z_1 z_2$.

Dot Product Properties

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Let U, V, and W be vectors.

Let a be a scalar.

Miscellaneous Dot Product Properties

$$(U + V) \cdot W = U \cdot W + V \cdot W$$

$$(aU) \cdot V = a(U \cdot V)$$

$$U \cdot V = V \cdot U$$

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Dot Product Properties

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Orthogonality

The more casual term is perpendicular.

Vectors U and V are called *orthogonal* iff $U \cdot V = 0$.

This is an important property for finding intercepts.

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Dot Product Properties

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Angle

Let U and V be two vectors.

Then $U \cdot V = \|U\|\|V\| \cos \phi \ldots$

... where ϕ is the smallest angle between the two vectors.

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Cross Product

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Cross Product

The cross product of two vectors results in a vector orthogonal to both.

The *cross product* of vectors V_1 and V_2, denoted $V_1 \times V_2$, is

$$V_1 \times V_2 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}.$$

Cross Product Properties

Let U and V be two vectors and let $W = U \times V$.

Then both U and V are orthogonal to W.

$\|U \times V\| = \|U\|\|V\| \sin\phi$.

$U \times V = -V \times U$.

$(aU + bV) \times W = a(U \times W) + b(V \times W)$.

Lines, Planes, Intercepts

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Lines, planes, and intercepts covered on the blackboard.

Transformation:

A mapping (conversion) from one coordinate set to another (*e.g.*, from feet to meters) or to a new location in an existing coordinate set.

Particular Transformations to be Covered

Translation: Moving things around.

Scale: Change size.

Rotation: Rotate around some axis.

Projection: Moving to a surface.

Transform by multiplying 4×4 matrix with coordinate.

$$P_{\text{new}} = M_{\text{transform}} P_{\text{old}}.$$

Transforms

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Scale Transform

$$S(s,t,u) = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$S(s,t,u)$ stretches an object s times along the x-axis, t times along the y-axis, and u times along the z-axis.

Scaling centered on the origin.

Rotation Transformations

$R_x(\theta)$ rotates around x axis by θ; likewise for R_y and R_z.

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Translation Transform

$$T(s,t,u) = \begin{pmatrix} 1 & 0 & 0 & s \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moves point s units along x axis, etc.

Miscellaneous Matrix Multiplication Math

Let M and N denote arbitrary 4×4 matrices.

Identity Matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$IM = MI = M.$$

Matrix Inverse

Matrix A is an inverse of M iff $AM = MA = I$.

Will use M^{-1} to denote inverse.

Not every matrix has an inverse.

Computing inverse of an arbitrary matrix expensive ...
... but inverse of some matrices are easy to compute ...
... for example, $T(x, y, z)^{-1} = T(-x, -y, -z)$.

Matrix Multiplication Rules

Is associative: $(LM)N = L(MN)$.

Is not commutative: $MN \neq NM$ for arbitrary M and N.

$(MN)^{-1} = N^{-1}M^{-1}$. (Note change in order.)

Projection Transformations

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Projection Transform:
A transform that maps a coordinate to a space with fewer dimensions.

A projection transform will map a 3D coordinate from our physical or graphical model ...
... to a 2D location on our monitor (or a window).

Projection Types

Vague definitions on this page.

Perspective Projection

Points appear to be in "correct" location,...
... as though monitor were just a window into the simulated world.

This projection used when realism is important.

Orthographic Projection

A projection without perspective foreshortening.

This projection used when a real ruler will be used to measure distances.

Perspective Projection Derivation

Given

A graphical (or physical) model.

Lets put user and user's monitor in graphical model:

Location of user's eye: E.

A point on the user's monitor: Q.

Normal to user's monitor pointing away from user: n.

Goal:

Find S, point where line from E to P intercepts monitor (plane Q, n).

Line from E to P called the *projector*.

The user's monitor is in the *projection plane*.

The point S is called the *projection* of point P on the projection plane.

Solution:

Projector equation: $S = E + t\overrightarrow{EP}$.

Projection plane equation: $\overrightarrow{QS} \cdot n = 0$.

Find point S that's on projector and projection plane:

$$\overrightarrow{Q(E + t\overrightarrow{EP})} \cdot n = 0$$

$$(E + t\overrightarrow{EP} - Q) \cdot n = 0$$

$$\overrightarrow{QE} \cdot n + t\overrightarrow{EP} \cdot n = 0$$

$$t = \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n}$$

$$S = E + \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n}\overrightarrow{EP}$$

Note: $\overrightarrow{EQ} \cdot n$ is distance from user to plane in direction n ...

... and $\overrightarrow{EP} \cdot n$ is distance from user to point in direction n.

Perspective Projection Derivation

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To simplify projection:

Fix $E = (0, 0, 0)$: Put user at origin.

Fix $n = (0, 0, -1)$: Make "monitor" parallel to xy plane.

Result:

$$S = E + \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n} \overrightarrow{EP}$$

$$S = \frac{q_z}{p_z} P$$

The key operation in perspective projection is dividing out by z (given our geometry).

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