# Mathematics for 3D Graphics

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## Topics

Points, Vectors, Vertices, Coordinates

Dot Products, Cross Products

Lines, Planes, Intercepts

## References

Many texts cover the linear algebra used for 3D graphics ...

... the texts below are good references, Akenine-Möller is more relevant to the class.

Appendix A in T. Akenine-Möller, E. Haines, N. Hoffman, "Real-Time Rendering," Third Edition, A. K. Peters Ltd.

Appendix A in Foley, van Dam, Feiner, Huges, "Computer Graphics: Principles and Practice," Second Edition, Addison Wesley.

# Points and Vectors

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*Point:*

Indivisible location in space.

$$E.g.,\ P_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\ P_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

*Vector:*

Difference between two points.

$$E.g.,\ V = P_2 - P_1 = \overrightarrow{P_1 P_2} = \begin{bmatrix} 4-1 \\ 5-2 \\ 6-3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

Equivalently: $P_2 = P_1 + V$.

**Don't confuse points and vectors!**

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## Point-Related Terminology

Will define several terms related to points.

At times they may be used interchangeably.

*Point:*
A location in space.

*Coordinate:*
A representation of location.

*Vertex:*
Term may mean point, coordinate, or part of graphical object.

As used in class, vertex is a less formal term.

It might refer to a point, its coordinate, and other info like color.



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*Coordinate:*
A representation of where a point is located.

Familiar representations:

3D Cartesian $P = (x, y, z)$.

2D Polar $P = (r, \theta)$.

In class we will use 3D *homogeneous coordinates*.



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# Homogeneous Coordinates

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*Homogeneous Coordinate:*

A coordinate representation for points in 3D space consisting of four components...

... each component is a real number...

... and the last component is non-zero.

Representation: $P = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$, where $w \neq 0$.

$P$ refers to same point as Cartesian coordinate $(x/w, y/w, z/w)$.

To save paper sometimes written as $(x, y, z, w)$.

# Homogeneous Coordinates

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Each point can be described by many homogeneous coordinates . . .

. . . for example, $(10, 20, 30) = \begin{bmatrix} 10 \\ 20 \\ 30 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 20 \\ 40 \\ 60 \\ 2 \end{bmatrix} = \begin{bmatrix} 10w \\ 20w \\ 30w \\ w \end{bmatrix} = \ldots$

. . . these are all equivalent so long as $w \neq 0$.

Column matrix $\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$ could not be a homogeneous coordinate . . .

. . . but it could be a vector.





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# Homogeneous Coordinates

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Why not just Cartesian coordinates like $(x, y, z)$?

The $w$ simplifies certain computations.

Confused?

Then for a little while pretend that $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$ is just $(x, y, z)$.

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## Homogenized Homogeneous Coordinate

A homogeneous coordinate is *homogenized* by dividing each element by the last.

For example, the homogenization of $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is $\begin{bmatrix} x/w \\ y/w \\ z/w \\ 1 \end{bmatrix}$

Homogenization is also known as *perspective divide*.

## Vector Arithmetic

*Points just sit there, it's vectors that do all the work.*

In other words, most operations defined on vectors.

### Point/Vector Sum

*The result of adding a point to a vector is a point.*

Consider point with homogenized coordinate $P = (x, y, z, 1)$ and vector $V = (i, j, k)$.

The sum $P + V$ is the point with coordinate $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} + \begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} x+i \\ y+j \\ z+k \\ 1 \end{bmatrix}$

This follows directly from the vector definition.

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## Scalar/Vector Multiplication

*The result of multiplying scalar $a$ with a vector is a vector...*

*... that is $a$ times longer but points in the same or opposite direction...*

*... if $a \neq 0$.*

Let $a$ denote a scalar real number and $V$ a vector.

The *scalar vector product* is $$aV = a \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax \\ ay \\ az \end{bmatrix}.$$



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## Vector/Vector Addition

*The result of adding two vectors is another vector.*

Let $V_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ denote two vectors.

The *vector sum*, denoted $U + V$, is $\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$

Vector subtraction could be defined similarly...

... but doesn't need to be because we can use scalar/vector multiplication: $V_1 - V_2 = V_1 + (-1 \times V_2)$.

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## Vector Addition Properties

Vector addition is associative:

$$U + (V + W) = (U + V) + W.$$

Vector addition is commutative:

$$U + V = V + U.$$



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# Vector Magnitude, Normalization

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## Vector Magnitude

*The magnitude of a vector is its length, a scalar.*

The *magnitude* of $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ denoted $\|V\|$, is $\sqrt{x^2 + y^2 + z^2}$.

The magnitude is also called the *length* and the *norm*.

Vector $V$ is called a *unit vector* if $\|V\| = 1$.

A vector is *normalized* by dividing each of its components by its length.

The notation $\hat{V}$ indicates $V/\|V\|$, the normalized version of $V$.

# Dot Product

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## The Vector Dot Product

*The dot product of two vectors is a scalar.*

Roughly, it indicates how much they point in the same direction.

Consider vectors $V_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$.

The *dot product* of $V_1$ and $V_2$, denoted $V_1 \cdot V_2$, is $\quad x_1x_2 + y_1y_2 + z_1z_2$.



EE 7700-1 Lecture Transparency. Formatted 10:16, 29 January 2010 from set-1-math.

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# Dot Product Properties

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Let $U$, $V$, and $W$ be vectors.

Let $a$ be a scalar.

Miscellaneous Dot Product Properties

$$(U + V) \cdot W = U \cdot W + V \cdot W$$

$$(aU) \cdot V = a(U \cdot V)$$

$$U \cdot V = V \cdot U$$

$$\text{abs}(U \cdot U) = \|U\|^2$$

# Dot Product Properties

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## Orthogonality

*The more casual term is perpendicular.*

Vectors $U$ and $V$ are called *orthogonal* iff $U \cdot V = 0$.

This is an important property for finding intercepts.

# Dot Product Properties

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## Angle

Let $U$ and $V$ be two vectors.

Then $U \cdot V = \|U\|\|V\| \cos\phi \ldots$

$\ldots$ where $\phi$ is the smallest angle between the two vectors.

## Cross Product

*The cross product of two vectors results in a vector orthogonal to both.*

The *cross product* of vectors $V_1$ and $V_2$, denoted $V_1 \times V_2$, is

$$V_1 \times V_2 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}.$$

# Cross Product Properties

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## Cross Product Properties

Let $U$ and $V$ be two vectors and let $W = U \times V$.

Then both $U$ and $V$ are orthogonal to $W$.

$\|U \times V\| = \|U\|\|V\| \sin \phi$.

$U \times V = -V \times U$.

$(aU + bV) \times W = a(U \times W) + b(V \times W)$.

If $U$ and $V$ define a parallelogram, its area is $\|U \times V\| \ldots$

$\ldots$ if they define a triangle its area is $\frac{1}{2}\|U \times V\|$.



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# Lines, Planes, Intercepts

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Lines, planes, and intercepts covered on the blackboard.

# Sample Problem

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Problem: *A light model specifies that in a scene with a light of brightness b (scalar) at location L (coordinate), and a point P on a surface with normal $\hat{n}$, the lighted color, c, of P (a scalar) will be the dot product of the surface normal with the direction to the light divided by the distance to the light.*

Restate this as a formula.

Estimate the number of floating point operations in a streamlined computation.

Solution:

Formula: $c = b\widehat{PL} \cdot \hat{n} \frac{1}{\|\overrightarrow{PL}\|}$.

# Transforms

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*Transformation:*

A mapping (conversion) from one coordinate set to another (*e.g.*, from feet to meters) or to a new location in an existing coordinate set.

Particular Transformations to be Covered

*Translation*: Moving things around.

*Scale*: Change size.

*Rotation*: Rotate around some axis.

*Projection*: Moving to a surface.

Transform by multiplying $4 \times 4$ matrix with coordinate.

$P_{\text{new}} = M_{\text{transform}} P_{\text{old}}$.

# Transforms

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## *Scale Transform*

$$S(s,t,u) = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$S(s,t,u)$ stretches an object $s$ times along the $x$-axis, $t$ times along the $y$-axis, and $u$ times along the $z$-axis.

Scaling centered on the origin.

## *Rotation Transformations*

$R_x(\theta)$ rotates around $x$ axis by $\theta$; likewise for $R_y$ and $R_z$.

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

# Transforms

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*Translation Transform*

$$T(s,t,u) = \begin{pmatrix} 1 & 0 & 0 & s \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moves point $s$ units along $x$ axis, etc.

## Miscellaneous Matrix Multiplication Math

Let $M$ and $N$ denote arbitrary $4 \times 4$ matrices.

*Identity Matrix*

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$IM = MI = M.$$

# Transforms and Matrix Arithmetic

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## *Matrix Inverse*

Matrix $A$ is an inverse of $M$ iff $AM = MA = I$.

Will use $M^{-1}$ to denote inverse.

Not every matrix has an inverse.

Computing inverse of an arbitrary matrix expensive . . .

. . . but inverse of some matrices are easy to compute . . .

. . . for example, $T(x, y, z)^{-1} = T(-x, -y, -z)$.

## Matrix Multiplication Rules

Is associative: $(LM)N = L(MN)$.

**Is not** commutative: $MN \neq NM$ for arbitrary $M$ and $N$.

$(MN)^{-1} = N^{-1}M^{-1}$. (Note change in order.)

# Projection Transformations

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*Projection Transform:*

A transform that maps a coordinate to a space with fewer dimensions.

*A projection transform will map a 3D coordinate from our physical or graphical model . . .*

*. . . to a 2D location on our monitor (or a window).*

## Projection Types

Vague definitions on this page.

### *Perspective Projection*

*Points appear to be in "correct" location,. . .*

*. . . as though monitor were just a window into the simulated world.*

This projection used when realism is important.

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*Orthographic Projection*

*A projection without perspective foreshortening.*

This projection used when a real ruler will be used to measure distances.



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# Perspective Projection Derivation

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Lets put user and user's monitor in world coordinate space:

Location of user's eye: $E$.

A point on the user's monitor: $Q$.

Normal to user's monitor pointing away from user: $n$.

Goal:

Find $S$, point where line from $E$ to $P$ intercepts monitor (plane $Q, n$).

Line from $E$ to $P$ called the *projector*.

The user's monitor is in the *projection plane*.

The point $S$ is called the *projection* of point $P$ on the projection plane.

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Solution:

Projector equation: $S = E + t\overrightarrow{EP}$.

Projection plane equation: $\overrightarrow{QS} \cdot n = 0$.

Find point $S$ that's on projector and projection plane:

$$\overrightarrow{Q(E + t\overrightarrow{EP})} \cdot n = 0$$

$$(E + t\overrightarrow{EP} - Q) \cdot n = 0$$

$$\overrightarrow{QE} \cdot n + t\overrightarrow{EP} \cdot n = 0$$

$$t = \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n}$$

$$S = E + \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n}\overrightarrow{EP}$$

Note: $\overrightarrow{EQ} \cdot n$ is distance from user to plane in direction $n$ ...

... and $\overrightarrow{EP} \cdot n$ is distance from user to point in direction $n$.



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# Perspective Projection Derivation

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To simplify projection:

Fix $E = (0, 0, 0)$: Put user at origin.

Fix $n = (0, 0, -1)$: Make "monitor" parallel to $xy$ plane.

Result:

$$S = E + \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n} \overrightarrow{EP}$$

$$S = \frac{q_z}{p_z} P,$$

where $q_z$ is the $z$ component of $Q$, and $p_z$ defined similarly.

The key operation in perspective projection is dividing out by $z$ (given our geometry).

# Simple Perspective Projection Transformation

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## Simple Projection Transform

Eye at origin, projection surface at $(x, y, -q_z)$, normal is $(0, 0, -1)$.

$$F_{q_z} = \begin{pmatrix} q_z & 0 & 0 & 0 \\ 0 & q_z & 0 & 0 \\ 0 & 0 & 0 & q_z \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Note: $F_{q_z} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} q_z x \\ q_z y \\ q_z \\ -z \end{bmatrix} = \begin{bmatrix} \frac{q_z}{-z} x \\ \frac{q_z}{-z} y \\ \frac{q_z}{-z} \\ 1 \end{bmatrix}$

This maps $z$ coordinates to $q_z/z$, which will be useful.

## Frustum Perspective Transform

Given six values: $l, r, t, b, n, f$ (left, right, top, bottom, near, far).

Eye at origin, projection surface at $(x, y, n)$, normal is $(0, 0, -1)$.

Viewer screen is rectangle from $(l, b, -n)$ to $(r, t, -n)$.

Points with $z > -t$ and $z < -f$ are not of interest.

$$F_{l,r,t,b,n,f} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -2\frac{fn}{f-n} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$