**Essentials of Robust Control**

These slides will be updated when I have time.

Last updated on August 28, 2000

# Introduction

This introduction is adopted from some of John Doyle's lectures.

## Classical control in the 1930's and 1940's
## Bode, Nyquist, Nichols, . . .

- Feedback amplifier design
- Single input, single output (SISO)
- Frequency domain
- Graphical techniques
- Emphasized design tradeoffs
  - Effects of uncertainty
  - Nonminimum phase systems
  - Performance vs. robustness

## Problems with classical control

Overwhelmed by complex systems:

- Highly coupled multiple input, multiple output systems
- Nonlinear systems
- Time-domain performance specifications

## The origins of modern control theory

Early years

- Wiener (1930's - 1950's) Generalized harmonic analysis, cybernetics, filtering, prediction, smoothing
- Kolmogorov (1940's) Stochastic processes
- Linear and nonlinear programming (1940's - )

Optimal control

- Bellman's Dynamic Programming (1950's)
- Pontryagin's Maximum Principle (1950's)
- Linear optimal control (late 1950's and 1960's)
  - Kalman Filtering
  - Linear-Quadratic (LQ) regulator problem
  - Stochastic optimal control (LQG)

## The diversification of modern control in the 1960's and 1970's

- Applications of Maximum Principle and Optimization
  - Zoom maneuver for time-to-climb
  - Spacecraft guidance (e.g. Apollo)
  - Scheduling, resource management, etc.
- Linear optimal control
- Linear systems theory
  - Controllability, observability, realization theory
  - Geometric theory, disturbance decoupling
  - Pole assignment
  - Algebraic systems theory
- Nonlinear extensions
  - Nonlinear stability theory, small gain, Lyapunov
  - Geometric theory
  - Nonlinear filtering
- Extension of LQ theory to infinite-dimensional systems
- Adaptive control

## Modern control application: Shuttle reentry

The problem is to control the reentry of the shuttle, from orbit to landing. The modern control approach is to break the problem into two pieces:

- Trajectory optimization
- Flight control

- Trajectory optimization: tremendous use of modern control principles
  - State estimation (filtering) for navigation
  - Bang-bang control of thrusters
  - Digital autopilot
  - Nonlinear optimal trajectory selection
- Flight control: primarily used classical methods with lots of simulation
  - Gain scheduled linear designs
  - Uncertainty studied with ad-hoc methods

Modern control has had little impact on feedback design because it neglects fundamental feedback tradeoffs and the role of plant uncertainty.

## The 1970's and the return of the frequency domain

Motivated by the inadequacies of modern control, many researchers returned to the frequency domain for methods for MIMO feedback control.

- British school
  - Inverse Nyquist Array
  - Characteristic Loci
- Singular values
  - MIMO generalization of Bode gain plots
  - MIMO generalization of Bode design
  - Crude MIMO representations of uncertainty
- Multivariable loopshaping and LQG/LTR
  - Attempt to reconcile modern and classical methods
  - Popular, but hopelessly flawed
  - Too crude a representation of uncertainty

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While these methods allowed modern and classical methods to be blended to handle many MIMO design problems, it became clear that fundamentally new methods needed to be developed to handle complex, uncertain, interconnected MIMO systems.

## Postmodern Control

- Mostly for fun. Sick of "modern control," but wanted a name equally pretentious and self-absorbed.
- Other possible names are inadequate:
  - Robust ( too narrow, sounds too macho)
  - Neoclassical (boring, sounds vaguely fascist )
  - Cyberpunk ( too nihilistic )
- Analogy with postmodern movement in art, architecture, literature, social criticism, philosophy of science, feminism, etc. ( talk about pretentious ).

## The tenets of postmodern control theory

- Theories don't design control systems, engineers do.
- The application of any methodology to real problems will require some leap of faith on the part of the engineer (and some ad hoc fixes).
- The goal of the theoretician should be to make this leap smaller and the ad hoc fixes less dominant.

# Issues in postmodern control theory

- More connection with data
- Modeling
  - Flexible signal representation and performance objectives
  - Flexible uncertainty representations
  - Nonlinear nominal models
  - Uncertainty modeling in specific domains
- Analysis
- System Identification
  - Nonprobabilistic theory
  - System ID with plant uncertainty
  - Resolving ambiguity; "uncertainty about uncertainty"
  - Attributing residuals to perturbations, not just noise
  - Interaction with modeling and system design
- Optimal control and filtering
  - $H_\infty$ optimal control
  - More general optimal control with mixed norms
  - Robust performance for complex systems with structured uncertainty

# Chapter 2: Linear Algebra

- linear subspaces
- eigenvalues and eigenvectors
- matrix inversion formulas
- invariant subspaces
- vector norms and matrix norms
- singular value decomposition
- generalized inverses
- semidefinite matrices

## Linear Subspaces

- *linear combination*:

$$\alpha_1 x_1 + \ldots + \alpha_k x_k, \quad x_i \in \mathbb{F}^n, \quad \alpha \in \mathbb{F}$$

$$\text{span}\{x_1, x_2, \ldots, x_k\} := \{x = \alpha_1 x_1 + \ldots + \alpha_k x_k : \alpha_i \in \mathbb{F}\}.$$

- $x_1, x_2, \ldots, x_k \in \mathbb{F}^n$ *linearly dependent* if there exists $\alpha_1, \ldots, \alpha_k \in \mathbb{F}$ not all zero such that $\alpha_1 x_2 + \ldots + \alpha_k x_k = 0$; otherwise they are *linearly independent*.

- $\{x_1, x_2, \ldots, x_k\} \in S$ is a *basis* for $S$ if $x_1, x_2, \ldots, x_k$ are linearly independent and $S = \mathit{span}\{x_1, x_2, \ldots, x_k\}$.

- $\{x_1, x_2, \ldots, x_k\}$ in $\mathbb{F}^n$ are mutually *orthogonal* if $x_i^* x_j = 0$ for all $i \neq j$ and *orthonormal* if $x_i^* x_j = \delta_{ij}$.

- *orthogonal complement* of a subspace $S \subset \mathbb{F}^n$:

$$S^\perp := \{y \in \mathbb{F}^n : y^* x = 0 \text{ for all } x \in S\}.$$

- linear transformation

$$A : \mathbb{F}^n \longmapsto \mathbb{F}^m.$$

- *kernel* or *null space*

$$\text{Ker}A = N(A) := \{x \in \mathbb{F}^n : Ax = 0\},$$

  and the *image* or *range* of $A$ is

$$\text{Im}A = R(A) := \{y \in \mathbb{F}^m : y = Ax, \; x \in \mathbb{F}^n\}.$$

  Let $a_i$, $i = 1, 2, \ldots, n$ denote the columns of a matrix $A \in \mathbb{F}^{m \times n}$, then

$$\text{Im}A = \text{span}\{a_1, a_2, \ldots, a_n\}.$$

- The rank of a matrix $A$ is defined by

$$\text{rank}(A) = \dim(\text{Im}A).$$

$\text{rank}(A) = \text{rank}(A^*)$. $A \in \mathbb{F}^{m\times n}$ is *full row rank* if $m \le n$ and $\text{rank}(A) = m$. $A$ is *full column rank* if $n \le m$ and $\text{rank}(A) = n$.

- *unitary matrix* $U^*U = I = UU^*$.

- Let $D \in \mathbb{F}^{n\times k}$ $(n > k)$ be such that $D^*D = I$. Then there exists a matrix $D_\perp \in \mathbb{F}^{n\times(n-k)}$ such that $\begin{bmatrix} D & D_\perp \end{bmatrix}$ is a unitary matrix.

- Sylvester equation

$$AX + XB = C$$

with $A \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{m\times m}$, and $C \in \mathbb{F}^{n\times m}$ has a unique solution $X \in \mathbb{F}^{n\times m}$ if and only if $\lambda_i(A) + \lambda_j(B) \neq 0$, $\forall i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

"Lyapunov Equation": $B = A^*$.

- Let $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times k}$. Then

$$\text{rank}\,(A) + \text{rank}(B) - n \le \text{rank}(AB) \le \min\{\text{rank}\,(A),\ \text{rank}(B)\}.$$

- the *trace* of $A = [a_{ij}] \in \mathbb{C}^{n\times n}$

$$\text{Trace}(A) := \sum_{i=1}^{n} a_{ii}.$$

Trace has the following properties:

$$\text{Trace}(\alpha A) = \alpha\,\text{Trace}(A),\ \ \forall \alpha \in \mathbb{C},\ A \in \mathbb{C}^{n\times n}$$

$$\text{Trace}(A + B) = \text{Trace}(A) + \text{Trace}(B),\ \ \forall A,\ B \in \mathbb{C}^{n\times n}$$

$$\text{Trace}(AB) = \text{Trace}(BA),\ \ \forall A \in \mathbb{C}^{n\times m},\ B \in \mathbb{C}^{m\times n}.$$

# Eigenvalues and Eigenvectors

- The eigenvalues and eigenvectors of $A \in \mathbb{C}^{n\times n}$: $\lambda$, $x \in \mathbb{C}^n$

$$Ax = \lambda x$$

$x$ is a *right eigenvector*

$y$ is a *left eigenvector*:

$$y^*A = \lambda y^*.$$

- eigenvalues: the roots of $\det(\lambda I - A)$.
- the *spectral radius:* $\rho(A) := \max_{1\le i\le n} |\lambda_i|$
- Jordan canonical form: $A \in \mathbb{C}^{n\times n}$, $\exists\ T$

$$A = TJT^{-1}$$

where

$$J = \operatorname{diag}\{J_1, J_2, \ldots, J_l\}$$

$$J_i = \operatorname{diag}\{J_{i1}, J_{i2}, \ldots, J_{im_i}\}$$

$$J_{ij} = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix} \in \mathbb{C}^{n_{ij}\times n_{ij}}$$

The transformation $T$ has the following form:

$$T = \begin{bmatrix} T_1 & T_2 & \ldots & T_l \end{bmatrix}$$

$$T_i = \begin{bmatrix} T_{i1} & T_{i2} & \ldots & T_{im_i} \end{bmatrix}$$

$$T_{ij} = \begin{bmatrix} t_{ij1} & t_{ij2} & \ldots & t_{ijn_{ij}} \end{bmatrix}$$

where $t_{ij1}$ are the eigenvectors of $A$,

$$At_{ij1} = \lambda_i t_{ij1},$$

and $t_{ijk} \neq 0$ defined by the following linear equations for $k \geq 2$

$$(A - \lambda_i I)t_{ijk} = t_{ij(k-1)}$$

are called the *generalized eigenvectors* of $A$.

$A \in \mathbb{R}^{n\times n}$ with distinct eigenvalues can be diagonalized:

$$A\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

and has the following *spectral decomposition*:

$$A = \sum_{i=1}^{n} \lambda_i x_i y_i^*$$

where $y_i \in \mathbb{C}^n$ is given by

$$\begin{bmatrix} y_1^* \\ y_2^* \\ \vdots \\ y_n^* \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^{-1}.$$

- $A \in \mathbb{R}^{n\times n}$ with real eigenvalue $\lambda \in \mathbb{R} \Rightarrow$ real eigenvector $x \in \mathbb{R}^n$.
- $A$ is *Hermitian*, i.e., $A = A^* \Rightarrow \exists$ unitary $U$ such that $A = U\Lambda U^*$ and $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is real.

# Matrix Inversion Formulas

- $$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$
  $\Delta := A_{22} - A_{21}A_{11}^{-1}A_{12}$
- $$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{\Delta} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix}$$
  $\hat{\Delta} := A_{11} - A_{12}A_{22}^{-1}A_{21}$
- $$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}\Delta^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}\Delta^{-1} \\ -\Delta^{-1}A_{21}A_{11}^{-1} & \Delta^{-1} \end{bmatrix}$$
  and
  $$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \hat{\Delta}^{-1} & -\hat{\Delta}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}\hat{\Delta}^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}\hat{\Delta}^{-1}A_{12}A_{22}^{-1} \end{bmatrix}.$$

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

- $\det A = \det A_{11} \det(A_{22} - A_{21}A_{11}^{-1}A_{12}) = \det A_{22} \det(A_{11} - A_{12}A_{22}^{-1}A_{21})$.
  In particular, for any $B \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{n\times m}$, we have
  $$\det \begin{bmatrix} I_m & B \\ -C & I_n \end{bmatrix} = \det(I_n + CB) = \det(I_m + BC)$$
  and for $x, y \in \mathbb{C}^n \quad \det(I_n + xy^*) = 1 + y^*x$.
- *matrix inversion lemma:*
  $$(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} = A_{11}^{-1} + A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1}.$$

# Invariant Subspaces

- a subspace $S \subset \mathbb{C}^n$ is an $A$-*invariant* subspace if $Ax \in S$ for every $x \in S$.

  For example, $\{0\}$, $\mathbb{C}^n$, and $\mathrm{Ker}A$ are all $A$-invariant subspaces.

  Let $\lambda$ and $x$ be an eigenvalue and a corresponding eigenvector of $A \in \mathbb{C}^{n\times n}$. Then $S := \mathrm{span}\{x\}$ is an $A$-invariant subspace since $Ax = \lambda x \in S$.

  In general, let $\lambda_1, \ldots, \lambda_k$ (not necessarily distinct) and $x_i$ be a set of eigenvalues and a set of corresponding eigenvectors and the generalized eigenvectors. Then $S = \mathrm{span}\{x_1, \ldots, x_k\}$ is an $A$-invariant subspace provided that all the lower rank generalized eigenvectors are included.

- An $A$-invariant subspace $S \subset \mathbb{C}^n$ is called a *stable invariant subspace* if all the eigenvalues of $A$ constrained to $S$ have negative real parts.

  Stable invariant subspaces are used to compute the stabilizing solutions of the algebraic Riccati equations

- **Example**

$$A\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}\begin{bmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix}$$

  with $\mathrm{Re}\lambda_1 < 0$, $\lambda_3 < 0$, and $\lambda_4 > 0$. Then it is easy to verify that

$$\begin{array}{llllll} S_1 = \mathrm{span}\{x_1\} & S_{12} = \mathrm{span}\{x_1, x_2\} & S_{123} = \mathrm{span}\{x_1, x_2, x_3\} \\ S_3 = \mathrm{span}\{x_3\} & S_{13} = \mathrm{span}\{x_1, x_3\} & S_{124} = \mathrm{span}\{x_1, x_2, x_4\} \\ S_4 = \mathrm{span}\{x_4\} & S_{14} = \mathrm{span}\{x_1, x_4\} & S_{34} = \mathrm{span}\{x_3, x_4\} \end{array}$$

  are all $A$-invariant subspaces. Moreover, $S_1, S_3, S_{12}, S_{13}$, and $S_{123}$ are stable $A$-invariant subspaces.

However, the subspaces

$$S_2 = \text{span}\{x_2\}, \quad S_{23} = \text{span}\{x_2, x_3\}$$

$$S_{24} = \text{span}\{x_2, x_4\}, \quad S_{234} = \text{span}\{x_2, x_3, x_4\}$$

are not $A$-invariant subspaces since the lower rank generalized eigenvector $x_1$ of $x_2$ is not in these subspaces.

To illustrate, consider the subspace $S_{23}$. It is an $A$-invariant subspace if $Ax_2 \in S_{23}$. Since

$$Ax_2 = \lambda x_2 + x_1,$$

$Ax_2 \in S_{23}$ would require that $x_1$ be a linear combination of $x_2$ and $x_3$, but this is impossible since $x_1$ is independent of $x_2$ and $x_3$.

## Vector Norms and Matrix Norms

$X$ a vector space. $\|\cdot\|$ is a *norm* if

(i) $\|x\| \geq 0$ (positivity);

(ii) $\|x\| = 0$ if and only if $x = 0$ (positive definiteness);

(iii) $\|\alpha x\| = |\alpha| \, \|x\|$, for any scalar $\alpha$ (homogeneity);

(iv) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

for any $x \in X$ and $y \in X$.

Let $x \in \mathbb{C}^n$. Then we define the vector $p$-norm of $x$ as

$$\|x\|_p := \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \text{ for } 1 \leq p \leq \infty.$$

In particular, when $p = 1, 2, \infty$ we have

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|;$$

$$\|x\|_2 := \sqrt{\sum_{i=1}^{n} |x_i|^2};$$

$$\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|.$$

the matrix norm *induced* by a vector $p$-norm is defined as

$$\|A\|_p := \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}.$$

In particular, for $p = 1, 2, \infty$, the corresponding induced matrix norm can be computed as

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}| \quad \text{(column sum)} \; ;$$

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^* A)} \; ;$$

$$\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}| \quad \text{(row sum)} .$$

The Euclidean 2-norm has some very nice properties:
Let $x \in \mathbb{F}^n$ and $y \in \mathbb{F}^m$.

1. Suppose $n \ge m$. Then $\|x\| = \|y\|$ iff there is a matrix $U \in \mathbb{F}^{n\times m}$ such that $x = Uy$ and $U^*U = I$.
2. Suppose $n = m$. Then $|x^*y| \le \|x\| \|y\|$. Moreover, the equality holds iff $x = \alpha y$ for some $\alpha \in \mathbb{F}$ or $y = 0$.
3. $\|x\| \le \|y\|$ iff there is a matrix $\Delta \in \mathbb{F}^{n\times m}$ with $\|\Delta\| \le 1$ such that $x = \Delta y$. Furthermore, $\|x\| < \|y\|$ iff $\|\Delta\| < 1$.
4. $\|Ux\| = \|x\|$ for any appropriately dimensioned unitary matrices $U$.

*Frobenius norm*

$$\|A\|_F := \sqrt{\mathrm{Trace}(A^*A)} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2} .$$

Let $A$ and $B$ be any matrices with appropriate dimensions. Then

1. $\rho(A) \le \|A\|$ (This is also true for $F$ norm and any induced matrix norm).
2. $\|AB\| \le \|A\| \|B\|$. In particular, this gives $\|A^{-1}\| \ge \|A\|^{-1}$ if $A$ is invertible. (This is also true for any induced matrix norm.)
3. $\|UAV\| = \|A\|$, and $\|UAV\|_F = \|A\|_F$, for any appropriately dimensioned unitary matrices $U$ and $V$.
4. $\|AB\|_F \le \|A\| \|B\|_F$ and $\|AB\|_F \le \|B\| \|A\|_F$.

## Singular Value Decomposition

Let $A \in \mathbb{F}^{m\times n}$. There exist unitary matrices

$$\begin{aligned} U &= [u_1, u_2, \ldots, u_m] \in \mathbb{F}^{m\times m} \\ V &= [v_1, v_2, \ldots, v_n] \in \mathbb{F}^{n\times n} \end{aligned}$$

such that

$$A = U\Sigma V^*, \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$\Sigma_1 = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_p \end{bmatrix}$$

and

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0, \;\; p = \min\{m, n\}.$$

Singular values are good measures of the "size" of a matrix

Singular vectors are good indications of strong/weak input or output directions.

Note that

$$\begin{aligned} Av_i &= \sigma_i u_i \\ A^* u_i &= \sigma_i v_i. \end{aligned}$$

$$\begin{aligned} A^* A v_i &= \sigma_i^2 v_i \\ AA^* u_i &= \sigma_i^2 u_i. \end{aligned}$$

$$\overline{\sigma}(A) = \sigma_{max}(A) = \sigma_1 = \text{the largest singular value of } A;$$

and

$$\underline{\sigma}(A) = \sigma_{min}(A) = \sigma_p = \text{the smallest singular value of } A\ .$$

Geometrically, the singular values of a matrix $A$ are precisely the lengths of the semi-axes of the hyper-ellipsoid $E$ defined by

$$E = \{y : y = Ax, \ x \in \mathbb{C}^n, \ \|x\| = 1\}.$$

Thus $v_1$ is the direction in which $\|y\|$ is the largest for all $\|x\| = 1$; while $v_n$ is the direction in which $\|y\|$ is the smallest for all $\|x\| = 1$.

$v_1$ ($v_n$) is the *highest (lowest) gain input direction*

$u_1$ ($u_m$) is the *highest (lowest) gain observing direction*

e.g.,

$$A = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}.$$

$A$ maps a unit disk to an ellipsoid with semi-axes of $\sigma_1$ and $\sigma_2$.

alternative definitions:

$$\overline{\sigma}(A) := \max_{\|x\|=1} \|Ax\|$$

and for the smallest singular value $\underline{\sigma}$ of a *tall matrix*:

$$\underline{\sigma}(A) := \min_{\|x\|=1} \|Ax\|.$$

Suppose $A$ and $\Delta$ are square matrices. Then

(i) $|\underline{\sigma}(A + \Delta) - \underline{\sigma}(A)| \leq \overline{\sigma}(\Delta)$;

(ii) $\underline{\sigma}(A\Delta) \geq \underline{\sigma}(A)\underline{\sigma}(\Delta)$;

(iii) $\overline{\sigma}(A^{-1}) = \dfrac{1}{\underline{\sigma}(A)}$ if $A$ is invertible.

Some useful properties
Let $A \in \mathbb{F}^{m\times n}$ and

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = 0, \ \ r \leq \min\{m, n\}.$$

Then

1. $\mathrm{rank}(A) = r$;
2. $\mathrm{Ker}A = \mathrm{span}\{v_{r+1}, \ldots, v_n\}$ and $(\mathrm{Ker}A)^{\perp} = \mathrm{span}\{v_1, \ldots, v_r\}$;
3. $\mathrm{Im}A = \mathrm{span}\{u_1, \ldots, u_r\}$ and $(\mathrm{Im}A)^{\perp} = \mathrm{span}\{u_{r+1}, \ldots, u_m\}$;
4. $A \in \mathbb{F}^{m\times n}$ has a dyadic expansion:

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^* = U_r \Sigma_r V_r^*$$

   where $U_r = [u_1, \ldots, u_r]$, $V_r = [v_1, \ldots, v_r]$, and $\Sigma_r = \mathrm{diag}\,(\sigma_1, \ldots, \sigma_r)$;
5. $\|A\|_F^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2$;
6. $\|A\| = \sigma_1$;
7. $\sigma_i(U_0 A V_0) = \sigma_i(A), \ i = 1, \ldots, p$ for any appropriately dimensioned unitary matrices $U_0$ and $V_0$;
8. Let $k < r = \mathrm{rank}(A)$ and $A_k := \Sigma_{i=1}^{k} \sigma_i u_i v_i^*$, then

$$\min_{\mathrm{rank}(B) \leq k} \|A - B\| = \|A - A_k\| = \sigma_{k+1}.$$

## Generalized Inverses

Let $A \in \mathbb{C}^{m\times n}$. $X \in \mathbb{C}^{n\times m}$ is a *right inverse* if $AX = I$. one of the right inverses is given by $X = A^*(AA^*)^{-1}$.

$YA = I$ then $Y$ is a *left inverse* of $A$.

*pseudo-inverse*or *Moore-Penrose inverse* $A^+$:

(i) $AA^+A = A$;

(ii) $A^+AA^+ = A^+$;

(iii) $(AA^+)^* = AA^+$;

(iv) $(A^+A)^* = A^+A$.

pseudo-inverse is unique.

$$A = BC$$

$B$ has full column rank and $C$ has full row rank. Then

$$A^+ = C^*(CC^*)^{-1}(B^*B)^{-1}B^*.$$

or

$$A = U\Sigma V^*$$

with

$$\Sigma = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_r > 0.$$

Then $A^+ = V\Sigma^+U^*$ with

$$\Sigma^+ = \begin{bmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

## Semidefinite Matrices

- $A = A^*$ is *positive definite (semi-definite)* denoted by $A > 0$ $(\geq 0)$, if $x^*Ax > 0$ $(\geq 0)$ for all $x \neq 0$.
- $A \in \mathbb{F}^{n\times n}$ and $A = A^* \geq 0$, $\exists$ $B \in \mathbb{F}^{n\times r}$ with $r \geq \text{rank}(A)$ such that $A = BB^*$.
- Let $B \in \mathbb{F}^{m\times n}$ and $C \in \mathbb{F}^{k\times n}$. Suppose $m \geq k$ and $B^*B = C^*C$. $\exists$ $U \in \mathbb{F}^{m\times k}$ such that $U^*U = I$ and $B = UC$.
- *square root* for a positive semi-definite matrix $A$, $A^{1/2} = (A^{1/2})^* \geq 0$, by

$$A = A^{1/2}A^{1/2}.$$

Clearly, $A^{1/2}$ can be computed by using spectral decomposition or SVD: let $A = U\Lambda U^*$, then

$$A^{1/2} = U\Lambda^{1/2}U^*$$

where

$$\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_n\}, \quad \Lambda^{1/2} = \text{diag}\{\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}\}.$$

- $A = A^* > 0$ and $B = B^* \geq 0$. Then $A > B$ iff $\rho(BA^{-1}) < 1$.
- Let $X = X^* \geq 0$ be partitioned as

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}.$$

Then $\text{Ker}X_{22} \subset \text{Ker}X_{12}$. Consequently, if $X_{22}^+$ is the pseudo-inverse of $X_{22}$, then $Y = X_{12}X_{22}^+$ solves

$$YX_{22} = X_{12}$$

and

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} = \begin{bmatrix} I & X_{12}X_{22}^+ \\ 0 & I \end{bmatrix} \begin{bmatrix} X_{11} - X_{12}X_{22}^+X_{12}^* & 0 \\ 0 & X_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ X_{22}^+X_{12}^* & I \end{bmatrix}.$$

# Chapter 3: Linear Systems

- dynamical systems
- controllability and stabilizability
- observability and detectability
- observer theory
- system interconnections
- realizations
- poles and zeros

# Dynamical Systems

- Linear equations:

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(t_0) = x_0 \\ y &= Cx + Du \end{aligned}$$

- transfer matrix:

$$Y(s) = G(s)U(s)$$

$$G(s) = C(sI - A)^{-1}B + D.$$

- notation

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(sI - A)^{-1}B + D$$

- solution:

$$\begin{aligned} x(t) &= e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

- impulse matrix

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = Ce^{At}B1_{+}(t) + D\delta(t)$$

- input/output relationship:

$$y(t) = (g * u)(t) := \int_{-\infty}^{t} g(t - \tau)u(\tau)d\tau.$$

$\gg$ **G=pck(A, B, C, D)** % pack the realization in partitioned form

$\gg$ **seesys(G)** % display $G$ in partitioned format

$\gg$ **[A, B, C, D]=unpck(G)** % unpack the system matrix

$\gg$ **G=pck([], [], [], 10)** % create a constant system matrix

$\gg$ **[y, x, t]=step(A, B, C, D, Iu)** % Iu=i (step response of the $i$th channel)

$\gg$ **[y, x, t]=initial(A, B, C, D, $\mathbf{x_0}$)** % initial response with initial condition $x_0$

$\gg$ **[y, x, t]=impulse(A, B, C, D, Iu)** % impulse response of the $Iu$th channel

$\gg$ **[y,x]=lsim(A,B,C,D,U,T)** % U is a length($T$) $\times$ column($B$) matrix input; $T$ is the sampling points.

## Controllability

- Controllability: $(A, B)$ is *controllable* if, for any initial state $x(0) = x_0$, $t_1 > 0$ and final state $x_1$, there exists a (piecewise continuous) input $u(\cdot)$ such that satisfies $x(t_1) = x_1$.
- The matrix
$$W_c(t) := \int_0^t e^{A\tau} BB^* e^{A^*\tau} d\tau$$
is positive definite for any $t > 0$.
- The controllability matrix
$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$
has full row rank, i.e., $\langle A \,|\mathrm{Im} B\rangle := \sum_{i=1}^n \mathrm{Im}(A^{i-1}B) = \mathbb{R}^n$.
- The eigenvalues of $A + BF$ can be freely assigned by a suitable $F$.

**PBH test:**

- The matrix $[A - \lambda I,\ B]$ has full row rank for all $\lambda$ in $\mathbb{C}$.
- Let $\lambda$ and $x$ be any eigenvalue and *any* corresponding left eigenvector of $A$, i.e., $x^*A = x^*\lambda$, then $x^*B \neq 0$.

# Stability and Stabilizability

$A$ is *stable* if $\mathrm{Re}\lambda(A) < 0$.

- $(A,\ B)$ is stabilizable.
- $A + BF$ is stable for some $F$.

**PBH test:**

- The matrix $[A - \lambda I,\ B]$ has full row rank for all $\mathrm{Re}\lambda \geq 0$.
- For all $\lambda$ and $x$ such that $x^*A = x^*\lambda$ and $\mathrm{Re}\lambda \geq 0$, $x^*B \neq 0$.

## Observability

- $(C, A)$ is *observable* if, for any $t_1 > 0$, the initial state $x(0) = x_0$ can be determined from the time history of the input $u(t)$ and the output $y(t)$ in the interval of $[0, t_1]$.
- The matrix
$$W_o(t) := \int_0^t e^{A^*\tau} C^* C e^{A\tau} d\tau$$
is positive definite for any $t > 0$.
- The observability matrix
$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$
has full column rank, i.e., $\cap_{i=1}^n \mathrm{Ker}(CA^{i-1}) = 0$.
- The eigenvalues of $A + LC$ can be freely assigned by a suitable $L$.
- $(A^*, C^*)$ is controllable.

**PBH test:**

- The matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full column rank for all $\lambda$ in $\mathbb{C}$.
- Let $\lambda$ and $y$ be *any* eigenvalue and any corresponding right eigenvector of $A$, i.e., $Ay = \lambda y$, then $Cy \neq 0$.

## Detectability

The following are equivalent:

- $(C, A)$ is detectable.
- $A + LC$ is stable for a suitable $L$.
- $(A^*, C^*)$ is stabilizable.

**PBH test:**

- The matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full column rank for all $\mathrm{Re}\lambda \geq 0$.
- For all $\lambda$ and $x$ such that $Ax = \lambda x$ and $\mathrm{Re}\lambda \geq 0$, $Cx \neq 0$.

an example:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cccc|c} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 1 \\ 0 & 0 & \lambda_1 & 0 & \alpha \\ 0 & 0 & 0 & \lambda_2 & 1 \\ \hline 1 & 0 & 0 & \beta & 0 \end{array}\right]$$

$\gg$ $\mathcal{C}$= **ctrb(A, B);** $\mathcal{O}$= **obsv(A, C);**

$\gg$ $\mathbf{W_c}(\infty)$**=gram(A, B);** % if $A$ is stable.

$\gg$ **F=-place(A, B, P)** % $P$ is a vector of desired eigenvalues.

## Observers and Observer-Based Controllers

An observer is a dynamical system with input of $(u, y)$ and output of, say $\hat{x}$, which asymptotically estimates the state $x$, i.e., $\hat{x}(t) - x(t) \to 0$ as $t \to \infty$ for all initial states and for every input.

An observer exists iff $(C, A)$ is detectable. Further, if $(C, A)$ is detectable, then a full order Luenberger observer is given by

$$\begin{aligned} \dot{q} &= Aq + Bu + L(Cq + Du - y) && (0.1) \\ \hat{x} &= q && (0.2) \end{aligned}$$

where $L$ is any matrix such that $A + LC$ is stable.

Observer-based controller:

$$\begin{aligned} \dot{\hat{x}} &= (A + LC)\hat{x} + Bu + LDu - Ly \\ u &= F\hat{x}. \end{aligned}$$

$$u = K(s)y$$

and

$$K(s) = \left[\begin{array}{c|c} A + BF + LC + LDF & -L \\ \hline F & 0 \end{array}\right].$$

## Example

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$.
Design $u = Fx$ such that the closed-loop poles are at $\{-2, -3\}$
$F = \begin{bmatrix} -6 & -8 \end{bmatrix}$

$$\gg F = -\text{place}(A, B, [-2, -3]).$$

Suppose observer poles are at $\{-10, -10\}$
Then $L = \begin{bmatrix} -21 \\ -51 \end{bmatrix}$ can be obtained by using

$$\gg L = -\text{acker}(A', C', [-10, -10])'$$

and the observer-based controller is given by

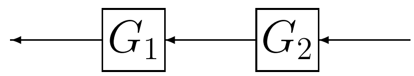
$$K(s) = \frac{-534(s + 0.6966)}{(s + 34.6564)(s - 8.6564)}.$$

stabilizing controller itself is unstable: this may not be desirable in practice.

# Operations on Systems

$$G_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \qquad G_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right].$$

- cascade:

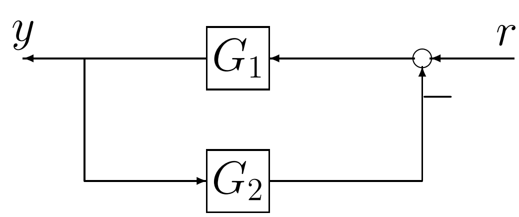



$$\begin{aligned} G_1 G_2 &= \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1 C_2 & A_1 & B_1 D_2 \\ \hline D_1 C_2 & C_1 & D_1 D_2 \end{array}\right]. \end{aligned}$$

- addition:

$$G_1 + G_2 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] + \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right].$$

- feedback:




$$T = \left[\begin{array}{cc|c} A_1 - B_1 D_2 R_{12}^{-1} C_1 & -B_1 R_{21}^{-1} C_2 & B_1 R_{21}^{-1} \\ B_2 R_{12}^{-1} C_1 & A_2 - B_2 D_1 R_{21}^{-1} C_2 & B_2 D_1 R_{21}^{-1} \\ \hline R_{12}^{-1} C_1 & -R_{12}^{-1} D_1 C_2 & D_1 R_{21}^{-1} \end{array}\right]$$

where $R_{12} = I + D_1 D_2$ and $R_{21} = I + D_2 D_1$.

- *transpose* or *dual system*

$$G \longmapsto G^T(s) = B^*(sI - A^*)^{-1}C^* + D^*$$

or equivalently

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \longmapsto \left[\begin{array}{c|c} A^* & C^* \\ \hline B^* & D^* \end{array}\right].$$

- *conjugate* system

$$G \longmapsto G^{\sim}(s) := G^T(-s) = B^*(-sI - A^*)^{-1}C^* + D^*$$

or equivalently

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \longmapsto \left[\begin{array}{c|c} -A^* & -C^* \\ \hline B^* & D^* \end{array}\right].$$

In particular, we have $G^*(j\omega) := [G(j\omega)]^* = G^{\sim}(j\omega)$.

- Let $D^\dagger$ denote a right (left) inverse of $D$ if $D$ has full row (column) rank. Then

$$G^\dagger = \left[\begin{array}{c|c} A - BD^\dagger C & -BD^\dagger \\ \hline D^\dagger C & D^\dagger \end{array}\right]$$

is a right (left) inverse of $G$.

$$G_1G_2 \iff \mathbf{mmult(G_1, G_2)}, \quad \begin{bmatrix} G_1 & G_2 \end{bmatrix} \iff \mathbf{sbs(G_1, G_2)}$$

$$G_1 + G_2 \iff \mathbf{madd(G_1, G_2)}, \quad G_1 - G_2 \iff \mathbf{msub(G_1, G_2)}$$

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \iff \mathbf{abv(G_1, G_2)}, \quad \begin{bmatrix} G_1 & \\ & G_2 \end{bmatrix} \iff \mathbf{daug(G_1, G_2)},$$

$$G^T(s) \iff \mathbf{transp(G)}, \quad G^{\sim}(s) \iff \mathbf{cjt(G)}, \quad G^{-1}(s) \iff \mathbf{minv(G)}$$

$$\alpha\, G(s) \iff \mathbf{mscl}(\mathbf{G}, \alpha), \quad \alpha \text{ is a scalar.}$$

## State Space Realizations

Given $G(s)$, find $(A, B, C, D)$ such that

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

which is a *state space realization* of $G(s)$.

- A state space realization $(A, B, C, D)$ of $G(s)$ is minimal if and only if $(A, B)$ is controllable and $(C, A)$ is observable.
- Let $(A_1, B_1, C_1, D)$ and $(A_2, B_2, C_2, D)$ be two minimal realizations of $G(s)$. Then there exists a unique nonsingular $T$ such that

  $$A_2 = TA_1T^{-1}, \quad B_2 = TB_1, \quad C_2 = C_1T^{-1}.$$

  Furthermore, $T$ can be specified as

  $$T = (\mathcal{O}_2^*\mathcal{O}_2)^{-1}\mathcal{O}_2^*\mathcal{O}_1$$

  or

  $$T^{-1} = \mathcal{C}_1\mathcal{C}_2^*(\mathcal{C}_2\mathcal{C}_2^*)^{-1}.$$

  where $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{O}_1$, and $\mathcal{O}_2$ are e the corresponding controllability and observability matrices, respectively.

# SIMO and MISO

SIMO Case: Let

$$G(s) = \begin{pmatrix} g_1(s) \\ g_2(s) \\ \vdots \\ g_m(s) \end{pmatrix} = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1} s + \beta_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} + d,$$

where $\beta_i \in \mathbb{R}^m$ and $d \in \mathbb{R}^m$. Then

$$G(s) = \left[\begin{array}{c|c} A & b \\ \hline C & d \end{array}\right], \quad b \in \mathbb{R}^n,\ C \in \mathbb{R}^{m \times n},\ d \in \mathbb{R}^m$$

where

$$A := \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad b := \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \end{bmatrix}$$

MISO Case: Let

$$\begin{aligned} G(s) &= \begin{pmatrix} g_1(s) & g_2(s) & \ldots & g_p(s) \end{pmatrix} \\ &= \frac{\eta_1 s^{n-1} + \eta_2 s^{n-2} + \cdots + \eta_{n-1} s + \eta_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} + d \end{aligned}$$

with $\eta_i^*, d^* \in \mathbb{R}^p$. Then

$$G(s) = \left[\begin{array}{ccccc|c} -a_1 & 1 & 0 & \cdots & 0 & \eta_1 \\ -a_2 & 0 & 1 & \cdots & 0 & \eta_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 & \eta_{n-1} \\ -a_n & 0 & 0 & \cdots & 0 & \eta_n \\ \hline 1 & 0 & 0 & \cdots & 0 & d \end{array}\right]$$

## Realizing Each Elements

To illustrate, consider a $2 \times 2$ (block) matrix

$$G(s) = \begin{bmatrix} G_1(s) & G_2(s) \\ G_3(s) & G_4(s) \end{bmatrix}$$

and assume that $G_i(s)$ has a state space realization of

$$G_i(s) = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], \quad i = 1, \ldots, 4.$$

Note that $G_i(s)$ may itself be a MIMO transfer matrix.

Then a realization for $G(s)$ can be given by

$$G(s) = \left[\begin{array}{cccc|cc} A_1 & 0 & 0 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & 0 & 0 & B_2 \\ 0 & 0 & A_3 & 0 & B_3 & 0 \\ 0 & 0 & 0 & A_4 & 0 & B_4 \\ \hline C_1 & C_2 & 0 & 0 & D_1 & D_2 \\ 0 & 0 & C_3 & C_4 & D_3 & D_4 \end{array}\right].$$

Problem: minimality.

$\gg$ **G=nd2sys(num, den, gain); G=zp2sys(zeros, poles, gain);**

# Gilbert's Realization

Let $G(s)$ be a $p \times m$ transfer matrix

$$G(s) = \frac{N(s)}{d(s)}$$

with $d(s)$ a scalar polynomial. For simplicity, we shall assume that $d(s)$ has only real and distinct roots $\lambda_i \neq \lambda_j$ if $i \neq j$ and

$$d(s) = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_r).$$

Then $G(s)$ has the following partial fractional expansion:

$$G(s) = D + \sum_{i=1}^{r} \frac{W_i}{s - \lambda_i}.$$

Suppose

$$\text{rank } W_i = k_i$$

and let $B_i \in \mathbb{R}^{k_i \times m}$ and $C_i \in \mathbb{R}^{p \times k_i}$ be two constant matrices such that

$$W_i = C_i B_i.$$

Then a realization for $G(s)$ is given by

$$G(s) = \left[\begin{array}{ccc|c} \lambda_1 I_{k_1} & & & B_1 \\ & \ddots & & \vdots \\ & & \lambda_r I_{k_r} & B_r \\ \hline C_1 & \cdots & C_r & D \end{array}\right].$$

This realization is controllable and observable (minimal) by PBH tests.

# Repeated Poles

Note that

$$G(s) = \left[\begin{array}{cc|c} \lambda & 1 & b_1 \\ 0 & \lambda & b_2 \\ \hline c_1 & c_2 & 0 \end{array}\right]$$

$$= \frac{c_1[b_2 + (s-\lambda)b_1]}{(s-\lambda)^2} + \frac{c_2 b_2}{s-\lambda}$$

$$= \frac{\mathbf{c_1 b_2}}{(s-\lambda)^2} + \frac{\mathbf{c_1} b_1 + c_2 \mathbf{b_2}}{s-\lambda}$$

A realization procedure:

- Let $G(s)$ be a $p \times q$ matrix and have the following partial fractional expansion:

$$G(s) = \frac{R_1}{(s-\lambda)^2} + \frac{R_2}{s-\lambda}$$

- Suppose $\mathrm{rank}(R_1) = 1$ and write

$$R_1 = c_1 b_1, \quad c_1 \in \mathbb{R}^p, \; b_1 \in \mathbb{R}^q$$

- Find $c_2$ and $b_1$ if possible such that

$$\mathbf{c_1} b_1 + c_2 \mathbf{b_2} = R_2$$

  Otherwise find also matrices $C_3$ and $B_3$ such that

$$\mathbf{c_1} b_1 + c_2 \mathbf{b_2} + C_3 B_3 = R_2$$

  and $[c_1 \; C_3]$ full column rank and $\begin{bmatrix} b_2 \\ B_3 \end{bmatrix}$ full row rank.

- if $\mathrm{rank}(R_1) > 1$ then write

$$R_1 = c_1 b_1 + \tilde{c}_1 \tilde{b}_1 + \ldots$$

  and repeated the above process.

Consider a $3 \times 3$ transfer matrix:

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{2s+1}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \\ \frac{1}{(s+1)^2} & \frac{s^2+5s+3}{(s+1)^2} & \frac{s}{(s+1)^2} \\ \frac{1}{(s+1)^2(s+2)} & \frac{2s+1}{(s+1)^2(s+2)} & \frac{s}{(s+1)^2(s+2)} \end{bmatrix}.$$

$$G(s) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{(s+1)^2} \overbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}^{c_1} \overbrace{\begin{bmatrix} 1 & -1 & -1 \end{bmatrix}}^{b_2}$$

$$+\frac{1}{s+1}\left( \overbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}^{c_1} \overbrace{\begin{bmatrix} 0 & 3 & 1 \end{bmatrix}}^{b_1} + \overbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}^{c_2} \overbrace{\begin{bmatrix} 1 & -1 & -1 \end{bmatrix}}^{b_2} \right)$$

$$+\frac{1}{s+1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \end{bmatrix} + \frac{1}{s+2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -3 & -2 \end{bmatrix}$$

So a 4-th order minimal state space realization is given by

$$G(s) = \left[\begin{array}{cccc|ccc} -1 & 1 & 0 & 0 & 0 & 3 & 1 \\ 0 & -1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 & -3 & -2 \\ \hline 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right].$$

Let

$$G_3(s) = \left[\begin{array}{ccc|c} \lambda & 1 & 0 & b_1 \\ 0 & \lambda & 1 & b_2 \\ 0 & 0 & \lambda & b_3 \\ \hline c_1 & c_2 & c_3 & 0 \end{array}\right]$$

$$= \frac{c_1[b_3 + (s-\lambda)b_2 + (s-\lambda)^2 b_1]}{(s-\lambda)^3}$$

$$+ \frac{c_2[b_3 + (s-\lambda)b_2]}{(s-\lambda)^2} + \frac{c_3 b_3}{s-\lambda}$$

$$= \frac{\mathbf{c_1 b_3}}{(s-\lambda)^3} + \frac{\mathbf{c_1} b_2 + c_2 \mathbf{b_3}}{(s-\lambda)^2} + \frac{\mathbf{c_1} b_1 + c_2 b_2 + c_3 \mathbf{b_3}}{s-\lambda}$$

Example: Let

$$G(s) = \begin{bmatrix} \frac{1}{(s+2)^3(s+5)} & \frac{1}{s+5} \\ \frac{1}{s+2} & 0 \end{bmatrix}$$

$$= \frac{1}{(s+2)^3} \overbrace{\begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}}^{c_1} \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{b_3} + \frac{1}{(s+2)^2} \overbrace{\begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix}}^{c_2} \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{b_3}$$

$$+ \frac{1}{s+2} \overbrace{\begin{bmatrix} \frac{1}{27} \\ 1 \end{bmatrix}}^{c_3} \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{b_3} + \frac{1}{s+5} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{27} & 1 \end{bmatrix}$$

Take $b_1 = 0$ and $b_2 = 0$, we get

$$G(s) = \left[\begin{array}{cccc|cc} -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -5 & -\frac{1}{27} & 1 \\ \hline \frac{1}{3} & -\frac{1}{9} & \frac{1}{27} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

Example: Let

$$G(s) = \frac{\overbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}^{c_1} \overbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}^{b_3}}{(s+p)^3} + \frac{\overbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}^{\tilde{c}_1} \overbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}^{\tilde{b}_3}}{(s+p)^3}$$

$$+ \frac{\overbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}^{c_2} \overbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}^{b_3}}{(s+p)^2} + \frac{\overbrace{\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}}^{2c_1+3\tilde{c}_1} \overbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}^{\frac{1}{2}b_2 \ or \ \frac{1}{3}\tilde{c}_1}}{(s+p)^2}$$

$$+ \frac{\overbrace{\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}}^{-c_3 \ or \ -\tilde{c}_3} \overbrace{\begin{bmatrix} -1 & -1 & 0 \end{bmatrix}}^{-b_3-\tilde{b}_3}}{s+p}$$

Hence

$$G(s) = \left[\begin{array}{cccccc|ccc} -p & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -p & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & -p & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -p & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -p & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & -p & 0 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

## System Poles and Zeros

An example:

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s+2} \\ \dfrac{2}{s+2} & \dfrac{1}{s+1} \end{bmatrix}$$

which is stable and each element of $G(s)$ has no finite zeros. Let

$$K = \begin{bmatrix} \dfrac{s+2}{s-\sqrt{2}} & -\dfrac{s+1}{s-\sqrt{2}} \\ 0 & 1 \end{bmatrix}$$

which is unstable. However,

$$KG = \begin{bmatrix} -\dfrac{s+\sqrt{2}}{(s+1)(s+2)} & 0 \\ \dfrac{2}{s+2} & \dfrac{1}{s+1} \end{bmatrix}$$

is stable. This implies that $G(s)$ must have an unstable zero at $\sqrt{2}$ that cancels the unstable pole of $K$.

## Smith Form

- a square polynomial matrix $Q(s)$ is *unimodular* if and only if $\det Q(s)$ is a constant.
- Let $Q(s)$ be a $(p \times m)$ polynomial matrix. Then the *normal rank* of $Q(s)$, denoted *normalrank* $(Q(s))$, is the maximally possible rank of $Q(s)$ for at least one $s \in \mathbb{C}$.

  an example:

  $$Q(s) = \begin{bmatrix} s & 1 \\ s^2 & 1 \\ s & 1 \end{bmatrix}.$$

  $Q(s)$ has normal rank 2 since rank $Q(2) = 2$. However, $Q(0)$ has rank 1.
- *Smith form:* Let $P(s)$ be any polynomial matrix, then there exist unimodular matrices $U(s), V(s) \in \mathbb{R}[s]$ such that

  $$U(s)P(s)V(s) = S(s) := \begin{bmatrix} \gamma_1(s) & 0 & \cdots & 0 & 0 \\ 0 & \gamma_2(s) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma_r(s) & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

  and $\gamma_i(s)$ divides $\gamma_{i+1}(s)$.

  $S(s)$ is called the *Smith form* of $P(s)$. $r$ is the normal rank of $P(s)$.

an example:

$$P(s) = \begin{bmatrix} s+1 & (s+1)(2s+1) & s(s+1) \\ s+2 & (s+2)(s^2+5s+3) & s(s+2) \\ 1 & 2s+1 & s \end{bmatrix}.$$

$P(s)$ has normal rank 2 since $\det(P(s)) \equiv 0$ and

$$\det \begin{bmatrix} s+1 & (s+1)(2s+1) \\ s+2 & (s+2)(s^2+5s+3) \end{bmatrix} = (s+1)^2(s+2)^2 \not\equiv 0.$$

Let

$$U = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -(s+2) \\ 1 & 0 & -(s+1) \end{bmatrix}.$$

$$V(s) = \begin{bmatrix} 1 & -(2s+1) & -s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$S(s) = U(s)P(s)V(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (s+1)(s+2)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

## Smith-McMillan Form

- Let $G(s)$ be any proper real rational transfer matrix, then there exist unimodular matrices $U(s), V(s) \in \mathbb{R}[s]$ such that

$$U(s)G(s)V(s) = M(s) := \begin{bmatrix} \frac{\alpha_1(s)}{\beta_1(s)} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\alpha_2(s)}{\beta_2(s)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\alpha_r(s)}{\beta_r(s)} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

  and $\alpha_i(s)$ divides $\alpha_{i+1}(s)$ and $\beta_{i+1}(s)$ divides $\beta_i(s)$.

- Write $G(s)$ as $G(s) = N(s)/d(s)$ such that $d(s)$ is a scalar polynomial and $N(s)$ is a $p \times m$ polynomial matrix.

  Let the Smith form of $N(s)$ be $S(s) = U(s)N(s)V(s)$.

  Then $M(s) = S(s)/d(s)$.

- *McMillan degree* of $G(s) = \sum_i \deg(\beta_i(s))$ where $\deg(\beta_i(s))$ denotes the degree of the polynomial $\beta_i(s)$.

- McMillan degree of $G(s) =$ the dimension of a minimal realization of $G(s)$.

- *poles* of $G =$ roots of $\beta_i(s)$

- *transmission zeros* of $G(s) =$ the roots of $\alpha_i(s)$

  $z_0 \in \mathbb{C}$ is a *blocking zero* of $G(s)$ if $G(z_0) = 0$.

An example:

$$G(s) = \begin{bmatrix} \dfrac{1}{(s+1)(s+2)} & \dfrac{2s+1}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \\ \dfrac{1}{(s+1)^2} & \dfrac{s^2+5s+3}{(s+1)^2} & \dfrac{s}{(s+1)^2} \\ \dfrac{1}{(s+1)^2(s+2)} & \dfrac{2s+1}{(s+1)^2(s+2)} & \dfrac{s}{(s+1)^2(s+2)} \end{bmatrix}.$$

Then $G(s)$ can be written as

$$G(s) = \frac{1}{(s+1)^2(s+2)} \begin{bmatrix} s+1 & (s+1)(2s+1) & s(s+1) \\ s+2 & (s+2)(s^2+5s+3) & s(s+2) \\ 1 & 2s+1 & s \end{bmatrix}.$$

$G(s)$ has the McMillan form

$$M(s) = \begin{bmatrix} \dfrac{1}{(s+1)^2(s+2)} & 0 & 0 \\ 0 & \dfrac{s+2}{s+1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

McMillan degree of $G(s) = 4$.
poles of the transfer matrix: $\{-1, -1, -1, -2\}$.
transmission zero: $\{-2\}$.

*The transfer matrix has pole and zero at the same location* $\{-2\}$*; this is the unique feature of multivariable systems.*

## Alternative Characterizations

- Let $G(s)$ have full column normal rank. Then $z_0 \in \mathbb{C}$ is a transmission zero of $G(s)$ if and only if there exists a vector $0 \neq u_0$ such that $G(z_0)u_0 = 0$.

  not true if $G(s)$ does not have full column normal rank.

  an example
  $$G(s) = \frac{1}{s+1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad u_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$
  $G$ has no transmission zero but $G(s)u_0 = 0$ for all $s$.

  $z_0$ can be a pole of $G(s)$ although $G(z_0)$ is not defined. (however $G(z_0)u_0$ may be well defined.) For example,
  $$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \frac{s+2}{s-1} \end{bmatrix}, \quad u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
  Then $G(1)u_0 = 0$. Therefore, 1 is a transmission zero.

- Let $G(s)$ have full row normal rank. Then $z_0 \in \mathbb{C}$ is a transmission zero of $G(s)$ if and only if there exists a vector $\eta_0 \neq 0$ such that $\eta_0^* G(z_0) = 0$.

- Suppose $z_0 \in \mathbb{C}$ is not a pole of $G(s)$. Then $z_0$ is a transmission zero if and only if $\operatorname{rank}(G(z_0)) < \operatorname{normalrank}(G(s))$.

- Let $G(s)$ be a square $m \times m$ matrix and $\det G(s) \not\equiv 0$. Suppose $z_o \in \mathbb{C}$ is not a pole of $G(s)$. Then $z_0 \in \mathbb{C}$ is a transmission zero of $G(s)$ if and only if $\det G(z_0) = 0$.
  $$\det \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s+2} \\ \dfrac{2}{s+2} & \dfrac{1}{s+1} \end{bmatrix} = \frac{2-s^2}{(s+1)^2(s+2)^2}.$$

## Invariant Zeros

The poles and zeros of a transfer matrix can also be characterized in terms of its state space realizations:

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

consider the following system matrix

$$Q(s) = \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}.$$

$z_0 \in \mathbb{C}$ is an *invariant zero* of the realization if it satisfies

$$\operatorname{rank} \begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} < normalrank \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}.$$

- Suppose $\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$ has full column normal rank. Then $z_0 \in \mathbb{C}$ is an invariant zero iff there exist $0 \neq x \in \mathbb{C}^n$ and $u \in \mathbb{C}^m$ such that

  $$\begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0.$$

  Moreover, if $u = 0$, then $z_0$ is also a non-observable mode.

- Suppose $\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$ has full row normal rank. Then $z_0 \in \mathbb{C}$ is an invariant zero iff there exist $0 \neq y \in \mathbb{C}^n$ and $v \in \mathbb{C}^p$ such that

  $$\begin{bmatrix} y^* & v^* \end{bmatrix} \begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} = 0.$$

  Moreover, if $v = 0$, then $z_0$ is also a non-controllable mode.

- $G(s)$ has full column (row) normal rank if and only if $\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$ has full column (row) normal rank.

This follows by noting that

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ C(A - sI)^{-1} & I \end{bmatrix} \begin{bmatrix} A - sI & B \\ 0 & G(s) \end{bmatrix}$$

and

$$\text{normalrank} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = n + \text{normalrank}(G(s)).$$

- Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal realization. Then $z_0$ is a transmission zero of $G(s)$ iff it is an invariant zero of the minimal realization.

- Let $G(s)$ be a $p \times m$ transfer matrix and let $(A, B, C, D)$ be a minimal realization. Let the input be $u(t) = u_0 e^{\lambda t}$, where $\lambda \in \mathbb{C}$ is not a pole of $G(s)$ and $u_0 \in \mathbb{C}^m$ is an arbitrary constant vector, then the output with the initial state $x(0) = (\lambda I - A)^{-1} B u_0$ is $y(t) = G(\lambda) u_0 e^{\lambda t}, \ \forall t \geq 0$.

- Let $G(s)$ be a $p \times m$ transfer matrix and let $(A, B, C, D)$ be a minimal realization. Suppose that $z_0 \in \mathbb{C}$ is a transmission zero of $G(s)$ and is not a pole of $G(s)$. Then for any nonzero vector $u_0 \in \mathbb{C}^m$ such that $G(z_0)u_0 = 0$, the output of the system due to the initial state $x(0) = (z_0 I - A)^{-1} B u_0$ and the input $u = u_0 e^{z_0 t}$ is identically zero: $y(t) = G(z_0) u_0 e^{z_0 t} = 0$.

$$\underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{M} \begin{bmatrix} x \\ u \end{bmatrix} = z_0 \underbrace{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}_{N} \begin{bmatrix} x \\ u \end{bmatrix}$$

MATLAB command: **eig(M, N)**.

## Example

Let

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccc|ccc} -1 & -2 & 1 & 1 & 2 & 3 \\ 0 & 2 & -1 & 3 & 2 & 1 \\ -4 & -3 & -2 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 4 & 0 & 0 & 0 \end{array}\right].$$

Then the invariant zeros of the system can be found using the MATLAB command

$\gg$ **G=pck(A, B, C, D)**, $\quad \mathbf{z_0} = \mathbf{szeros}(\mathbf{G})$, % or

$\gg \mathbf{z_0} = \mathbf{tzero}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$

which gives $z_0 = 0.2$. Since $G(s)$ is full-row rank, we can find $y$ and $v$ such that

$$\begin{bmatrix} y^* & v^* \end{bmatrix} \begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} = 0,$$

which can again be computed using a MATLAB command:

$$\gg \mathbf{null}([\mathbf{A} - \mathbf{z_0} * \mathbf{eye}(\mathbf{3}), \mathbf{B}; \mathbf{C}, \mathbf{D}]') \implies \begin{bmatrix} y \\ v \end{bmatrix} = \left[\begin{array}{c} 0.0466 \\ 0.0466 \\ -0.1866 \\ \hdashline -0.9702 \\ 0.1399 \end{array}\right].$$

# Chapter 4: $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Spaces

- Hilbert space
- $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Functions
- State Space Computation of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms

## Hilbert Spaces

Inner product on $\mathbb{C}^n$:

$$\langle x, y\rangle := x^* y = \sum_{i=1}^n \bar{x}_i y_i \quad \forall x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},\ y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{C}^n.$$

$$\|x\| := \sqrt{\langle x, x\rangle},$$

$$\cos\angle(x, y) = \frac{\langle x, y\rangle}{\|x\|\,\|y\|}, \quad \angle(x, y) \in [0, \pi].$$

*orthogonal* if $\angle(x, y) = \frac{\pi}{2}$.

**Definition 0.1** Let $V$ be a vector space over $\mathbb{C}$. An *inner product* on $V$ is a complex valued function,

$$\langle\cdot,\cdot\rangle :\ V \times V \longmapsto \mathbb{C}$$

such that for any $x, y, z \in V$ and $\alpha, \beta \in \mathbb{C}$

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$

(ii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

(iii) $\langle x, x\rangle > 0$ if $x \neq 0$.

A vector space $V$ with an inner product is called an *inner product space*.

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inner product induced norm $\|x\| := \sqrt{\langle x, x\rangle}$
distance between vectors $x$ and $y$: $d(x, y) = \|x - y\|$.
Two vectors $x$ and $y$ *orthogonal* if $\langle x, y\rangle = 0$, denoted $x \perp y$.

- $|\langle x, y\rangle| \leq \|x\| \, \|y\|$ (Cauchy-Schwarz inequality). Equality holds iff $x = \alpha y$ for some constant $\alpha$ or $y = 0$.
- $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ (Parallelogram law) .
- $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ if $x \perp y$.

*Hilbert space:* a complete inner product space.
Examples:

- $\mathbb{C}^n$ with the usual inner product.
- $\mathbb{C}^{n\times m}$ with the inner product

$$\langle A, B\rangle := \operatorname{Trace} A^*B = \sum_{i=1}^{n}\sum_{j=1}^{m} \bar{a}_{ij} b_{ij} \quad \forall A, B \in \mathbb{C}^{n\times m}$$

- $\mathcal{L}_2[a, b]$: all square integrable and Lebesgue measurable functions defined on an interval $[a, b]$ with the inner product

$$\langle f, g\rangle := \int_a^b f(t)^* g(t) dt$$

  Matrix form: $\langle f, g\rangle := \int_a^b \operatorname{Trace}\left[f(t)^* g(t)\right] dt$.
- $\mathcal{L}_2 = \mathcal{L}_2(-\infty, \infty)$: $\langle f, g\rangle := \int_{-\infty}^{\infty} \operatorname{Trace}\left[f(t)^* g(t)\right] dt$.
- $\mathcal{L}_{2+} = \mathcal{L}_2[0, \infty)$: subspace of $\mathcal{L}_2(-\infty, \infty)$.
- $\mathcal{L}_{2-} = \mathcal{L}_2(-\infty, 0]$: subspace of $\mathcal{L}_2(-\infty, \infty)$.

## Analytic Functions

Let $S \subset \mathbb{C}$ be an open set, and let $f(s)$ be a complex valued function defined on $S$:

$$f(s) : S \longmapsto \mathbb{C}.$$

Then $f(s)$ is *analytic at a point* $z_0$ in $S$ if it is differentiable at $z_0$ and also at each point in some neighborhood of $z_0$.

It is a fact that if $f(s)$ is analytic at $z_0$ then $f$ has continuous derivatives of all orders at $z_0$. Hence, a function analytic at $z_0$ has a power series representation at $z_0$.

A function $f(s)$ is said to be *analytic in* $S$ if it has a derivative or is analytic at each point of $S$.

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*Maximum Modulus Theorem:* If $f(s)$ is defined and continuous on a closed-bounded set $S$ and analytic on the interior of $S$, then

$$\max_{s \in S} |f(s)| = \max_{s \in \partial S} |f(s)|$$

where $\partial S$ denotes the boundary of $S$.

## $\mathcal{L}_2$ and $\mathcal{H}_2$ Spaces

**$\mathcal{L}_2(j\mathbb{R})$ Space:** all complex matrix functions $F$ such that the integral below is bounded:

$$\int_{-\infty}^{\infty} \text{Trace}\left[F^*(j\omega)F(j\omega)\right] d\omega < \infty$$

with the inner product

$$\langle F, G\rangle := \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left[F^*(j\omega)G(j\omega)\right] d\omega$$

and the inner product induced norm is given by

$$\|F\|_2 := \sqrt{\langle F, F\rangle}.$$

$\mathcal{RL}_2(j\mathbb{R})$ or simply $\mathcal{RL}_2$: all real rational strictly proper transfer matrices with no poles on the imaginary axis.

**$\mathcal{H}_2$ Space:** a (closed) subspace of $\mathcal{L}_2(j\mathbb{R})$ with functions $F(s)$ analytic in $Re(s) > 0$.

$$\begin{aligned}\|F\|_2^2 &:= \sup_{\sigma>0}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left[F^*(\sigma+j\omega)F(\sigma+j\omega)\right] d\omega\right\} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left[F^*(j\omega)F(j\omega)\right] d\omega.\end{aligned}$$

$\mathcal{RH}_2$ (real rational subspace of $\mathcal{H}_2$): all strictly proper and real rational stable transfer matrices.

**$\mathcal{H}_2^\perp$ Space:** the orthogonal complement of $\mathcal{H}_2$ in $\mathcal{L}_2$, i.e., the (closed) subspace of functions in $\mathcal{L}_2$ that are analytic in $Re(s) < 0$. .

$\mathcal{RH}_2^\perp$ (the real rational subspace of $\mathcal{H}_2^\perp$): all strictly proper rational antistable transfer matrices.

*Parseval's relations*:

$$\mathcal{L}_2(-\infty,\infty) \cong \mathcal{L}_2(j\mathbb{R}) \qquad \mathcal{L}_2[0,\infty) \cong \mathcal{H}_2 \qquad \mathcal{L}_2(-\infty,0] \cong \mathcal{H}_2^\perp.$$

$$\|G\|_2 = \|g\|_2 \quad \text{where } G(s) = \mathcal{L}[g(t)] \in \mathcal{L}_2(j\mathbb{R})$$

# $\mathcal{L}_\infty$ and $\mathcal{H}_\infty$ Spaces

## $\mathcal{L}_\infty(j\mathbb{R})$ Space

$\mathcal{L}_\infty(j\mathbb{R})$ or simply $\mathcal{L}_\infty$ is a Banach space of matrix-valued (or scalar-valued) functions that are (essentially) bounded on $j\mathbb{R}$, with norm

$$\|F\|_\infty := \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \overline{\sigma}\left[F(j\omega)\right].$$

$\mathcal{RL}_\infty(j\mathbb{R})$ or simply $\mathcal{RL}_\infty$: all proper and real rational transfer matrices with no poles on the imaginary axis.

## $\mathcal{H}_\infty$ Space

$\mathcal{H}_\infty$ is a (closed) subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in the open right-half plane. The $\mathcal{H}_\infty$ norm is defined as

$$\|F\|_\infty := \sup_{\mathrm{Re}(s)>0} \overline{\sigma}\left[F(s)\right] = \sup_{\omega\in\mathbb{R}} \overline{\sigma}\left[F(j\omega)\right].$$

The second equality can be regarded as a generalization of the maximum modulus theorem for matrix functions. See Boyd and Desoer [1985] for a proof.

$\mathcal{RH}_\infty$: all proper and real rational stable transfer matrices.

## $\mathcal{H}_\infty^-$ Space

$\mathcal{H}_\infty^-$ is a (closed) subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in the open left-half plane. The $\mathcal{H}_\infty^-$ norm is defined as

$$\|F\|_\infty := \sup_{\mathrm{Re}(s)<0} \overline{\sigma}\left[F(s)\right] = \sup_{\omega\in\mathbb{R}} \overline{\sigma}\left[F(j\omega)\right].$$

$\mathcal{RH}_\infty^-$: all proper real rational antistable transfer matrices.

# $\mathcal{H}_\infty$ Norm as Induced $\mathcal{H}_2$ Norm

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Let $G(s) \in \mathcal{L}_\infty$ be a $p \times q$ transfer matrix. Then a multiplication operator is defined as

$$M_G : \mathcal{L}_2 \longmapsto \mathcal{L}_2$$

$$M_G f := Gf.$$

Then $\|M_G\| = \sup_{f \in \mathcal{L}_2} \dfrac{\|Gf\|_2}{\|f\|_2} = \|G\|_\infty$.

---

$$\begin{aligned} \|Gf\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(j\omega)G^*(j\omega)G(j\omega)f(j\omega)\ d\omega \\ &\le \|G\|_\infty^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} \|f(j\omega)\|^2\ d\omega \\ &= \|G\|_\infty^2 \|f\|_2^2. \end{aligned}$$

To show that $\|G\|_\infty$ is the least upper bound, first choose a frequency $\omega_0$ where $\overline{\sigma}\left[G(j\omega)\right]$ is maximum, i.e.,

$$\overline{\sigma}\left[G(j\omega_0)\right] = \|G\|_\infty$$

and denote the singular value decomposition of $G(j\omega_0)$ by

$$G(j\omega_0) = \overline{\sigma} u_1(j\omega_0)v_1^*(j\omega_0) + \sum_{i=2}^{r} \sigma_i u_i(j\omega_0)v_i^*(j\omega_0)$$

where $r$ is the rank of $G(j\omega_0)$ and $u_i, v_i$ have unit length.

If $\omega_0 < \infty$, write $v_1(j\omega_0)$ as

$$v_1(j\omega_0) = \begin{bmatrix} \alpha_1 e^{j\theta_1} \\ \alpha_2 e^{j\theta_2} \\ \vdots \\ \alpha_q e^{j\theta_q} \end{bmatrix}$$

where $\alpha_i \in \mathbb{R}$ is such that $\theta_i \in (-\pi, 0]$. Now let $0 \leq \beta_i \leq \infty$ be such that

$$\theta_i = \angle\left(\frac{\beta_i - j\omega_0}{\beta_i + j\omega_0}\right)$$

(with $\beta_i = \infty$ if $\theta_i = 0$) and let $f$ be given by

$$f(s) = \begin{bmatrix} \alpha_1 \frac{\beta_1 - s}{\beta_1 + s} \\ \alpha_2 \frac{\beta_2 - s}{\beta_2 + s} \\ \vdots \\ \alpha_q \frac{\beta_q - s}{\beta_q + s} \end{bmatrix} \hat{f}(s)$$

(with 1 replacing $\frac{\beta_i - s}{\beta_i + s}$ if $\theta_i = 0$) where a scalar function $\hat{f}$ is chosen so that

$$|\hat{f}(j\omega)| = \begin{cases} c & \text{if } |\omega - \omega_0| < \epsilon \text{ or } |\omega + \omega_0| < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

where $\epsilon$ is a small positive number and $c$ is chosen so that $\hat{f}$ has unit 2-norm, i.e., $c = \sqrt{\pi/2\epsilon}$. This in turn implies that $f$ has unit 2-norm. Then

$$\begin{aligned} \|Gf\|_2^2 &\approx \frac{1}{2\pi}\left[\overline{\sigma}\left[G(-j\omega_0)\right]^2 \pi + \overline{\sigma}\left[G(j\omega_0)\right]^2 \pi\right] \\ &= \overline{\sigma}\left[G(j\omega_0)\right]^2 = \|G\|_\infty^2. \end{aligned}$$

Similarly, if $\omega_0 = \infty$, the conclusion follows by letting $\omega_0 \to \infty$ in the above.

## Computing $\mathcal{L}_2$ and $\mathcal{H}_2$ Norms

Let $G(s) \in \mathcal{L}_2$ and $g(t) = \mathcal{L}^{-1}[G(s)]$. Then

$$\begin{aligned}
\|G\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\{G^*(j\omega)G(j\omega)\}\, d\omega = \frac{1}{2\pi j}\oint \text{Trace}\{G^{\sim}(s)G(s)\}\, ds. \\
&= \sum \begin{array}{c}\text{the residues of Trace}\{G^{\sim}(s)G(s)\} \\ \text{at its poles in the left half plane.}\end{array} \\
&= \int_{-\infty}^{\infty} \text{Trace}\{g^*(t)g(t)\}\, dt = \|g\|_2^2
\end{aligned}$$

Consider $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \in \mathcal{RH}_2$. Then we have

$$\|G\|_2^2 = \text{trace}(B^* L_o B) = \text{trace}(C L_c C^*)$$

where $L_o$ and $L_c$ are observability and controllability Gramians:

$$AL_c + L_c A^* + BB^* = 0 \qquad A^* L_o + L_o A + C^* C = 0.$$

---

Note that $g(t) = \mathcal{L}^{-1}(G) = \begin{cases} Ce^{At}B, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$$L_o = \int_0^\infty e^{A^*t} C^* C e^{At}\, dt, \quad L_c = \int_0^\infty e^{At} BB^* e^{A^*t}\, dt,$$

$$\begin{aligned}
\|G\|_2^2 &= \int_0^\infty \text{Trace}\{g^*(t)g(t)\}\, dt = \int_0^\infty \text{Trace}\{B^* e^{A^*t} C^* C e^{At} B\}\, dt \\
&= \text{Trace}\{B^* \int_0^\infty e^{A^*t} C^* C e^{At} dt B\} = \text{trace}(B^* L_o B) \\
&= \int_0^\infty \text{Trace}\{g(t)g^*(t)\}\, dt = \int_0^\infty \text{Trace}\{Ce^{At}BB^*e^{A^*t}C^*\}\, dt.
\end{aligned}$$

---

hypothetical input-output experiments: Apply the impulsive input $\delta(t)e_i$ ($\delta(t)$ is the unit impulse and $e_i$ is the $i^{\text{th}}$ standard basis vector) and denote the output by $z_i(t) (= g(t)e_i)$. Then $z_i \in \mathcal{L}_{2+}$ (assuming $D = 0$) and

$$\|G\|_2^2 = \sum_{i=1}^{m} \|z_i\|_2^2.$$

Can be used for nonlinear time varying systems.

## Example

Consider a transfer matrix

$$G = \begin{bmatrix} \dfrac{3(s+3)}{(s-1)(s+2)} & \dfrac{2}{s-1} \\ \dfrac{s+1}{(s+2)(s+3)} & \dfrac{1}{s-4} \end{bmatrix} = G_s + G_u$$

with

$$G_s = \left[\begin{array}{cc|cc} -2 & 0 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right], \quad G_u = \left[\begin{array}{cc|cc} 1 & 0 & 4 & 2 \\ 0 & 4 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

Then the command $\mathbf{h2norm}(\mathbf{G_s})$ gives $\|G_s\|_2 = 0.6055$ and $\mathbf{h2norm}(\mathbf{cjt}(\mathbf{G_u}))$ gives $\|G_u\|_2 = 3.182$. Hence $\|G\|_2 = \sqrt{\|G_s\|_2^2 + \|G_u\|_2^2} = 3.2393$.

$\gg$ $\mathbf{P} = \mathbf{gram}(\mathbf{A}, \mathbf{B})$; $\mathbf{Q} = \mathbf{gram}(\mathbf{A}', \mathbf{C}')$; **or** $\mathbf{P} = \mathbf{lyap}(\mathbf{A}, \mathbf{B} * \mathbf{B}')$;

$\gg$ $[\mathbf{G_s}, \mathbf{G_u}] = \mathbf{sdecomp}(\mathbf{G})$; % decompose into stable and antistable parts.

# Computing $\mathcal{L}_\infty$ and $\mathcal{H}_\infty$ Norms

Let $G(s) \in \mathcal{L}_\infty$

$$\|G\|_\infty := \text{ess } \sup_\omega \overline{\sigma}\{G(j\omega)\}.$$

- the farthest distance the Nyquist plot of $G$ from the origin
- the peak on the Bode magnitude plot
- estimation: set up a fine grid of frequency points, $\{\omega_1, \cdots, \omega_N\}$.

$$\|G\|_\infty \approx \max_{1 \le k \le N} \overline{\sigma}\{G(j\omega_k)\}.$$

---

Let $\gamma > 0$ and $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{RL}_\infty$.

$$\|G\|_\infty < \gamma \Longleftrightarrow \overline{\sigma}(D) < \gamma \ \ \& \ \ H \ \ \text{has no j}\omega \text{ eigenvalues}$$

where $H := \begin{bmatrix} A + BR^{-1}D^*C & BR^{-1}B^* \\ -C^*(I + DR^{-1}D^*)C & -(A + BR^{-1}D^*C)^* \end{bmatrix}$

and $R = \gamma^2 I - D^* D$.

---

Let $\Phi(s) = \gamma^2 I - G^\sim(s)G(s)$.

$\|G\|_\infty < \gamma$

$\Longleftrightarrow \Phi(j\omega) > 0, \ \forall \omega \in \mathbb{R}$.

$\Longleftrightarrow \det \Phi(j\omega) \neq 0$ since $\Phi(\infty) = R > 0$ and $\Phi(j\omega)$ is continuous

$\Longleftrightarrow \Phi(s)$ has no imaginary axis zero.

$\Longleftrightarrow \Phi^{-1}(s)$ has no imaginary axis pole.

$$\Phi^{-1}(s) = \left[\begin{array}{c|c} H & \begin{bmatrix} BR^{-1} \\ -C^*DR^{-1} \end{bmatrix} \\ \hline \begin{bmatrix} R^{-1}D^*C & R^{-1}B^* \end{bmatrix} & R^{-1} \end{array}\right].$$

$\Longleftrightarrow$ $H$ has no $j\omega$ axis eigenvalues if the above realization has neither uncontrollable modes nor unobservable modes on the imaginary axis.

---

Assume that $j\omega_0$ is an eigenvalue of $H$ but not a pole of $\Phi^{-1}(s)$. Then $j\omega_0$ must be either an unobservable mode of $\left(\begin{bmatrix} R^{-1}D^*C & R^{-1}B^* \end{bmatrix}, H\right)$ or an uncontrollable mode of $\left(H, \begin{bmatrix} BR^{-1} \\ -C^*DR^{-1} \end{bmatrix}\right)$. Suppose $j\omega_0$ is an unobservable mode of $\left(\begin{bmatrix} R^{-1}D^*C & R^{-1}B^* \end{bmatrix}, H\right)$. Then there exists an $x_0 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq 0$ such that

$$Hx_0 = j\omega_0 x_0, \ \begin{bmatrix} R^{-1}D^*C & R^{-1}B^* \end{bmatrix} x_0 = 0.$$

$$\Updownarrow$$

$$\begin{aligned} (j\omega_0 I - A)x_1 &= 0 \\ (j\omega_0 I + A^*)x_2 &= -C^*Cx_1 \\ D^*Cx_1 + B^*x_2 &= 0. \end{aligned}$$

Since $A$ has no imaginary axis eigenvalues, we have $x_1 = 0$ and $x_2 = 0$. Contradiction!!!

Similarly, a contradiction will also be arrived if $j\omega_0$ is assumed to be an uncontrollable mode of $\left(H, \begin{bmatrix} BR^{-1} \\ -C^*DR^{-1} \end{bmatrix}\right)$.

## Bisection Algorithm

(a) select an upper bound $\gamma_u$ and a lower bound $\gamma_l$ such that $\gamma_l \leq \|G\|_\infty \leq \gamma_u$;

(b) if $(\gamma_u - \gamma_l)/\gamma_l \leq$specified level, stop; $\|G\| \approx (\gamma_u + \gamma_l)/2$. Otherwise go to next step;

(c) set $\gamma = (\gamma_l + \gamma_u)/2$;

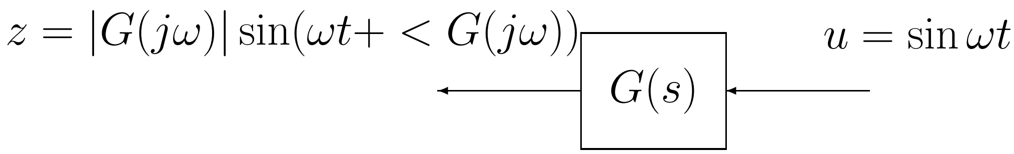
(d) test if $\|G\|_\infty < \gamma$ by calculating the eigenvalues of $H$ for the given $\gamma$;

(e) if $H$ has an eigenvalue on $j\mathbb{R}$ set $\gamma_l = \gamma$; otherwise set $\gamma_u = \gamma$; go back to step (b).

WLOG assume $\gamma = 1$ since $\|G\|_\infty < \gamma$ iff $\|\gamma^{-1}G\|_\infty < 1$

# Estimating the $\mathcal{H}_\infty$ norm

Estimating the $\mathcal{H}_\infty$ norm experimentally: the maximum magnitude of the steady-state response to all possible unit amplitude sinusoidal input signals.




Let the sinusoidal inputs be

$$u(t) = \begin{bmatrix} u_1 \sin(\omega_0 t + \phi_1) \\ u_2 \sin(\omega_0 t + \phi_2) \\ \vdots \\ u_q \sin(\omega_0 t + \phi_q) \end{bmatrix}, \quad \hat{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_q \end{bmatrix}.$$

Then the steady-state response of the system can be written as

$$y(t) = \begin{bmatrix} y_1 \sin(\omega_0 t + \theta_1) \\ y_2 \sin(\omega_0 t + \theta_2) \\ \vdots \\ y_p \sin(\omega_0 t + \theta_p) \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

for some $y_i,\ \theta_i,\ i = 1, 2, \ldots, p$, and furthermore,

$$\|G\|_\infty = \sup_{\phi_i, \omega_o, \hat{u}} \frac{\|\hat{y}\|}{\|\hat{u}\|}$$

where $\|\cdot\|$ is the Euclidean norm.

# Examples

Consider a mass/spring/damper system as shown in Figure 0.1.

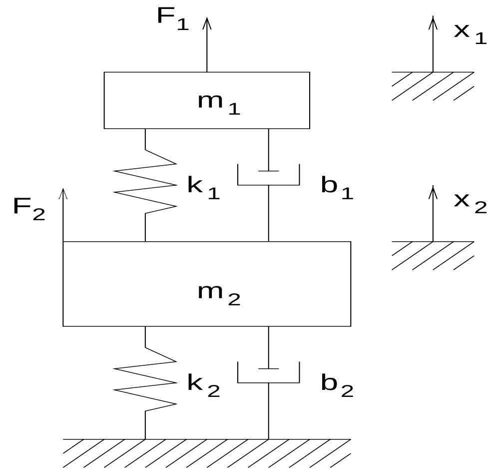



Figure 0.1: A two-mass/spring/damper system

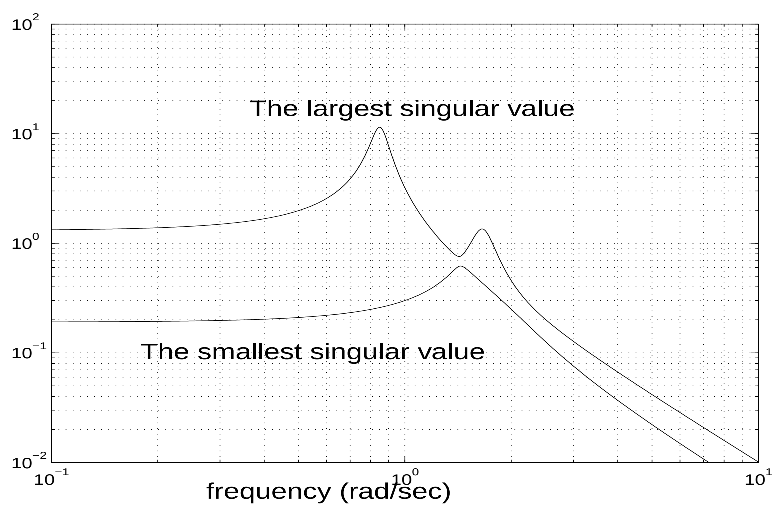



Figure 0.2: $\|G\|_\infty$ is the peak of the largest singular value of $G(j\omega)$

The dynamical system can be described by the following differential equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + B \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & \frac{k_1}{m_1} & -\frac{b_1}{m_1} & \frac{b_1}{m_1} \\ \frac{k_1}{m_2} & -\frac{k_1+k_2}{m_2} & \frac{b_1}{m_2} & -\frac{b_1+b_2}{m_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}.$$

Suppose that $G(s)$ is the transfer matrix from $(F_1, F_2)$ to $(x_1, x_2)$; that is,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = 0,$$

and suppose $k_1 = 1$, $k_2 = 4$, $b_1 = 0.2$, $b_2 = 0.1$, $m_1 = 1$, and $m_2 = 2$ with appropriate units.

$\gg$ **G=pck(A,B,C,D);**

$\gg$ **hinfnorm(G,0.0001)** or **linfnorm(G,0.0001)** % relative error $\leq 0.0001$

$\gg$ **w=logspace(-1,1,200);** % **200** points between $1 = 10^{-\mathbf{1}}$ and $10 = 10^{\mathbf{1}}$;

$\gg$ **Gf=frsp(G,w);** % computing frequency response;

$\gg$ **[u,s,v]=vsvd(Gf);** % SVD at each frequency;

$\gg$ **vplot('liv, lm', s),** **grid** % plot both singular values and grid.

$\|G(s)\|_\infty = 11.47 =$ the peak of the largest singular value Bode plot in Figure 0.2.

Since the peak is achieved at $\omega_{\max} = 0.8483$, exciting the system using the following sinusoidal input

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 0.9614 \sin(0.8483t) \\ 0.2753 \sin(0.8483t - 0.12) \end{bmatrix}$$

gives the steady-state response of the system as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 11.47 \times 0.9614 \sin(0.8483t - 1.5483) \\ 11.47 \times 0.2753 \sin(0.8483t - 1.4283) \end{bmatrix}.$$

This shows that the system response will be amplified 11.47 times for an input signal at the frequency $\omega_{\max}$, which could be undesirable if $F_1$ and $F_2$ are disturbance force and $x_1$ and $x_2$ are the positions to be kept steady.

---

Consider a two-by-two transfer matrix

$$G(s) = \begin{bmatrix} \dfrac{10(s+1)}{s^2+0.2s+100} & \dfrac{1}{s+1} \\ \dfrac{s+2}{s^2+0.1s+10} & \dfrac{5(s+1)}{(s+2)(s+3)} \end{bmatrix}.$$

A state-space realization of $G$ can be obtained using the following MATLAB commands:

$\gg$ **G11=nd2sys([10,10],[1,0.2,100]);**

$\gg$ **G12=nd2sys(1,[1,1]);**

$\gg$ **G21=nd2sys([1,2],[1,0.1,10]);**

$\gg$ **G22=nd2sys([5,5],[1,5,6]);**

$\gg$ **G=sbs(abv(G11,G21),abv(G12,G22));**

Next, we set up a frequency grid to compute the frequency response of $G$ and the singular values of $G(j\omega)$ over a suitable range of frequency.

$\gg$ **w=logspace(0,2,200);** % **200** points between $1 = 10^{\mathbf{0}}$ and $100 = 10^{\mathbf{2}}$;

$\gg$ **Gf=frsp(G,w);** % computing frequency response;

$\gg$ **[u,s,v]=vsvd(Gf);** % SVD at each frequency;

$\gg$ **vplot**('**liv**, **lm**', **s**), **grid** % plot both singular values and grid;

$\gg$ **pkvnorm(s)** % find the norm from the frequency response of the singular values.

The singular values of $G(j\omega)$ are plotted in Figure 0.3, which gives an estimate of $\|G\|_\infty \approx 32.861$. The state-space bisection algorithm described previously leads to $\|G\|_\infty = 50.25 \pm 0.01$ and the corresponding MATLAB command is

$\gg$ **hinfnorm(G,0.0001)** or **linfnorm(G,0.0001)** % relative error $\leq 0.0001$.

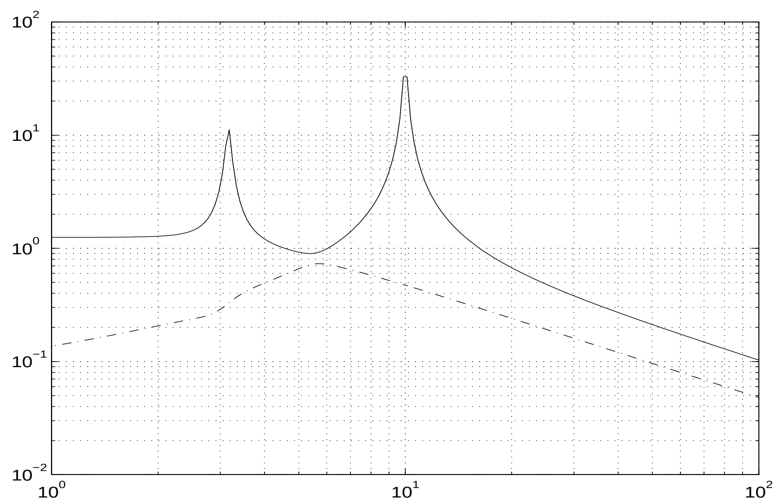



Figure 0.3: The largest and the smallest singular values of $G(j\omega)$

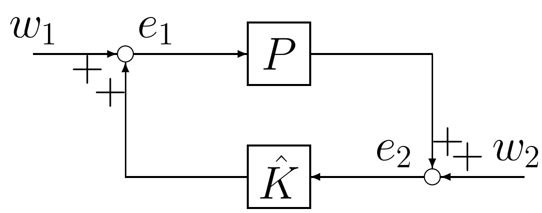
The preceding computational results show clearly that the graphical method can lead to a wrong answer for a lightly damped system if the frequency grid is not sufficiently dense. Indeed, we would get $\|G\|_\infty \approx 43.525, 48.286$ and $49.737$ from the graphical method if $400, 800,$ and $1600$ frequency points are used, respectively.

# Chapter 5: Internal Stability

- internal stability
- coprime factorization over $\mathcal{RH}_\infty$
- performance

# Internal Stability

Consider the following feedback system:




- well-posed if $I - \hat{K}(\infty)P(\infty)$ is invertible.
- **Internal Stability:** if

$$\begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} (I-\hat{K}P)^{-1} & \hat{K}(I-P\hat{K})^{-1} \\ P(I-\hat{K}P)^{-1} & (I-P\hat{K})^{-1} \end{bmatrix} \in \mathcal{RH}_\infty$$

- Need to check all **Four** transfer matrices. For example,

$$P = \frac{s-1}{s+1}, \qquad \hat{K} = -\frac{1}{s-1}.$$

$$\begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{s+1}{s+2} & -\dfrac{s+1}{(s-1)(s+2)} \\ \dfrac{s-1}{s+2} & \dfrac{s+1}{s+2} \end{bmatrix}$$

- Suppose $\hat{K} \in \mathcal{H}_\infty$. Internal stability $\Longleftrightarrow P(I-\hat{K}P)^{-1} \in \mathcal{H}_\infty$.
- Suppose $P \in \mathcal{H}_\infty$. Internal stability $\Longleftrightarrow \hat{K}(I-P\hat{K})^{-1} \in \mathcal{H}_\infty$.
- Suppose $P, \hat{K} \in \mathcal{H}_\infty$. Internal stability $\Longleftrightarrow (I-P\hat{K})^{-1} \in \mathcal{H}_\infty$.
- Suppose no unstable pole-zero cancellation in $PK$.
  Internal stability $\Longleftrightarrow (I-P(s)\hat{K}(s))^{-1} \in \mathcal{H}_\infty$

## Example

Let $P$ and $\hat{K}$ be two-by-two transfer matrices

$$P = \begin{bmatrix} \dfrac{1}{s-1} & 0 \\ 0 & \dfrac{1}{s+1} \end{bmatrix}, \quad \hat{K} = \begin{bmatrix} \dfrac{1-s}{s+1} & -1 \\ 0 & -1 \end{bmatrix}.$$

Then

$$P\hat{K} = \begin{bmatrix} \dfrac{-1}{s+1} & \dfrac{-1}{s-1} \\ 0 & \dfrac{-1}{s+1} \end{bmatrix}, \quad (I - P\hat{K})^{-1} = \begin{bmatrix} \dfrac{s+1}{s+2} & -\dfrac{(s+1)^2}{(s+2)^2(s-1)} \\ 0 & \dfrac{s+1}{s+2} \end{bmatrix}.$$

So the closed-loop system is not stable even though

$$\det(I - P\hat{K}) = \frac{(s+2)^2}{(s+1)^2}$$

has no zero in the closed right-half plane and the number of unstable poles of $P\hat{K} = n_k + n_p = 1$. Hence, in general, $\det(I - P\hat{K})$ having no zeros in the closed right-half plane does not necessarily imply $(I - P\hat{K})^{-1} \in \mathcal{RH}_\infty$.

## Coprime Factorization over $\mathcal{RH}_\infty$

- two polynomials $m(s)$ and $n(s)$ are coprime if the only common factors are constants.

- two transfer functions $m(s)$ and $n(s)$ in $\mathcal{RH}_\infty$ are *coprime over* $\mathcal{RH}_\infty$ if the only common factors are stable and invertible transfer functions (units):

$$h, mh^{-1}, nh^{-1} \in \mathcal{RH}_\infty \Longrightarrow h^{-1} \in \mathcal{RH}_\infty.$$

  Equivalent, there exists $x, y \in \mathcal{RH}_\infty$ such that

$$xm + yn = 1.$$

- Matrices $M$ and $N$ in $\mathcal{RH}_\infty$ are *right coprime over* $\mathcal{RH}_\infty$ if there exist matrices $X_r$ and $Y_r$ in $\mathcal{RH}_\infty$ such that

$$\begin{bmatrix} X_r & Y_r \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = X_r M + Y_r N = I.$$

- Matrices $\tilde{M}$ and $\tilde{N}$ in $\mathcal{RH}_\infty$ are *left coprime over* $\mathcal{RH}_\infty$ if there exist matrices $X_l$ and $Y_l$ in $\mathcal{RH}_\infty$ such that

$$\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} \begin{bmatrix} X_l \\ Y_l \end{bmatrix} = \tilde{M} X_l + \tilde{N} Y_l = I.$$

Let $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ and $\hat{K} = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ be *rcf* and *lcf*, respectively. Then the following conditions are equivalent:

1. The feedback system is internally stable.
2. $\begin{bmatrix} M & U \\ N & V \end{bmatrix}$ is invertible in $\mathcal{RH}_\infty$.
3. $\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}$ is invertible in $\mathcal{RH}_\infty$.
4. $\tilde{M}V - \tilde{N}U$ is invertible in $\mathcal{RH}_\infty$.
5. $\tilde{V}M - \tilde{U}N$ is invertible in $\mathcal{RH}_\infty$.

Let $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a stabilizable and detectable realization, and let $F$ and $L$ be such that $A + BF$ and $A + LC$ are both stable.

Define

$$\begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} = \left[\begin{array}{c|cc} A + BF & B & -L \\ \hline F & I & 0 \\ C + DF & D & I \end{array}\right]$$

$$\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & -(B + LD) & L \\ \hline F & I & 0 \\ C & -D & I \end{array}\right].$$

Then

$$\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} = I.$$

## Example

Let $P(s) = \dfrac{s-2}{s(s+3)}$ and $\alpha = (s+1)(s+3)$. Then $P(s) = n(s)/m(s)$ with $n(s) = \dfrac{s-2}{(s+1)(s+3)}$ and $m(s) = \dfrac{s}{s+1}$ forms a coprime factorization. To find an $x(s) \in \mathcal{H}_\infty$ and a $y(s) \in \mathcal{H}_\infty$ such that $x(s)n(s) + y(s)m(s) = 1$, consider a stabilizing controller for $P$: $\hat{K} = -\dfrac{s-1}{s+10}$. Then $\hat{K} = u/v$ with $u = \hat{K}$ and $v = 1$ is a coprime factorization and

$$m(s)v(s) - n(s)u(s) = \frac{(s+11.7085)(s+2.214)(s+0.077)}{(s+1)(s+3)(s+10)} =: \beta(s)$$

Then we can take

$$x(s) = -u(s)/\beta(s) = \frac{(s-1)(s+1)(s+3)}{(s+11.7085)(s+2.214)(s+0.077)}$$

$$y(s) = v(s)/\beta(s) = \frac{(s+1)(s+3)(s+10)}{(s+11.7085)(s+2.214)(s+0.077)}$$

MATLAB programs can be used to find the appropriate $F$ and $L$ matrices in state-space so that the desired coprime factorization can be obtained. Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. Then an $F$ and an $L$ can be obtained from

$\gg$ **F=-lqr(A, B, eye(n), eye(m));** % or

$\gg$ **F=-place(A, B, Pf);** % Pf= poles of A+BF

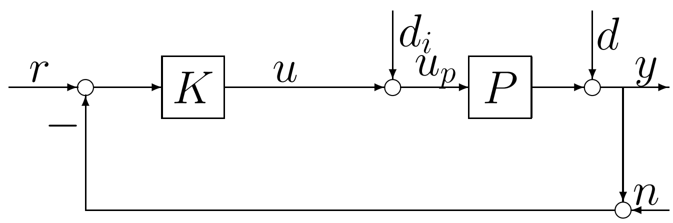
$\gg$ $\mathbf{L} = -\mathbf{lqr}(\mathbf{A}', \mathbf{C}', \mathbf{eye}(\mathbf{n}), \mathbf{eye}(\mathbf{p}))$; % or

$\gg$ $\mathbf{L} = -\mathbf{place}(\mathbf{A}', \mathbf{C}', \mathbf{Pl})$; % Pl=poles of A+LC.

# Chapter 6: Performance Specifications and Limitations

- Feedback Properties
- Weighted $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Performance
- Selection of Weighting Performance
- Bode's Gain and Phase Relation
- Bode's Sensitivity Integral
- Analyticity Constraints

# Feedback Properties




$$S_i = (I + KP)^{-1}, \quad S_o = (I + PK)^{-1}.$$

$$T_i = I - S_i = KP(I + KP)^{-1}, \quad T_o = I - S_o = PK(I + PK)^{-1}$$

$$y = T_o(r - n) + S_o P d_i + S_o d$$

$$u_p = KS_o(r - n) - KS_o d + S_i d_i.$$

Disturbance rejection at the plant output (low frequency):

$$\begin{aligned}\overline{\sigma}(S_o) &= \overline{\sigma}\left((I + PK)^{-1}\right) = \frac{1}{\underline{\sigma}(I + PK)} \quad (\ll 1) \\ \overline{\sigma}(S_o P) &= \overline{\sigma}\left((I + PK)^{-1}P\right) = \overline{\sigma}(PS_i) \quad (\ll 1)\end{aligned}$$

Disturbance rejection at the plant input (low frequency):

$$\begin{aligned}\overline{\sigma}(S_i) &= \overline{\sigma}\left((I + KP)^{-1}\right) = \frac{1}{\underline{\sigma}(I + KP)} \quad (\ll 1) \\ \overline{\sigma}(S_i K) &= \overline{\sigma}\left(K(I + PK)^{-1}\right) = \overline{\sigma}(KS_o) \quad (\ll 1)\end{aligned}$$

Sensor noise rejection and robust stability (high frequency) :

$$\overline{\sigma}(T_o) = \overline{\sigma}\left(PK(I + PK)^{-1}\right) \quad (\ll 1)$$

Note that

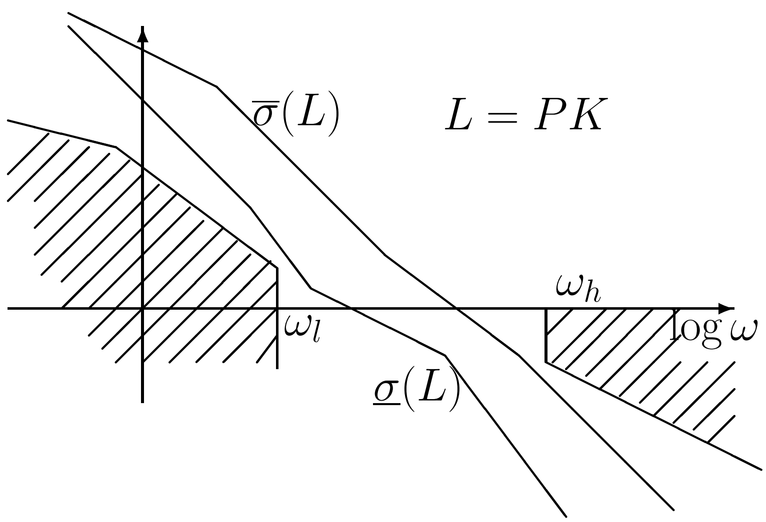
$$\overline{\sigma}(S_o) \ll 1 \iff \underline{\sigma}(PK) \gg 1$$
$$\overline{\sigma}(S_i) \ll 1 \iff \underline{\sigma}(KP) \gg 1$$
$$\overline{\sigma}(T_o) \ll 1 \iff \overline{\sigma}(PK) \ll 1.$$

Now suppose $P$ and $K$ are invertible, then

$$\underline{\sigma}(PK) \gg 1 \text{ or } \underline{\sigma}(KP) \gg 1$$

$$\iff \begin{cases} \overline{\sigma}(S_oP) = \overline{\sigma}\left((I+PK)^{-1}P\right) \approx \overline{\sigma}(K^{-1}) = \frac{1}{\underline{\sigma}(K)} \\ \overline{\sigma}(KS_o) = \overline{\sigma}\left(K(I+PK)^{-1}\right) \approx \overline{\sigma}(P^{-1}) = \frac{1}{\underline{\sigma}(P)}. \end{cases}$$

Desired Loop Shape



# Weighted $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Performance

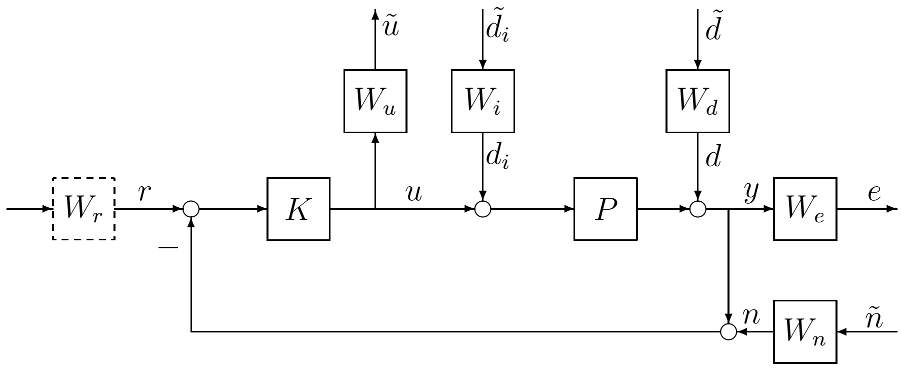



Figure 0.4: Standard feedback configuration with weights

$\mathcal{H}_2$ **Performance:** Assume $\tilde{d}(t) = \eta\delta(t)$ and $E(\eta\eta^*) = I$

Minimize the expected energy of the error $e$:

$$E\left\{\|e\|_2^2\right\} = E\left\{\int_0^\infty \|e\|^2 \, dt\right\} = \|W_e S_o W_d\|_2^2$$

Include the control signal $u$ in the cost function:

$$E\left\{\|e\|_2^2 + \rho^2 \|\tilde{u}\|_2^2\right\} = \left\|\begin{bmatrix} W_e S_o W_d \\ \rho W_u K S_o W_d \end{bmatrix}\right\|_2^2$$

Robustness problem?????

$\mathcal{H}_\infty$ **Performance:** under worst possible case

$$\sup_{\|\tilde{d}\|_2 \le 1} \|e\|_2 = \|W_e S_o W_d\|_\infty$$

restrictions on the control energy or control bandwidth:

$$\sup_{\|\tilde{d}\|_2 \le 1} \|\tilde{u}\|_2 = \|W_u K S_o W_d\|_\infty$$

Combined cost:

$$\sup_{\|\tilde{d}\|_2 \le 1} \left\{\|e\|_2^2 + \rho^2 \|\tilde{u}\|_2^2\right\} = \left\|\begin{bmatrix} W_e S_o W_d \\ \rho W_u K S_o W_d \end{bmatrix}\right\|_\infty^2$$

# Selection of Weighting Functions: SISO

Let $L = PK$ be a standard second-order system

$$L = \frac{\omega_n^2}{s(s+2\xi\omega_n)}$$

$$t_r \approx \frac{0.6+2.16\xi}{\omega_n},\ 0.3 \le \xi \le 0.8;\ t_s \approx \frac{4}{\xi\omega_n};\ M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}},\ 0 < \xi < 1$$

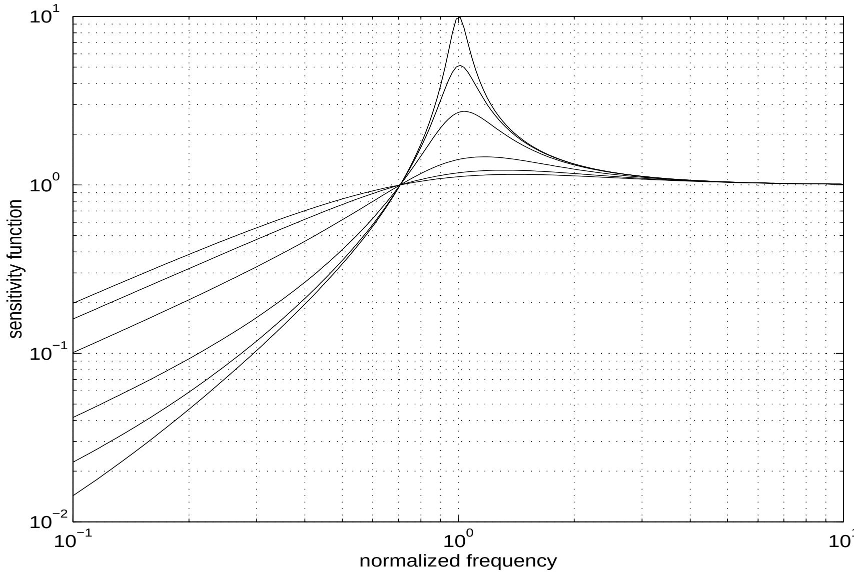



Figure 0.5: Sensitivity function $S$ for $\xi = 0.05, 0.1, 0.2, 0.5, 0.8$, and 1 with normalized frequency $(\omega/\omega_n)$

$$S = \frac{1}{1+L} = \frac{s(s+2\xi\omega_n)}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$|S(j\omega_n/\sqrt{2})| = 1$$

closed-loop bandwidth $\omega_b \approx \omega_n/\sqrt{2}$ since $|S(j\omega)| \ge 1,\ \forall\omega \ge \omega_b$

A good control design: $M_s := \|S\|_\infty$ not too large.

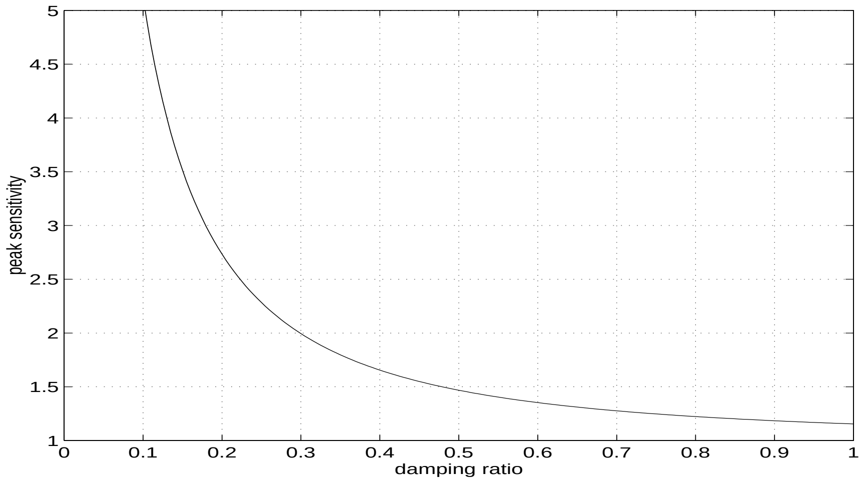



Figure 0.6: Peak sensitivity $M_s$ versus damping ratio $\xi$

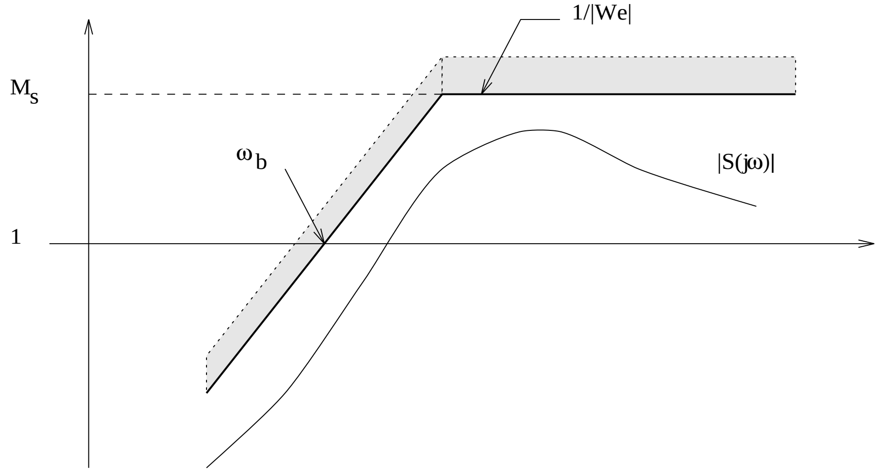



Figure 0.7: Performance weight $W_e$ and desired $S$

We require

$$|S(s)| \leq \left| \frac{s}{s/M_s + \omega_b} \right|, \; s = j\omega, \; \forall \; \omega$$

$$\iff \quad |W_e S| \leq 1, \quad W_e = \frac{s/M_s + \omega_b}{s}$$

Practical consideration: $W_e = \frac{s/M_s + \omega_b}{s + \omega_b \varepsilon}$

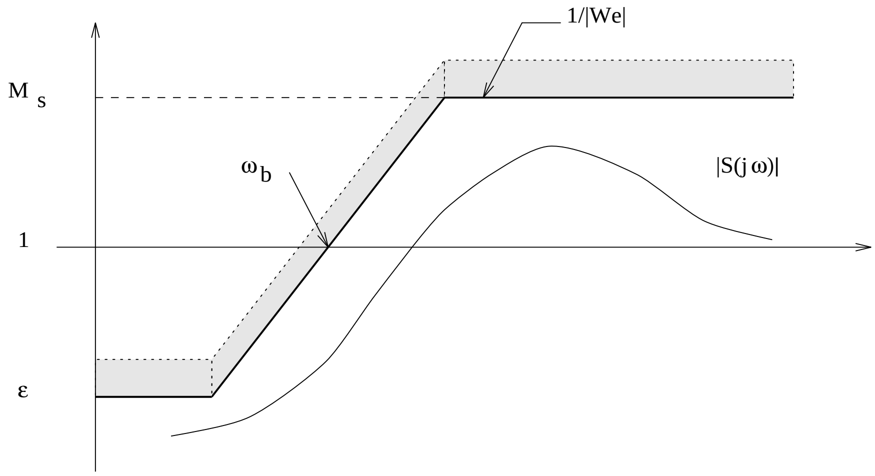



Figure 0.8: Practical performance weight $W_e$ and desired $S$

## Control weighting function $W_u$:

$$W_u = \frac{s + \omega_{bc}/M_u}{\varepsilon_1 s + \omega_{bc}}$$

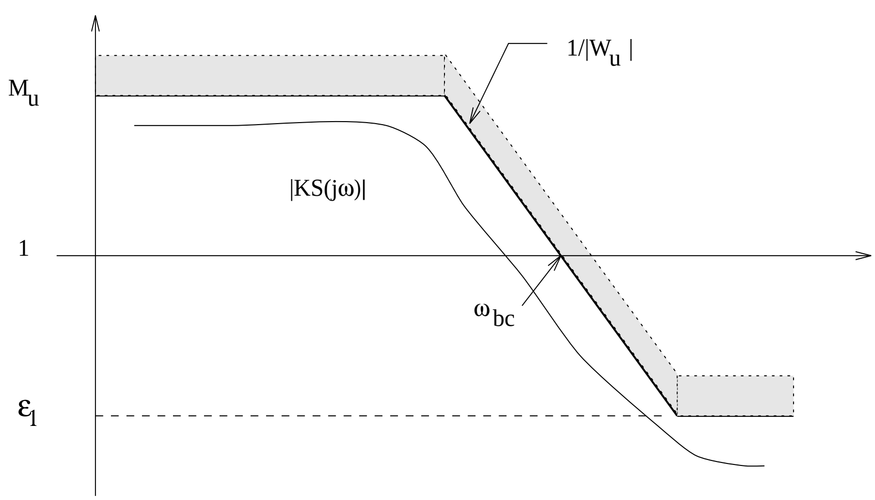



Figure 0.9: Control weight $W_u$ and desired $KS$

# Bode's Gain and Phase Relation

$L$ stable and minimum phase:

$$\angle L(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\ln|L|}{d\nu} \ln\coth\frac{|\nu|}{2} d\nu \qquad \nu := \ln(\omega/\omega_0)$$

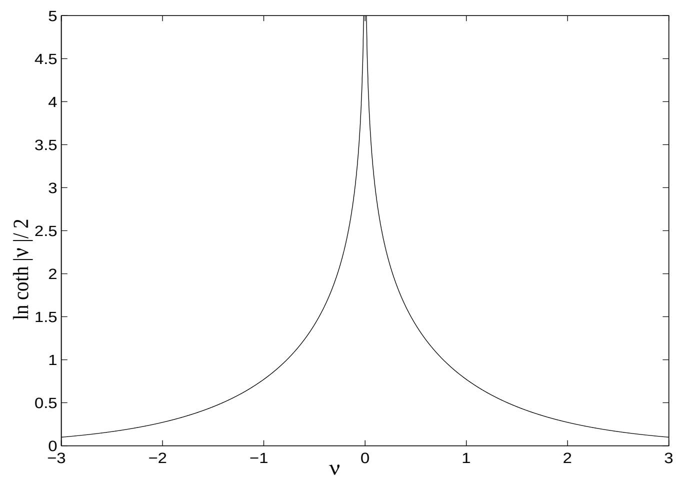



Figure 0.10: The function $\ln\coth\frac{|\nu|}{2}$ vs $\nu$

$\angle L(j\omega_0)$ depends mostly on the behavior of $\dfrac{d\ln|L(j\omega)|}{d\nu}$ near $\omega_0$:

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha} \ln\coth\frac{|\nu|}{2} d\nu = \begin{cases} 1.1406 \text{ (rad)}, & \alpha = \ln 3 \\ 1.3146 \text{ (rad)}, & \alpha = \ln 5 \\ 1.443 \text{ (rad)}, & \alpha = \ln 10 \end{cases} = \begin{cases} 65.3^{\text{o}}, & \alpha = \ln 3 \\ 75.3^{\text{o}}, & \alpha = \ln 5 \\ 82.7^{\text{o}}, & \alpha = \ln 10. \end{cases}$$

$\angle L(j\omega_0)$ large if $|L|$ attenuates slowly near $\omega_0$ and small if it attenuates rapidly near $\omega_0$. For example, it is reasonable to expect

$$\angle L(j\omega_0) < \begin{cases} -\ell \times 65.3^{\text{o}}, & \text{if the slope of } L = -\ell \text{ for } \frac{1}{3} \le \frac{\omega}{\omega_0} \le 3 \\ -\ell \times 75.3^{\text{o}}, & \text{if the slope of } L = -\ell \text{ for } \frac{1}{5} \le \frac{\omega}{\omega_0} \le 5 \\ -\ell \times 82.7^{\text{o}}, & \text{if the slope of } L = -\ell \text{ for } \frac{1}{10} \le \frac{\omega}{\omega_0} \le 10. \end{cases}$$

The behavior of $\angle L(j\omega)$ is particularly important near the crossover frequency $\omega_c$, where $|L(j\omega_c)| = 1$ since $\pi + \angle L(j\omega_c)$ is the phase margin of the feedback system. Further, the return difference is given by

$$|1+L(j\omega_c)| = |1+L^{-1}(j\omega_c)| = 2\left|\sin\frac{\pi+\angle L(j\omega_c)}{2}\right|,$$

which must not be too small for good stability robustness.

It is important to keep the slope of $L$ near $\omega_c$ not much smaller than $-1$ for a reasonably wide range of frequencies in order to guarantee some reasonable performance.

$L$ stable and nonminimum phase with RHP zeros: $z_1, z_2, \ldots, z_k$:

$$L(s) = \frac{-s+z_1}{s+z_1}\frac{-s+z_2}{s+z_2}\cdots\frac{-s+z_k}{s+z_k}L_{\mathrm{mp}}(s)$$

where $L_{\mathrm{mp}}$ is stable and minimum phase and $|L(j\omega)| = |L_{\mathrm{mp}}(j\omega)|$. Hence

$$\angle L(j\omega_0) = \angle L_{\mathrm{mp}}(j\omega_0) + \angle\prod_{i=1}^{k}\frac{-j\omega_0+z_i}{j\omega_0+z_i}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\ln|L_{\mathrm{mp}}|}{d\nu}\ln\coth\frac{|\nu|}{2}d\nu + \sum_{i=1}^{k}\angle\frac{-j\omega_0+z_i}{j\omega_0+z_i},$$

which gives

$$\angle L(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\ln|L|}{d\nu}\ln\coth\frac{|\nu|}{2}d\nu + \sum_{i=1}^{k}\angle\frac{-j\omega_0+z_i}{j\omega_0+z_i}.$$

Since $\angle\frac{-j\omega_0+z_i}{j\omega_0+z_i} \leq 0$ for each $i$, a nonminimum phase zero contributes an additional phase lag and imposes limitations on the rolloff rate of the open-loop gain. For example, suppose $L$ has a zero at $z>0$; then

$$\phi_1(\omega_0/z) := \angle\frac{-j\omega_0+z}{j\omega_0+z}\bigg|_{\omega_0=z,z/2,z/4} = -90^{\mathrm{o}}, -53.13^{\mathrm{o}}, -28^{\mathrm{o}},$$

Since the slope of $|L|$ near the crossover frequency is, in general, no greater than $-1$, which means that the phase due to the minimum phase part, $L_{\mathrm{mp}}$, of $L$ will, in general, be no greater than $-90^{\mathrm{o}}$, the crossover frequency (or the closed-loop bandwidth) must satisfy

$$\omega_c < z/2$$

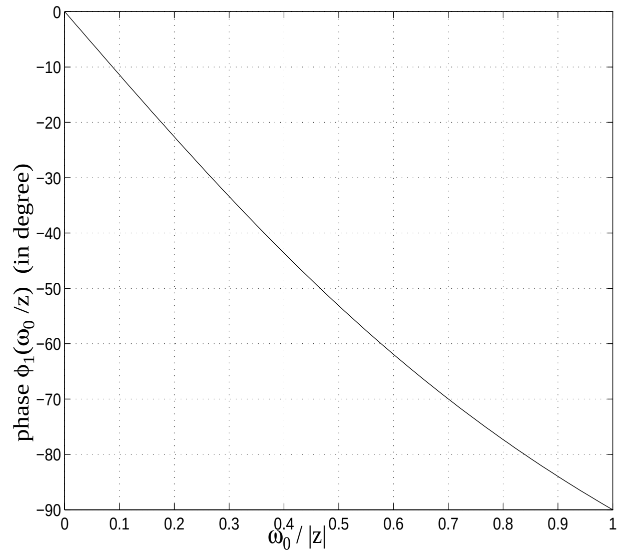



Figure 0.11: Phase $\phi_1(\omega_0/z)$ due to a real zero $z > 0$

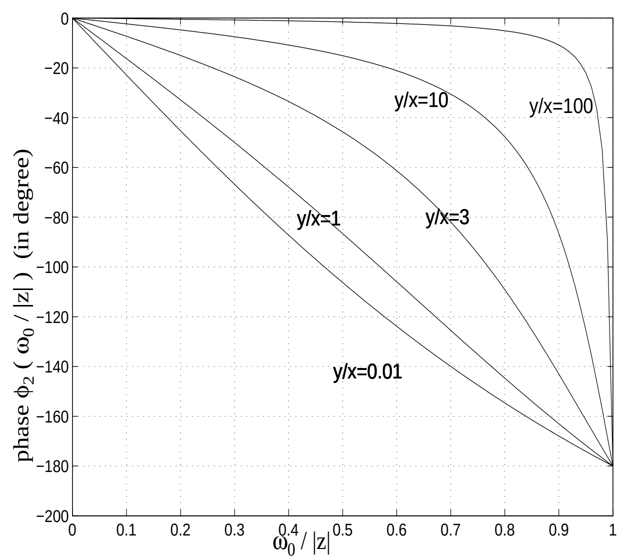



Figure 0.12: Phase $\phi_2(\omega_0/|z|)$ due to a pair of complex zeros: $z = x \pm jy$ and $x > 0$

for closed-loop stability and some reasonable closed-loop performance.

Next suppose $L$ has a pair of complex right-half zeros at $z = x \pm jy$ with $x > 0$; then

$$\phi_2(\omega_0/|z|) := \angle \frac{-j\omega_0 + z}{j\omega_0 + z} \, \frac{-j\omega_0 + \bar{z}}{j\omega_0 + \bar{z}}\Big|_{\omega_0 = |z|, |z|/2, |z|/3, |z|/4}$$

$$\approx \begin{cases} -180^{\mathrm{o}}, & -106.26^{\mathrm{o}}, & -73.7^{\mathrm{o}}, & -56^{\mathrm{o}}, & \Re(z) \gg \Im(z) \\ -180^{\mathrm{o}}, & -86.7^{\mathrm{o}}, & -55.9^{\mathrm{o}}, & -41.3^{\mathrm{o}}, & \Re(z) \approx \Im(z) \\ -360^{\mathrm{o}}, & 0^{\mathrm{o}}, & 0^{\mathrm{o}}, & 0^{\mathrm{o}}, & \Re(z) \ll \Im(z) \end{cases}$$

In this case we conclude that the crossover frequency must satisfy

$$\omega_c < \begin{cases} |z|/4, & \Re(z) \gg \Im(z) \\ |z|/3, & \Re(z) \approx \Im(z) \\ |z|, & \Re(z) \ll \Im(z) \end{cases}$$

in order to guarantee the closed-loop stability and some reasonable closed-loop performance.

## Bode's Sensitivity Integral

Let $p_1, p_2, \ldots, p_m$ be the open right-half plane poles of $L$

$$\int_0^\infty \ln |S(j\omega)| d\omega = \pi \sum_{i=1}^{m} \Re(p_i) \tag{0.3}$$

In the case where $L$ is stable, the integral simplifies to

$$\int_0^\infty \ln |S(j\omega)| d\omega = 0 \tag{0.4}$$

water bed effect:

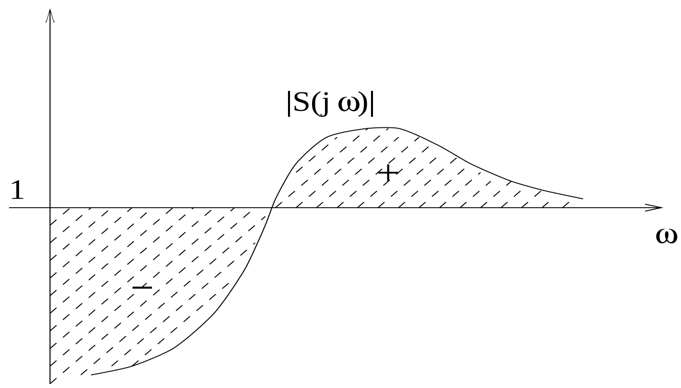



Figure 0.13: Water bed effect of sensitivity function

Suppose

$$|S(j\omega)| \leq \epsilon < 1, \quad \forall \omega \in [0,\ \omega_l]$$

Bandwidth constraints and stability robustness:

$$|L(j\omega)| \leq \frac{M_h}{\omega^{1+\beta}} \leq \tilde{\epsilon} < 1, \quad \forall \omega \in [\omega_h,\ \infty)$$

$$\max_{\omega \in [\omega_l, \omega_h]} |S(j\omega)| \geq e^{\alpha} \left(\frac{1}{\epsilon}\right)^{\frac{\omega_l}{\omega_h - \omega_l}} (1 - \tilde{\epsilon})^{\frac{\omega_h}{\beta(\omega_h - \omega_l)}}$$

where

$$\alpha = \frac{\pi \sum_{i=1}^{m} \Re(p_i)}{\omega_h - \omega_l}.$$

The above lower bound shows that the sensitivity can be very significant in the transition band.

Poisson integral relation: Suppose $L$ has at least one more poles than zeros and suppose $z = x_0 + jy_0$ with $x_0 > 0$ is a right-half plane zero of $L$. Then

$$\int_{-\infty}^{\infty} \ln |S(j\omega)| \frac{x_0}{x_0^2 + (\omega - y_0)^2} d\omega = \pi \ln \prod_{i=1}^{m} \left| \frac{z + p_i}{z - p_i} \right| \tag{0.5}$$

Define

$$\theta(z) := \int_{-\omega_l}^{\omega_l} \frac{x_0}{x_0^2 + (\omega - y_0)^2} d\omega$$

Then

$$\begin{aligned} \pi \ln \prod_{i=1}^{m} \left| \frac{z + p_i}{z - p_i} \right| &= \int_{-\infty}^{\infty} \ln |S(j\omega)| \frac{x_0}{x_0^2 + (\omega - y_0)^2} d\omega \\ &\leq (\pi - \theta(z)) \ln \|S(j\omega)\|_\infty + \theta(z) \ln(\epsilon), \end{aligned}$$

which gives

$$\|S(s)\|_\infty \geq \left( \frac{1}{\epsilon} \right)^{\frac{\theta(z)}{\pi - \theta(z)}} \left( \prod_{i=1}^{m} \left| \frac{z + p_i}{z - p_i} \right| \right)^{\frac{\pi}{\pi - \theta(z)}}$$

Let $p_1, p_2, \ldots, p_m$ and $z_1, z_2, \ldots, z_k$ be the open right-half plane poles and zeros of $L$, respectively.

$$S(p_i) = 0, \quad T(p_i) = 1, \; i = 1, 2, \ldots, m$$

and

$$S(z_j) = 1, \quad T(z_j) = 0, \; j = 1, 2, \ldots, k$$

Suppose $S = (I+L)^{-1}$ and $T = L(I+L)^{-1}$ are stable. Then $p_1, p_2, \ldots, p_m$ are the right-half plane zeros of $S$ and $z_1, z_2, \ldots, z_k$ are the right-half plane zeros of $T$. Let

$$B_p(s) = \prod_{i=1}^{m} \frac{s - p_i}{s + p_i}, \quad B_z(s) = \prod_{j=1}^{k} \frac{s - z_j}{s + z_j}$$

Then $|B_p(j\omega)| = 1$ and $|B_z(j\omega)| = 1$ for all frequencies and, moreover,

$$B_p^{-1}(s)S(s) \in \mathcal{H}_\infty, \quad B_z^{-1}(s)T(s) \in \mathcal{H}_\infty.$$

Hence, by the maximum modulus theorem, we have

$$\|S(s)\|_\infty = \left\|B_p^{-1}(s)S(s)\right\|_\infty \geq |B_p^{-1}(z)S(z)| = |B_p^{-1}(z)|$$

for any $z$ with $\Re(z) > 0$. Let $z$ be a right-half plane zero of $L$; then

$$\|S(s)\|_\infty \geq |B_p^{-1}(z)| = \prod_{i=1}^{m} \left| \frac{z + p_i}{z - p_i} \right|$$

Similarly, one can obtain

$$\|T(s)\|_\infty \geq |B_z^{-1}(p)| = \prod_{j=1}^{k} \left| \frac{p + z_j}{p - z_j} \right|$$

where $p$ is a right-half plane pole of $L$.

The weighted problem can be considered in the same fashion. Let $W_e$ be a weight such that $W_e S$ is stable. Then

$$\|W_e(s)S(s)\|_\infty \geq |W_e(z)| \; \prod_{i=1}^{m} \left| \frac{z + p_i}{z - p_i} \right|$$

Now suppose $W_e(s) = \dfrac{s/M_s + \omega_b}{s + \omega_b \epsilon}$, $\|W_e S\|_\infty \leq 1$, and $z$ is a real right-half plane zero. Then

$$\frac{z/M_s + \omega_b}{z + \omega_b \epsilon} \leq \prod_{i=1}^{m} \left| \frac{z - p_i}{z + p_i} \right| =: \alpha \leq 1,$$

which gives

$$\omega_b \leq \frac{z}{1 - \alpha\epsilon}(\alpha - \frac{1}{M_s}) \approx z(\alpha - \frac{1}{M_s})$$

bandwidth must be much smaller than the right-half plane zero.

# Chapter 7: Balanced Model Reduction

- Balanced Realization
- Balanced Model Reduction
- Frequency Weighted Balanced Model Reduction
- Relative Reduction

## Balanced Realization

Consider the following Lyapunov equation

$$A^*X + XA + Q = 0$$

Assume that $A$ is stable, then the following statements hold:

- $X = \int_0^\infty e^{A^*t} Q e^{At} dt$.
- $X > 0$ if $Q > 0$ and $X \geq 0$ if $Q \geq 0$.
- if $Q \geq 0$, then $(Q, A)$ is observable iff $X > 0$.

Suppose $X$ is the solution of the Lyapunov equation, then

- $\mathrm{Re}\lambda_i(A) \leq 0$ if $X > 0$ and $Q \geq 0$.
- $A$ is stable if $X > 0$ and $Q > 0$.
- $A$ is stable if $X \geq 0$, $Q \geq 0$ and $(Q, A)$ is detectable.

Let $A$ be stable. Then a pair $(C, A)$ is observable iff the *observability Gramian* $Q > 0$

$$A^*Q + QA + C^*C = 0.$$

Similarly, $(A, B)$ is controllable iff the *controllability Gramian* $P > 0$

$$AP + PA^* + BB^* = 0$$

- Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a state space realization of a (not necessarily stable) transfer matrix $G(s)$. Suppose that there exists a symmetric matrix

  $$P = P^* = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$$

  with $P_1$ nonsingular such that

  $$AP + PA^* + BB^* = 0.$$

Now partition the realization $(A, B, C, D)$ compatibly with $P$ as

$$\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right].$$

Then

$$\left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

is also a realization of $G$. Moreover, $(A_{11}, B_1)$ is controllable if $A_{11}$ is stable.

**Proof** Using

$$0 = AP + PA^* + BB^*$$

to get $B_2 = 0$ and $A_{21} = 0$. Hence, part of the realization is not controllable:

$$\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & 0 \\ \hline C_1 & C_2 & D \end{array}\right] = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right].$$

- Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a state space realization of a (not necessarily stable) transfer matrix $G(s)$. Suppose that there exists a symmetric matrix

$$Q = Q^* = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix}$$

with $Q_1$ nonsingular such that

$$QA + A^*Q + C^*C = 0.$$

Now partition the realization $(A, B, C, D)$ compatibly with $Q$ as

$$\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right].$$

Then

$$\left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

is also a realization of $G$. Moreover, $(C_1, A_{11})$ is observable if $A_{11}$ is stable.

- Let $P$ and $Q$ be the controllability and observability Gramians,

$$AP + PA^* + BB^* = 0$$

$$A^*Q + QA + C^*C = 0.$$

Suppose

$$P = Q = \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$$

Then the state space realization is called internally *balanced realization* and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$, are called the *Hankel singular values* of the system.

Two other closely related realizations are called *input normal realization* with $P = I$ and $Q = \Sigma^2$, and *output normal realization* with $P = \Sigma^2$ and $Q = I$. Both realizations can be obtained easily from the balanced realization by a suitable scaling on the states.

- Let $P$ and $Q$ be two positive semidefinite matrices. Then there exists a nonsingular matrix $T$ such that

$$TPT^* = \begin{bmatrix} \Sigma_1 & & & \\ & \Sigma_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

$$(T^{-1})^*QT^{-1} = \begin{bmatrix} \Sigma_1 & & & \\ & 0 & & \\ & & \Sigma_3 & \\ & & & 0 \end{bmatrix}$$

respectively, with $\Sigma_1, \ \Sigma_2, \ \Sigma_3$ diagonal and positive definite.

In the special case where $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a minimal realization, a balanced realization can be obtained through the following simplified procedure:

1. Compute $P > 0$ and $Q > 0$.
2. Find a matrix $R$ such that $P = R^*R$.
3. Diagonalize $RQR^*$ to get $RQR^* = U\Sigma^2U^*$.
4. Let $T^{-1} = R^*U\Sigma^{-1/2}$. Then $TPT^* = (T^*)^{-1}QT^{-1} = \Sigma$ and $\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right]$ is balanced.

Suppose $\sigma_r \gg \sigma_{r+1}$ for some $r$ then the balanced realization implies that those states corresponding to the singular values of $\sigma_{r+1}, \ldots, \sigma_n$ are less controllable and observable than those states corresponding to $\sigma_1, \ldots, \sigma_r$. Therefore, truncating those less controllable and observable states will not lose much information about the system.

*input normal realization*: $P = I$ and $Q = \Sigma^2$
*output normal realization*: $P = \Sigma^2$ and $Q = I$.

Suppose

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \in \mathcal{RH}_\infty$$

is a balanced realization; that is, there exists

$$\Sigma = \operatorname{diag}(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_N I_{s_N}) \geq 0$$

with $\sigma_1 > \sigma_2 > \ldots > \sigma_N \geq 0$, such that

$$A\Sigma + \Sigma A^* + BB^* = 0 \qquad A^*\Sigma + \Sigma A + C^*C = 0$$

Then

$$\sigma_1 \leq \|G\|_\infty \leq \int_0^\infty \|g(t)\|\, dt \leq 2\sum_{i=1}^N \sigma_i$$

where $g(t) = Ce^{At}B$.

**Proof**.

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx. \end{aligned}$$

$(A, B)$ is controllable and $(C, A)$ is observable.

$$\frac{d}{dt}(x^*\Sigma^{-1}x) = \dot{x}^*\Sigma^{-1}x + x^*\Sigma^{-1}\dot{x} = x^*(A^*\Sigma^{-1} + \Sigma^{-1}A)x + 2\langle w, B^*\Sigma^{-1}x\rangle$$

$$\frac{d}{dt}(x^*\Sigma^{-1}x) = \|w\|^2 - \|w - B^*\Sigma^{-1}x\|^2$$

Integration from $t = -\infty$ to $t = 0$ with $x(-\infty) = 0$ and $x(0) = x_0$ gives

$$x_0^*\Sigma^{-1}x_0 = \|w\|_2^2 - \|w - B^*\Sigma^{-1}x\|_2^2 \leq \|w\|_2^2$$

$$\inf_{w \in \mathcal{L}_2[-\infty,0)} \left\{ \|w\|_2^2 \;\middle|\; x(0) = x_0 \right\} = x_0^*\Sigma^{-1}x_0.$$

Given $x(0) = x_0$ and $w = 0$ for $t \geq 0$, the norm of $z(t) = Ce^{At}x_0$ can be found from

$$\int_0^\infty \|z(t)\|^2\, dt = \int_0^\infty x_0^* e^{A^*t} C^* C e^{At} x_0 dt = x_0^* \Sigma x_0$$

To show $\sigma_1 \leq \|G\|_\infty$, note that

$$\|G\|_\infty = \sup_{w\in\mathcal{L}_2(-\infty,\infty)} \frac{\|g*w\|_2}{\|w\|_2} = \sup_{w\in\mathcal{L}_2(-\infty,\infty)} \frac{\sqrt{\int_{-\infty}^{\infty}\|z(t)\|^2\,dt}}{\sqrt{\int_{-\infty}^{\infty}\|w(t)\|^2\,dt}}$$

$$\geq \sup_{w\in\mathcal{L}_2(-\infty,0]} \frac{\sqrt{\int_0^{\infty}\|z(t)\|^2\,dt}}{\sqrt{\int_{-\infty}^{0}\|w(t)\|^2\,dt}} = \sup_{x_0\neq 0}\sqrt{\frac{x_0^*\Sigma x_0}{x_0^*\Sigma^{-1}x_0}} = \sigma_1$$

We shall now show the other inequalities. Since

$$G(s) := \int_0^\infty g(t)e^{-st}dt,\ \operatorname{Re}(s) > 0,$$

by the definition of $\mathcal{H}_\infty$ norm, we have

$$\begin{aligned}
\|G\|_\infty &= \sup_{\operatorname{Re}(s)>0}\left\|\int_0^\infty g(t)e^{-st}dt\right\| \\
&\leq \sup_{\operatorname{Re}(s)>0}\int_0^\infty \left\|g(t)e^{-st}\right\|dt \\
&\leq \int_0^\infty \|g(t)\|\,dt.
\end{aligned}$$

To prove the last inequality, let $e_i$ be the $i$th unit vector and define

$$E_1 = \begin{bmatrix} e_1 & \cdots & e_{s_1}\end{bmatrix}, \quad \ldots,$$

$$E_N = \begin{bmatrix} e_{s_1+\cdots+s_{N-1}+1} & \cdots & e_{s_1+\cdots+s_N}\end{bmatrix}.$$

Then $\sum_{i=1}^N E_iE_i^* = I$ and

$$\begin{aligned}
\int_0^\infty \|g(t)\|\,dt &= \int_0^\infty \left\|Ce^{At/2}\sum_{i=1}^N E_iE_i^*e^{At/2}B\right\|dt \\
&\leq \sum_{i=1}^N \int_0^\infty \left\|Ce^{At/2}E_iE_i^*e^{At/2}B\right\|dt \\
&\leq \sum_{i=1}^N \int_0^\infty \left\|Ce^{At/2}E_i\right\|\left\|E_i^*e^{At/2}B\right\|dt \\
&\leq \sum_{i=1}^N \sqrt{\int_0^\infty \left\|Ce^{At/2}E_i\right\|^2dt}\sqrt{\int_0^\infty \left\|E_i^*e^{At/2}B\right\|^2dt} \\
&\leq 2\sum_{i=1}^N \sigma_i
\end{aligned}$$

where we have used Cauchy-Schwarz inequality and the following relations:

$$
\begin{aligned}
\int_0^\infty \left\| C e^{At/2} E_i \right\|^2 dt &= \int_0^\infty \lambda_{\max}\left( E_i^* e^{A^*t/2} C^* C e^{At/2} E_i \right) dt \\
&= 2\lambda_{\max}\left( E_i^* \Sigma E_i \right) = 2\sigma_i \\
= \int_0^\infty \left\| E_i^* e^{At/2} B \right\|^2 dt &= \int_0^\infty \lambda_{\max}\left( E_i^* e^{At/2} B B^* e^{A^*t/2} E_i \right) dt
\end{aligned}
$$

□

$\gg$ **[Ab, Bb, Cb, sig, Tinv]=balreal(A, B, C);** % sig is a vector of Hankel singular values and Tinv $= T^{-1}$;

$\gg$ $[\mathbf{G_b}, \mathbf{sig}] = \mathbf{sysbal}(\mathbf{G});$

$\gg$ $\mathbf{G_r} = \mathbf{strunc}(\mathbf{G_b}, \mathbf{2});$ % truncate to the second-order.

$$G = G_r + \Delta_a, \quad \Longrightarrow \quad \inf_{\deg(G_r) \le r} \|G - G_r\|_\infty .$$

- Suppose

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

is a balanced realization with Gramian $\Sigma = \mathrm{diag}(\Sigma_1, \Sigma_2)$

$$A\Sigma + \Sigma A^* + BB^* = 0 \qquad A^*\Sigma + \Sigma A + C^*C = 0.$$

where

$$\begin{aligned} \Sigma_1 &= \mathrm{diag}(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_r I_{s_r}) \\ \Sigma_2 &= \mathrm{diag}(\sigma_{r+1} I_{s_{r+1}}, \sigma_{r+2} I_{s_{r+2}}, \ldots, \sigma_N I_{s_N}) \end{aligned}$$

and

$$\sigma_1 > \sigma_2 > \cdots > \sigma_r > \sigma_{r+1} > \sigma_{r+2} > \cdots > \sigma_N$$

where $\sigma_i$ has multiplicity $s_i,\ i = 1, 2, \ldots, N$ and $s_1 + s_2 + \cdots + s_N = n$. Then the truncated system

$$G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

is balanced and asymptotically stable. Furthermore

$$\|G(s) - G_r(s)\|_\infty \le 2(\sigma_{r+1} + \sigma_{r+2} + \cdots + \sigma_N).$$

- $\|G(s) - G(\infty)\|_\infty \le 2(\sigma_1 + \ldots + \sigma_N)$.
- $\|G(s) - G_{n-1}(s)\|_\infty = 2\sigma_N$.

**Proof.** We shall first show the one step model reduction. Hence we shall assume $\Sigma_2 = \sigma_N I_{s_N}$. Define the approximation error

$$
\begin{aligned}
E_{11} &:= \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] - \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right] \\
&= \left[\begin{array}{ccc|c} A_{11} & 0 & 0 & B_1 \\ 0 & A_{11} & A_{12} & B_1 \\ 0 & A_{21} & A_{22} & B_2 \\ \hline -C_1 & C_1 & C_2 & 0 \end{array}\right]
\end{aligned}
$$

Apply a similarity transformation $T$ to the preceding state-space realization with

$$
T = \begin{bmatrix} I/2 & I/2 & 0 \\ I/2 & -I/2 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} I & I & 0 \\ I & -I & 0 \\ 0 & 0 & I \end{bmatrix}
$$

to get

$$
E_{11} = \left[\begin{array}{ccc|c} A_{11} & 0 & A_{12}/2 & B_1 \\ 0 & A_{11} & -A_{12}/2 & 0 \\ A_{21} & -A_{21} & A_{22} & B_2 \\ \hline 0 & -2C_1 & C_2 & 0 \end{array}\right]
$$

Consider a dilation of $E_{11}(s)$:

$$
\begin{aligned}
E(s) &= \begin{bmatrix} E_{11}(s) & E_{12}(s) \\ E_{21}(s) & E_{22}(s) \end{bmatrix} \\
&= \left[\begin{array}{ccc|cc} A_{11} & 0 & A_{12}/2 & B_1 & 0 \\ 0 & A_{11} & -A_{12}/2 & 0 & \sigma_N \Sigma_1^{-1} C_1^* \\ A_{21} & -A_{21} & A_{22} & B_2 & -C_2^* \\ \hline 0 & -2C_1 & C_2 & 0 & 2\sigma_N I \\ -2\sigma_N B_1^* \Sigma_1^{-1} & 0 & -B_2^* & 2\sigma_N I & 0 \end{array}\right] \\
&=: \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]
\end{aligned}
$$

Then it is easy to verify that

$$\tilde{P} = \begin{bmatrix} \Sigma_1 & 0 & \\ 0 & \sigma_N^2 \Sigma_1^{-1} & 0 \\ 0 & 0 & 2\sigma_N I_{s_N} \end{bmatrix}$$

satisfies

$$\begin{aligned} \tilde{A}\tilde{P} + \tilde{P}\tilde{A}^* + \tilde{B}\tilde{B}^* &= 0 \\ \tilde{P}\tilde{C}^* + \tilde{B}\tilde{D}^* &= 0 \end{aligned}$$

Using these two equations, we have

$$\begin{aligned} E(s)E^{\sim}(s) &= \left[\begin{array}{cc|c} \tilde{A} & -\tilde{B}\tilde{B}^* & \tilde{B}\tilde{D}^* \\ 0 & -\tilde{A}^* & \tilde{C}^* \\ \hline \tilde{C} & -\tilde{D}\tilde{B}^* & \tilde{D}\tilde{D}^* \end{array}\right] \\ &= \left[\begin{array}{cc|c} \tilde{A} & -\tilde{A}\tilde{P} - \tilde{P}\tilde{A}^* - \tilde{B}\tilde{B}^* & \tilde{P}\tilde{C}^* + \tilde{B}\tilde{D}^* \\ 0 & -\tilde{A}^* & \tilde{C}^* \\ \hline \tilde{C} & -\tilde{C}\tilde{P} - \tilde{D}\tilde{B}^* & \tilde{D}\tilde{D}^* \end{array}\right] \\ &= \left[\begin{array}{cc|c} \tilde{A} & 0 & 0 \\ 0 & -\tilde{A}^* & \tilde{C}^* \\ \hline \tilde{C} & 0 & \tilde{D}\tilde{D}^* \end{array}\right] \\ &= \tilde{D}\tilde{D}^* = 4\sigma_N^2 I \end{aligned}$$

where the second equality is obtained by applying a similarity transformation

$$T = \begin{bmatrix} I & \tilde{P} \\ 0 & I \end{bmatrix}$$

Hence $\|E_{11}\|_\infty \leq \|E\|_\infty = 2\sigma_N$, which is the desired result.

The remainder of the proof is achieved by using the order reduction by one-step results and by noting that $G_k(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$ obtained by the "$k$th" order partitioning is internally balanced with balanced Gramian given by

$$\Sigma_1 = \operatorname{diag}(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_k I_{s_k})$$

Let $E_k(s) = G_{k+1}(s) - G_k(s)$ for $k = 1, 2, \ldots, N-1$ and let $G_N(s) = G(s)$. Then

$$\overline{\sigma}\left[E_k(j\omega)\right] \le 2\sigma_{k+1}$$

since $G_k(s)$ is a reduced-order model obtained from the internally balanced realization of $G_{k+1}(s)$ and the bound for one-step order reduction holds.

Noting that

$$G(s) - G_r(s) = \sum_{k=r}^{N-1} E_k(s)$$

by the definition of $E_k(s)$, we have

$$\overline{\sigma}\left[G(j\omega) - G_r(j\omega)\right] \le \sum_{k=r}^{N-1} \overline{\sigma}\left[E_k(j\omega)\right] \le 2\sum_{k=r}^{N-1} \sigma_{k+1}$$

This is the desired upper bound. $\square$

- bound can be tight. For example,

$$G(s) = \sum_{j=1}^{n} \frac{b_i}{s+a_i} == \left[\begin{array}{cccc|c} -a_1 & & & & \sqrt{b_1} \\ & -a_2 & & & \sqrt{b_2} \\ & & \ddots & & \vdots \\ & & & -a_n & \sqrt{b_n} \\ \hline \sqrt{b_1} & \sqrt{b_2} & \cdots & \sqrt{b_n} & 0 \end{array}\right]$$

with $a_i > 0$ and $b_i > 0$. Then $P = Q = \left[\frac{\sqrt{b_i b_j}}{a_i + a_j}\right]$ and

$$\|G(s)\|_\infty = G(0) = \sum_{i=1}^{n} \frac{b_i}{a_i} = 2\text{trace}(P) = 2\sum_{i=1}^{n} \sigma_i$$

- bound can also be loose for systems with Hankel singular values close to each other. For example,

$$G(s) = \left[\begin{array}{cccc|c} -19.9579 & -5.4682 & 9.6954 & 0.9160 & -6.3180 \\ 5.4682 & 0 & 0 & 0.2378 & 0.0020 \\ -9.6954 & 0 & 0 & -4.0051 & -0.0067 \\ 0.9160 & -0.2378 & 4.0051 & -0.0420 & 0.2893 \\ \hline -6.3180 & -0.0020 & 0.0067 & 0.2893 & 0 \end{array}\right]$$

with Hankel singular values given by

$$\sigma_1 = 1, \quad \sigma_2 = 0.9977, \quad \sigma_3 = 0.9957, \sigma_4 = 0.9952.$$

| $r$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\|G - G_r\|_\infty$ | 2 | 1.996 | 1.991 | 1.9904 |
| Bounds: $2\sum_{i=r+1}^{4} \sigma_i$ | 7.9772 | 5.9772 | 3.9818 | 1.9904 |
| $2\sigma_{r+1}$ | 2 | 1.9954 | 1.9914 | 1.9904 |

# Frequency-Weighted Balanced Model Reduction

$$\text{General Case:} \quad \inf_{\deg(G_r)\leq r} \|W_o(G-G_r)W_i\|_\infty$$

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right], \quad W_i = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], \quad W_o = \left[\begin{array}{c|c} A_o & B_o \\ \hline C_o & D_o \end{array}\right]$$

$$W_o G W_i = \left[\begin{array}{ccc|c} A & 0 & BC_i & BD_i \\ B_oC & A_o & 0 & 0 \\ 0 & 0 & A_i & B_i \\ \hline D_oC & C_o & 0 & 0 \end{array}\right] =: \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & 0 \end{array}\right].$$

Let $\bar{P}$ and $\bar{Q}$ be the solutions to the following Lyapunov equations

$$\begin{aligned} \bar{A}\bar{P} + \bar{P}\bar{A}^* + \bar{B}\bar{B}^* &= 0 \\ \bar{Q}\bar{A} + \bar{A}^*\bar{Q} + \bar{C}^*\bar{C} &= 0. \end{aligned}$$

The input/output weighted Gramians $P$ and $Q$ are defined by

$$P := \begin{bmatrix} I_n & 0 \end{bmatrix} \bar{P} \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad Q := \begin{bmatrix} I_n & 0 \end{bmatrix} \bar{Q} \begin{bmatrix} I_n \\ 0 \end{bmatrix}.$$

$P$ and $Q$ satisfy the following lower order equations

$$\begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix} \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} + \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} \begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix}^* + \begin{bmatrix} BD_i \\ B_i \end{bmatrix} \begin{bmatrix} BD_i \\ B_i \end{bmatrix}^* = 0$$

$$\begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} \begin{bmatrix} A & 0 \\ B_oC & A_o \end{bmatrix} + \begin{bmatrix} A & 0 \\ B_oC & A_o \end{bmatrix}^* \begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} + \begin{bmatrix} C^*D_o^* \\ C_o^* \end{bmatrix} \begin{bmatrix} C^*D_o^* \\ C_o^* \end{bmatrix}^* = 0.$$

$W_i = I \Longrightarrow P$ can be obtained from

$$PA^* + AP + BB^* = 0$$

$W_o = I \Longrightarrow Q$ can be obtained from

$$QA + A^*Q + C^*C = 0.$$

Now let $T$ be a nonsingular matrix such that

$$TPT^* = (T^{-1})^* Q T^{-1} = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}$$

(i.e., balanced) and partition the system accordingly as

$$\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right].$$

Then a reduced order model $G_r$ is obtained as

$$G_r = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & 0 \end{array}\right].$$

Works well but with guarantee.

## Relative Reduction

$$G_r = G(I + \Delta_{\text{rel}}), \quad \Longrightarrow \quad \inf_{\deg(G_r)\le r} \left\| G^{-1}(G - G_r) \right\|_\infty$$

and a related problem is

$$G = G_r(I + \Delta_{mul})$$

Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{RH}_\infty$ be minimum phase and $D$ be nonsingular.

Then $W_o = G^{-1}(s) = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]$.

(a) Then the input/output weighted Gramians $P$ and $Q$ are given by

$$PA^* + AP + BB^* = 0$$

$$Q(A - BD^{-1}C) + (A - BD^{-1}C)^* Q + C^*(D^{-1})^* D^{-1} C = 0.$$

(b) Suppose $P$ and $Q$ are balanced:

$$P = Q = \text{diag}(\sigma_1 I_{s_1}, \ldots, \sigma_r I_{s_r}, \sigma_{r+1} I_{s_{r+1}}, \ldots, \sigma_N I_{s_N}) = \text{diag}(\Sigma_1, \Sigma_2)$$

and let $G$ be partitioned compatibly with $\Sigma_1$ and $\Sigma_2$ as

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right].$$

Then

$$G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

is stable and minimum phase. Furthermore

$$\|\Delta_{rel}\|_\infty \le \prod_{i=r+1}^{N} \left(1 + 2\sigma_i(\sqrt{1+\sigma_i^2} + \sigma_i)\right) - 1$$

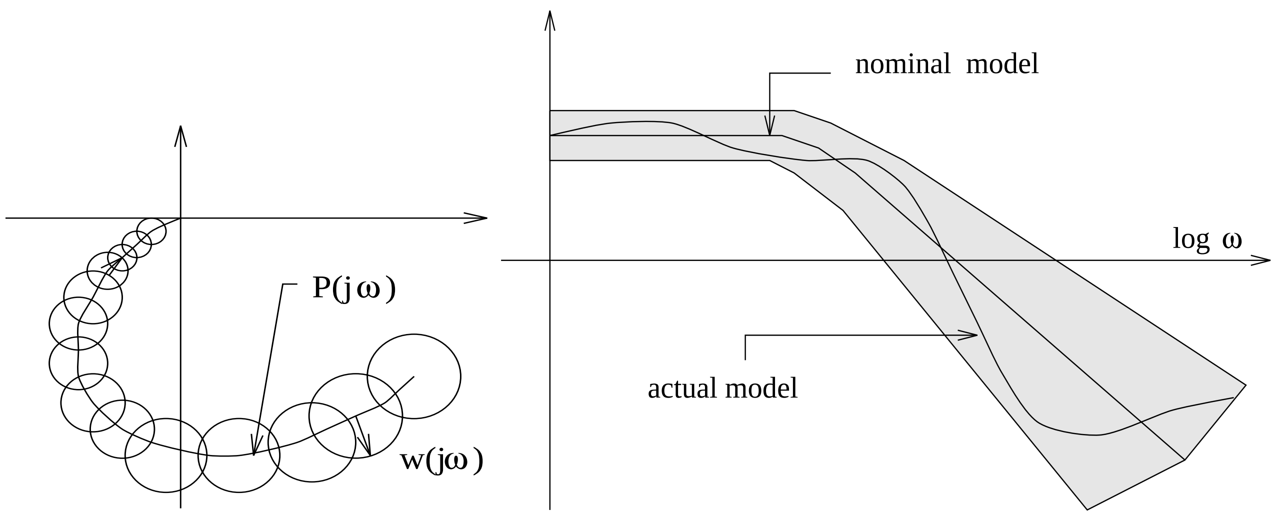
$$\|\Delta_{mul}\|_\infty \le \prod_{i=r+1}^{N} \left(1 + 2\sigma_i(\sqrt{1+\sigma_i^2} + \sigma_i)\right) - 1.$$

# Chapter 8: Uncertainty and Robustness

- model uncertainty
- small gain theorem
- additive uncertainty
- multiplicative uncertainty
- coprime factor uncertainty
- other tests
- robust performance
- skewed specifications
- example: siso vs mimo

# Model Uncertainty

$$
\begin{aligned}
P_\Delta(s) &= P(s) + w(s)\Delta(s), \quad \overline{\sigma}[\Delta(j\omega)] < 1, \ \forall\omega \\
P_\Delta(s) &= (I + w(s)\Delta(s))P(s).
\end{aligned}
$$




Suppose $P \in \mathbf{\Pi}$ is the nominal model and $K$ is a controller.

**Nominal Stability (NS):** if $K$ stabilizes the nominal $P$.

**Robust Stability (RS):** if $K$ stabilizes every plant in $\mathbf{\Pi}$.

**Nominal Performance (NP):** if the performance objectives are satisfied for the nominal plant $P$.

**Robust Performance (RP):** if the performance objectives are satisfied for every plant in $\mathbf{\Pi}$.

$$P(s,\alpha,\beta) = \frac{10\left((2+0.2\alpha)s^2 + (2+0.3\alpha+0.4\beta)s + (1+0.2\beta)\right)}{(s^2+0.5s+1)(s^2+2s+3)(s^2+3s+6)}$$

$$\alpha,\ \beta \in [-1,1]$$

$$P(s,\alpha,\beta) \in \{P_0 + W\Delta \mid \|\Delta\| \le 1\}$$

with $P_0 := P(s,0,0)$ and

$$W(s) = P(s,1,1) - P(s,0,0) = \frac{10\left(0.2s^2 + 0.7s + 0.2\right)}{(s^2+0.5s+1)(s^2+2s+3)(s^2+3s+6)}$$

The frequency response $P_0 + W\Delta$ is shown in Figure 0.14 as circles.

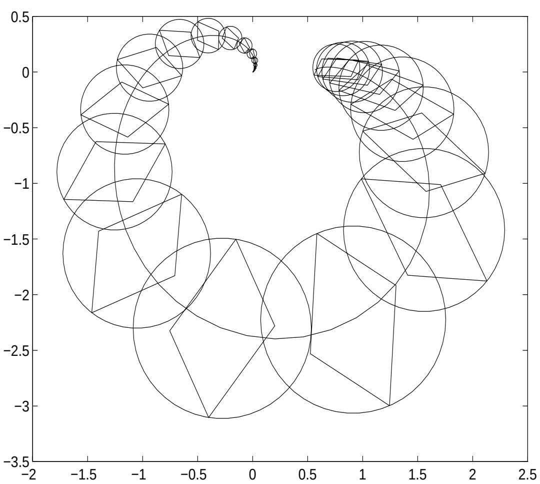



Figure 0.14: Nyquist diagram of uncertain system and disk covering

Another way to bound the frequency response is to treat $\alpha$ and $\beta$ as norm bounded uncertainties; that is,

$$P(s,\alpha,\beta) \in \{P_0 + W_1\Delta_1 + W_2\Delta_2 \mid \|\Delta_i\|_\infty \le 1\}$$

with $P_0 = P(s,0,0)$ and

$$W_1 = \frac{10(0.2s^2 + 0.3s)}{(s^2+0.5s+1)(s^2+2s+3)(s^2+3s+6)},$$

$$W_2 = \frac{10(0.4s + 0.2)}{(s^2 + 0.5s + 1)(s^2 + 2s + 3)(s^2 + 3s + 6)}$$

It is in fact easy to show that

$$\{P_0 + W_1\Delta_1 + W_2\Delta_2 \mid \|\Delta_i\|_\infty \le 1\} = \{P_0 + W\Delta \mid \|\Delta\|_\infty \le 1\}$$

with $|W| = |W_1| + |W_2|$. The frequency response $P_0 + W\Delta$ is shown in Figure 0.15. This bounding is clearly more conservative.

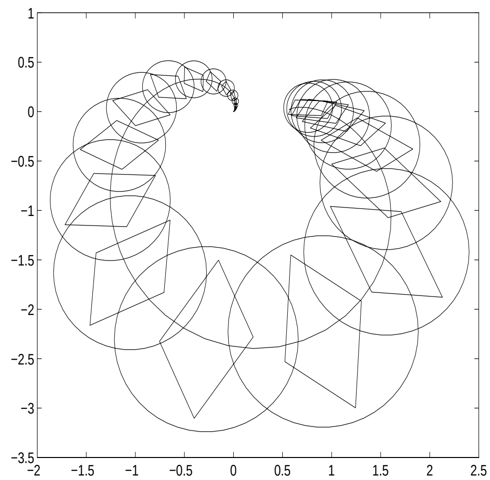



Figure 0.15: A conservative covering

---

Consider a process control model

$$G(s) = \frac{ke^{-\tau s}}{Ts + 1}, \quad 4 \le k \le 9, \ 2 \le T \le 3, \ 1 \le \tau \le 2.$$

Take the nominal model as

$$G_0(s) = \frac{6.5}{(2.5s + 1)(1.5s + 1)}$$

Then for each frequency, all possible frequency responses are in a box, as shown in Figure 0.16.

$$\Delta_a(j\omega) = G(j\omega) - G_0(j\omega)$$

$\gg$ **mf= ginput(50)** % pick 50 points: the first column of mf is the frequency points and the second column of mf is the corresponding magnitude responses.

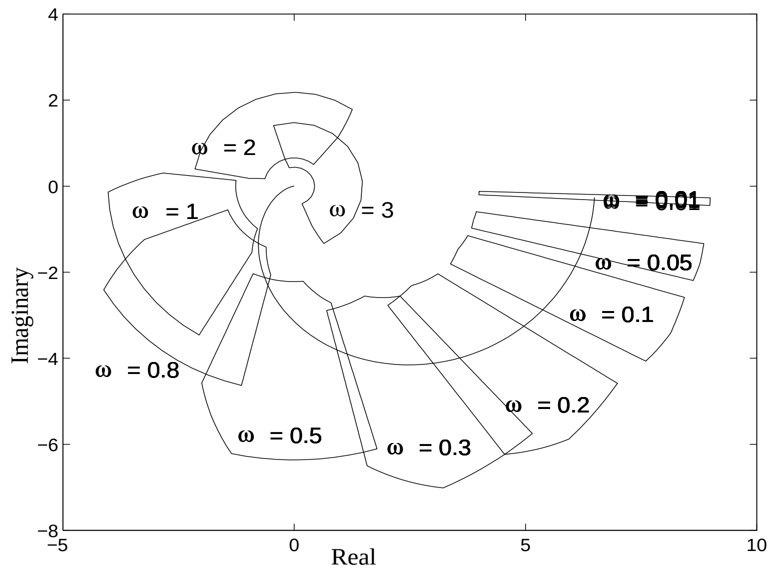



Figure 0.16: Uncertain delay system and $G_0$

≫ **magg=vpck(mf(:,2),mf(:,1));** % pack them as a varying matrix.

≫ $\mathbf{W_a}$ =**fitmag(magg);** % choose the order of $W_a$ online. A third-order $W_a$ is sufficient for this example.

≫ **[A,B,C,D]=unpck($\mathbf{W_a}$)** % converting into state-space.

≫ **[Z, P, K]=ss2zp(A,B,C,D)** % converting into zero/pole/gain form.

We get

$$W_a(s) = \frac{0.0376(s+116.4808)(s+7.4514)(s+0.2674)}{(s+1.2436)(s+0.5575)(s+4.9508)}$$

and the frequency response of $W_a$ is also plotted in Figure 0.17. Similarly, define the multiplicative uncertainty

$$\Delta_m(s) := \frac{G(s) - G_0(s)}{G_0(s)}$$

and a $W_m$ can be found such that $|\Delta_m(j\omega)| \leq |W_m(j\omega)|$, as shown in Figure 0.18. A $W_m$ is given by

$$W_m = \frac{2.8169(s+0.212)(s^2+2.6128s+1.732)}{s^2+2.2425s+2.6319}$$

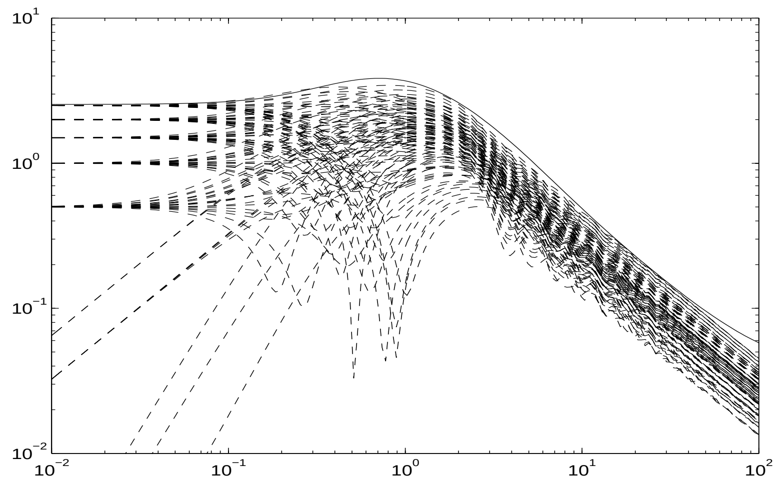



Figure 0.17: $\Delta_a$ (dashed line) and a bound $W_a$ (solid line)

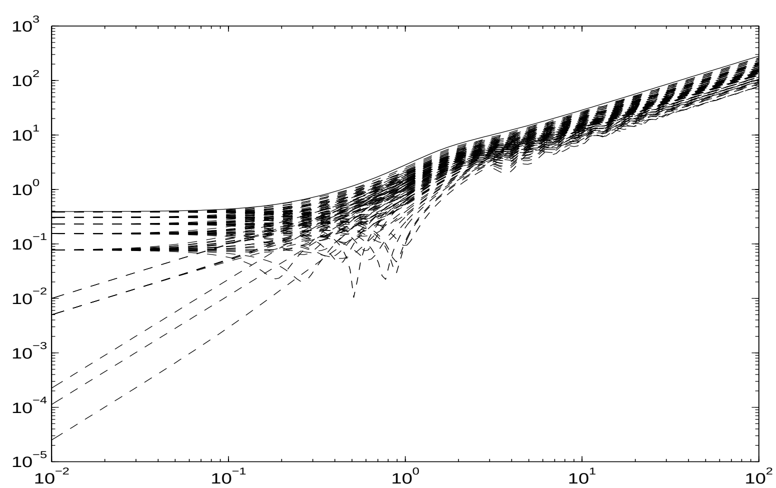
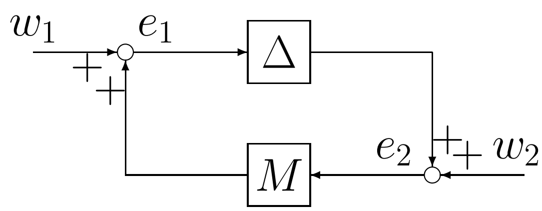



Figure 0.18: $\Delta_m$ (dashed line) and a bound $W_m$ (solid line)

# Small Gain Theorem




**Small Gain Theorem:** Suppose $M \in (\mathcal{RH}_\infty)^{p\times q}$. Then the system is well-posed and internally stable for all $\Delta(s) \in \mathcal{RH}_\infty$ with

(a) $\|\Delta\|_\infty \leq 1/\gamma$ if and only if $\|M(s)\|_\infty < \gamma$;

(b) $\|\Delta\|_\infty < 1/\gamma$ if and only if $\|M(s)\|_\infty \leq \gamma$.

**Proof.** Assume $\gamma = 1$. System is stable iff $\det(I - M\Delta)$ has no zero in the closed right-half plane for all $\Delta \in \mathcal{RH}_\infty$ and $\|\Delta\|_\infty \leq 1$.

($\Longleftarrow$) $\det(I - M\Delta) \neq 0$ for all $\Delta \in \mathcal{RH}_\infty$ and $\|\Delta\|_\infty \leq 1$ since

$$|\lambda(I - M\Delta)| \geq 1 - \max|\lambda(M\Delta)| \geq 1 - \|M\|_\infty > 0$$

($\Rightarrow$) Suppose $\|M\|_\infty \geq 1$. There exists a $\Delta \in \mathcal{RH}_\infty$ with $\|\Delta\|_\infty \leq 1$ such that $\det(I - M(s)\Delta(s))$ has a zero on the imaginary axis, so the system is unstable. Suppose $\omega_0 \in \mathbb{R}_+ \cup \{\infty\}$ is such that $\bar\sigma(M(j\omega_0)) \geq 1$. Let $M(j\omega_0) = U(j\omega)\Sigma(j\omega_0)V^*(j\omega_0)$ be a singular value decomposition with

$$U(j\omega_0) = \begin{bmatrix} u_1 & u_2 & \cdots & u_p \end{bmatrix}$$

$$V(j\omega_0) = \begin{bmatrix} v_1 & v_2 & \cdots & v_q \end{bmatrix}$$

$$\Sigma(j\omega_0) = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \end{bmatrix}$$

We shall construct a $\Delta \in \mathcal{RH}_\infty$ such that $\Delta(j\omega_0) = \frac{1}{\sigma_1} v_1 u_1^*$ and $\|\Delta\|_\infty \leq 1$. Indeed, for such $\Delta(s)$,

$$\det(I - M(j\omega_0)\Delta(j\omega_0)) = \det(I - U\Sigma V^* v_1 u_1^*/\sigma_1) = 1 - u_1^* U\Sigma V^* v_1/\sigma_1 = 0$$

and thus the closed-loop system is either not well-posed (if $\omega_0 = \infty$) or unstable (if $\omega \in \mathbb{R}$). There are two different cases:

(1) $\omega_0 = 0$ or $\infty$: then $U$ and $V$ are real matrices. Chose

$$\Delta = \frac{1}{\sigma_1} v_1 u_1^* \in \mathbb{R}^{q \times p}.$$

(2) $0 < \omega_0 < \infty$: write $u_1$ and $v_1$ in the following form:

$$u_1^* = \begin{bmatrix} u_{11}e^{j\theta_1} & u_{12}e^{j\theta_2} & \cdots & u_{1p}e^{j\theta_p} \end{bmatrix}, \qquad v_1 = \begin{bmatrix} v_{11}e^{j\phi_1} \\ v_{12}e^{j\phi_2} \\ \vdots \\ v_{1q}e^{j\phi_q} \end{bmatrix}$$

where $u_{1i}, v_{1j} \in \mathbb{R}$ are chosen so that $\theta_i, \phi_j \in [-\pi, 0)$.

Choose $\beta_i \geq 0$ and $\alpha_j \geq 0$ so that

$$\angle\left(\frac{\beta_i - j\omega_0}{\beta_i + j\omega_0}\right) = \theta_i, \quad \angle\left(\frac{\alpha_j - j\omega_0}{\alpha_j + j\omega_0}\right) = \phi_j$$

Let

$$\Delta(s) = \frac{1}{\sigma_1} \begin{bmatrix} v_{11}\frac{\alpha_1 - s}{\alpha_1 + s} \\ \vdots \\ v_{1q}\frac{\alpha_q - s}{\alpha_q + s} \end{bmatrix} \begin{bmatrix} u_{11}\frac{\beta_1 - s}{\beta_1 + s} & \cdots & u_{1p}\frac{\beta_p - s}{\beta_p + s} \end{bmatrix} \in \mathcal{RH}_\infty.$$

Then $\|\Delta\|_\infty = 1/\sigma_1 \leq 1$ and $\Delta(j\omega_0) = \frac{1}{\sigma_1} v_1 u_1^*$.

$\square$

The theorem still holds even if $\Delta$ and $M$ are infinite dimensional. This is summarized as the following corollary.

---

The following statements are equivalent:

(i) The system is well-posed and internally stable for all $\Delta \in \mathcal{H}_\infty$ with $\|\Delta\|_\infty < 1/\gamma$;

(ii) The system is well-posed and internally stable for all $\Delta \in \mathcal{RH}_\infty$ with $\|\Delta\|_\infty < 1/\gamma$;

(iii) The system is well-posed and internally stable for all $\Delta \in \mathbb{C}^{q\times p}$ with $\|\Delta\| < 1/\gamma$;

(iv) $\|M\|_\infty \leq \gamma$.

---

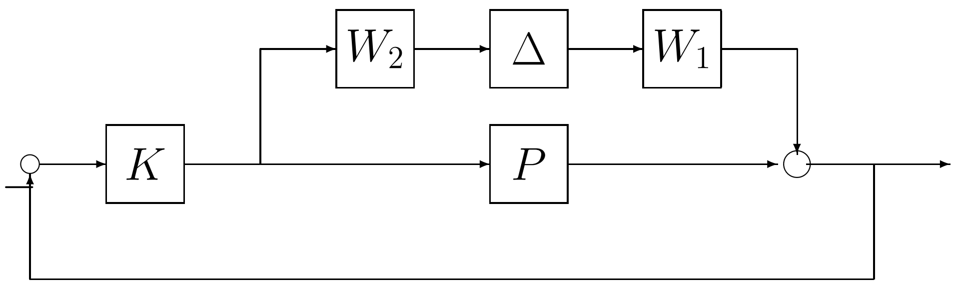
It can be shown that the small gain condition is sufficient to guarantee internal stability even if $\Delta$ is a nonlinear and time varying "stable" operator with an appropriately defined stability notion, see Desoer and Vidyasagar [1975].

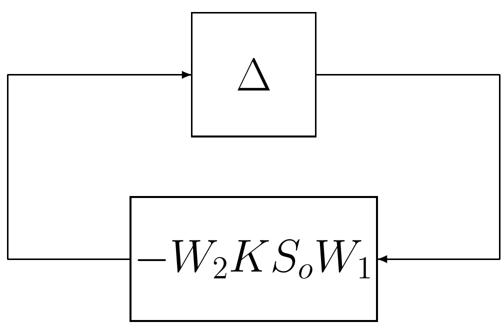
## Additive Uncertainty

$$S_o = (I + PK)^{-1}, \quad T_o = PK(I + PK)^{-1}$$

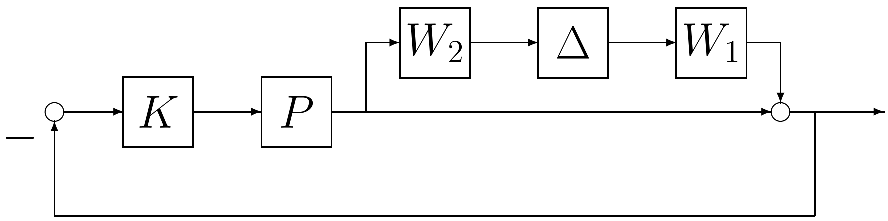
$$S_i = (I + KP)^{-1}, \quad T_i = KP(I + KP)^{-1}.$$

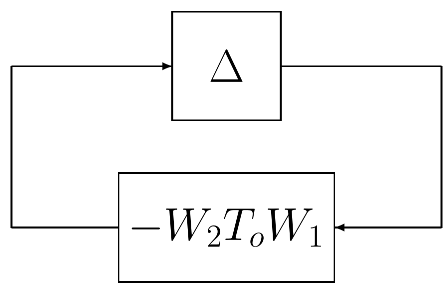
Let $\mathbf{\Pi} = \{P + W_1 \Delta W_2 : \quad \Delta \in \mathcal{RH}_\infty\}$ and let $K$ stabilize $P$. Then the closed-loop system is well-posed and internally stable for all $\|\Delta\|_\infty < 1$ if and only if $\|W_2 K S_o W_1\|_\infty \leq 1$.



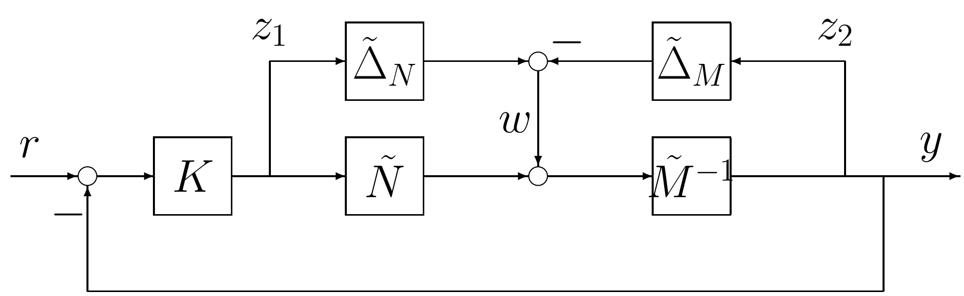
## Multiplicative Uncertainty




Let $\mathbf{\Pi} = \{(I + W_1 \Delta W_2)P : \quad \Delta \in \mathcal{RH}_\infty\}$ and let $K$ stabilize $P$. Then the closed-loop system is well-posed and internally stable for all $\Delta \in \mathcal{RH}_\infty$ with $\|\Delta\|_\infty < 1$ if and only if $\|W_2 T_o W_1\|_\infty \leq 1$.

# Coprime Factor Uncertainty




Let $P = \tilde{M}^{-1}\tilde{N}$ be stable left coprime factorization and $K$ stabilize $P$. Suppose

$$\Pi = (\tilde{M} + \tilde{\Delta}_M)^{-1}(\tilde{N} + \tilde{\Delta}_N), \quad \Delta := \begin{bmatrix} \tilde{\Delta}_N & \tilde{\Delta}_M \end{bmatrix}$$

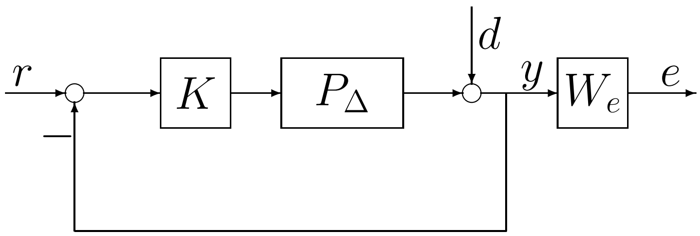
with $\tilde{\Delta}_M,\, \tilde{\Delta}_N \in \mathcal{RH}_\infty$. Then the closed-loop system is well-posed and internally stable for all $\|\Delta\|_\infty < 1$ if and only if

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\|_\infty \leq 1.$$

# Other Tests

| $W_1 \in \mathcal{RH}_\infty \quad W_2 \in \mathcal{RH}_\infty \quad \Delta \in \mathcal{RH}_\infty \quad \|\Delta\|_\infty < 1$ | | |
|---|---|---|
| Perturbed Model Sets $\mathbf{\Pi}$ | Representative Types of Uncertainty Characterized | Robust Stability Tests |
| $(I + W_1 \Delta W_2)P$ | output (sensor) errors<br>neglected HF dynamics<br>uncertain rhp zeros | $\|W_2 T_o W_1\|_\infty \leq 1$ |
| $P(I + W_1 \Delta W_2)$ | input (actuators) errors<br>neglected HF dynamics<br>uncertain rhp zeros | $\|W_2 T_i W_1\|_\infty \leq 1$ |
| $(I + W_1 \Delta W_2)^{-1} P$ | LF parameter errors<br>uncertain rhp poles | $\|W_2 S_o W_1\|_\infty \leq 1$ |
| $P(I + W_1 \Delta W_2)^{-1}$ | LF parameter errors<br>uncertain rhp poles | $\|W_2 S_i W_1\|_\infty \leq 1$ |
| $P + W_1 \Delta W_2$ | additive plant errors<br>neglected HF dynamics<br>uncertain rhp zeros | $\|W_2 K S_o W_1\|_\infty \leq 1$ |
| $P(I + W_1 \Delta W_2 P)^{-1}$ | LF parameter errors<br>uncertain rhp poles | $\|W_2 S_o P W_1\|_\infty \leq 1$ |
| $(\tilde{M} + \tilde{\Delta}_M)^{-1}(\tilde{N} + \tilde{\Delta}_N)$<br>$P = \tilde{M}^{-1}\tilde{N}$<br>$\Delta = \begin{bmatrix} \tilde{\Delta}_N & \tilde{\Delta}_M \end{bmatrix}$ | LF parameter errors<br>neglected HF dynamics<br>uncertain rhp poles & zeros | $\left\| \begin{bmatrix} K \\ I \end{bmatrix} S_o \tilde{M}^{-1} \right\|_\infty \leq 1$ |
| $(N + \Delta_N)(M + \Delta_M)^{-1}$<br>$P = NM^{-1}$<br>$\Delta = \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}$ | LF parameter errors<br>neglected HF dynamics<br>uncertain rhp poles & zeros | $\|M^{-1} S_i [K \ I]\|_\infty \leq 1$ |

# Robust Performance




$$\sup_{\|d\|_2 \le 1} \|e\|_2 \le 1$$

$$T_{ed} = W_e(I + P_\Delta K)^{-1}, \quad P_\Delta \in \mathbf{\Pi}.$$

Suppose $P_\Delta \in \{(I + \Delta W_2)P : \ \Delta \in \mathcal{RH}_\infty, \ \|\Delta\|_\infty < 1\}$ and $K$ internally stabilizes $P$. Then robust performance is guaranteed if

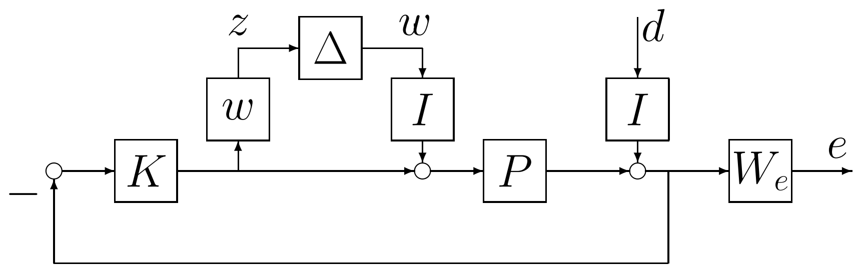
$$\overline{\sigma}(W_e S_o) + \overline{\sigma}(W_2 T_o) \le 1.$$

$$\overline{\sigma}(T_{ed}) \le \overline{\sigma}(W_e S_o)\overline{\sigma}[(I + \Delta W_2 T_o)^{-1}] = \frac{\overline{\sigma}(W_e S_o)}{\underline{\sigma}(I + \Delta W_2 T_o)}$$

$$\le \frac{\overline{\sigma}(W_e S_o)}{1 - \overline{\sigma}(\Delta W_2 T_o)} \le \frac{\overline{\sigma}(W_e S_o)}{1 - \overline{\sigma}(W_2 T_o)}.$$

## Skewed Specifications

$$\mathbf{\Pi} := \left\{P(I + w\Delta) : \ \Delta \in \mathcal{RH}_\infty, \ \|\Delta\|_\infty < 1\right\}.$$




robust stability:

$$\|wT_i\|_\infty \le 1,$$

nominal performance:

$$\|W_e S_o\|_\infty \le 1.$$

$$\tilde{T}_{ed} = W_e S_o (I + P\Delta w K S_o)^{-1} = W_e S_o \left[I + P\Delta P^{-1}(wT_o)\right]^{-1}.$$

---

robust performance is guaranteed if

$$\overline{\sigma}(W_e S_o) + \kappa(P)\overline{\sigma}(wT_i) \le 1$$

or

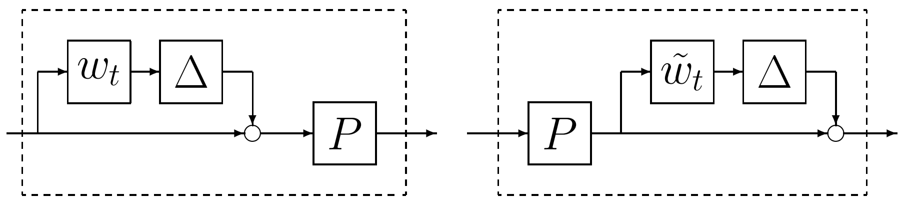
$$\overline{\sigma}(W_e S_o) + \kappa(P)\overline{\sigma}(wT_o) \le 1.$$

# Why Condition Number?

$$\begin{aligned}\mathbf{\Pi}_1 &:= \{P(I+w_t\Delta):\ \Delta\in\mathcal{RH}_\infty,\ \|\Delta\|_\infty<1\}\\ \mathbf{\Pi}_2 &:= \{(I+\tilde{w}_t\Delta)P:\ \Delta\in\mathcal{RH}_\infty,\ \|\Delta\|_\infty<1\}.\end{aligned}$$

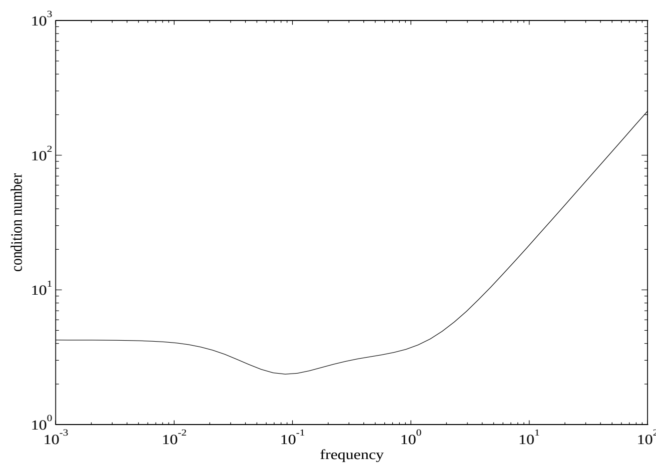
$$\mathbf{\Pi}_2 \supseteq \mathbf{\Pi}_1 \quad \text{if} \quad |\tilde{w}_t| \geq |w_t|\kappa(P) \quad \forall\omega$$

since $P(I+w_t\Delta) = (I + w_t P\Delta P^{-1})P$.

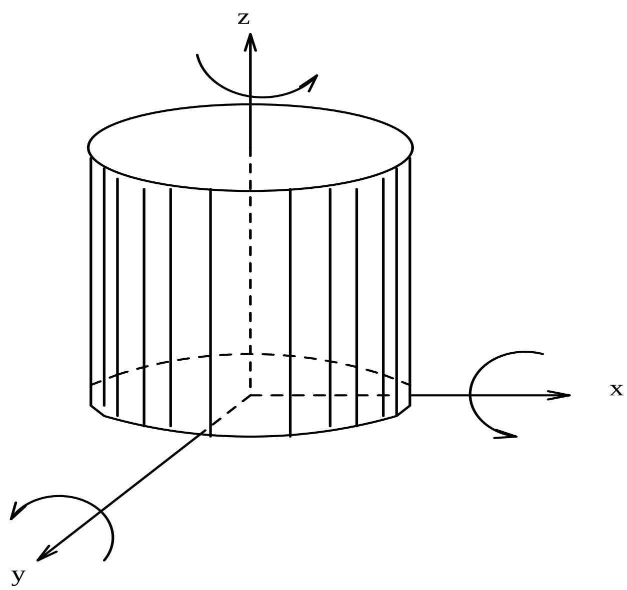



$$P(s) = \left[\begin{array}{ccc|cc} -0.2 & 0.1 & 1 & 0 & 1\\ -0.05 & 0 & 0 & 0 & 0.7\\ 0 & 0 & -1 & 1 & 0\\ \hline 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\end{array}\right] = \frac{1}{a(s)}\left[\begin{array}{cc} s & (s+1)(s+0.07)\\ -0.05 & 0.7(s+1)(s+0.13)\end{array}\right]$$

where $a(s) = (s+1)(s+0.1707)(s+0.02929)$.



# Example: SISO vs MIMO



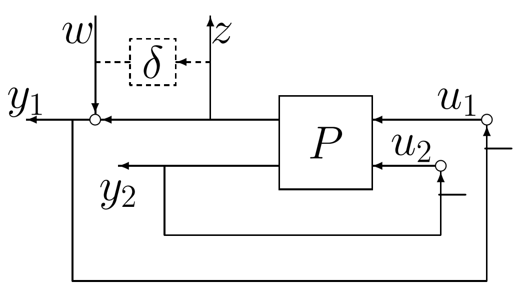

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

$$P(s) = \frac{1}{s^2 + a^2} \begin{bmatrix} s - a^2 & a(s+1) \\ -a(s+1) & s - a^2 \end{bmatrix}.$$

$$S = (I + P)^{-1} = \frac{1}{s+1} \begin{bmatrix} s & -a \\ a & s \end{bmatrix}, \quad T = P(I+P)^{-1} = \frac{1}{s+1} \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}.$$

Each loop has the open-loop transfer function as $\frac{1}{s}$ so each loop has phase margin $\phi_{\max} = -\phi_{\min} = 90^o$ and gain margin $k_{\min} = 0$, $k_{\max} = \infty$.

Suppose one loop transfer function is perturbed



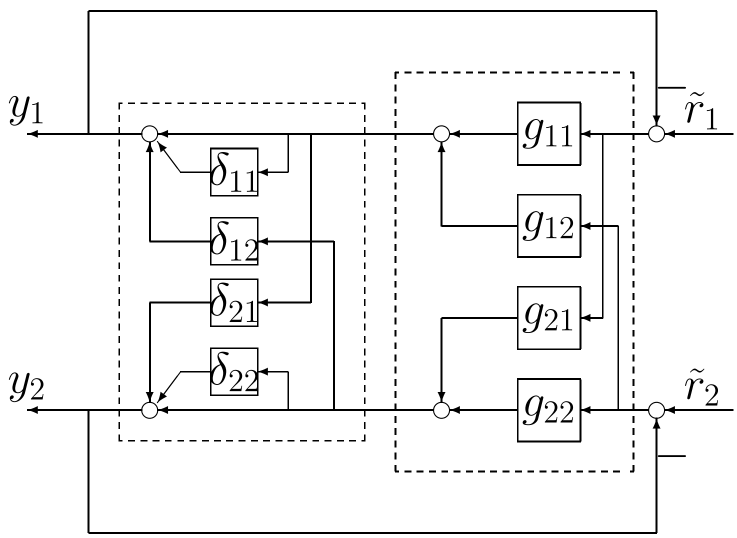

Denote

$$\frac{z(s)}{w(s)} = -T_{11} = -\frac{1}{s+1}.$$

Then the maximum allowable perturbation is given by

$$\|\delta\|_\infty < \frac{1}{\|T_{11}\|_\infty} = 1$$

which is independent of $a$. However, if both loops are perturbed at the same time, then the maximum allowable perturbation is much smaller, as shown below.




$$P_\Delta = (I + \Delta)P, \quad \Delta = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} \in \mathcal{RH}_\infty$$

$\|\Delta\|_\infty < \gamma$. The system is robustly stable for every such $\Delta$ iff

$$\gamma \le \frac{1}{\|T\|_\infty} = \frac{1}{\sqrt{1+a^2}} \quad (\ll 1 \text{ if } a \gg 1).$$

In particular, consider

$$\Delta = \Delta_d = \begin{bmatrix} \delta_{11} & \\ & \delta_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Then the closed-loop system is stable for every such $\Delta$ iff

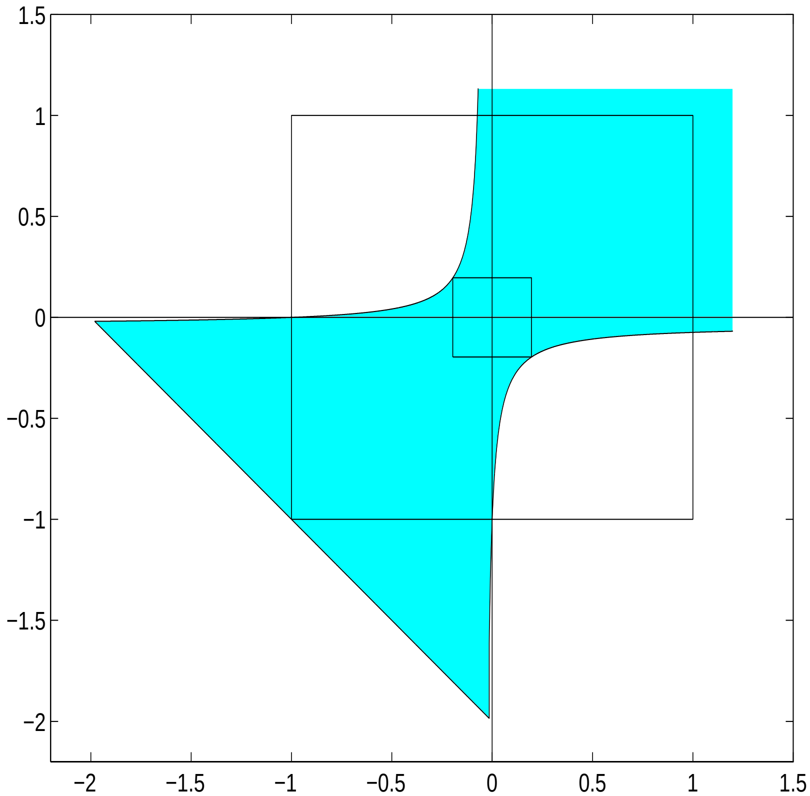
$$\det(I + T\Delta_d) = \frac{(s^2 + (2 + \delta_{11} + \delta_{22})s + 1 + \delta_{11} + \delta_{22} + (1 + a^2)\delta_{11}\delta_{22})}{(s+1)^2}$$

has no zero in the closed right-half plane. Hence the stability region is given by

$$\begin{aligned} 2 + \delta_{11} + \delta_{22} &> 0 \\ 1 + \delta_{11} + \delta_{22} + (1 + a^2)\delta_{11}\delta_{22} &> 0. \end{aligned}$$

The system is unstable with

$$\delta_{11} = -\delta_{22} = \frac{1}{\sqrt{1 + a^2}}.$$



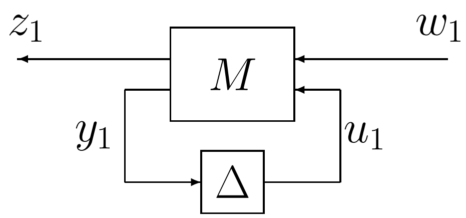
# Chapter 9: Linear Fractional Transformation

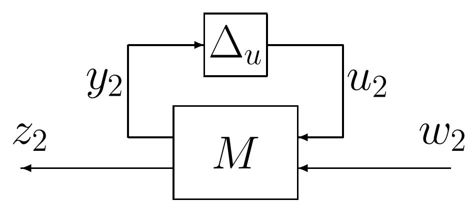
A (lower) LFT of $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ over $\Delta$ is defined as

$$\mathcal{F}_\ell(M, \Delta) := M_{11} + M_{12}\Delta(I - M_{22}\Delta)^{-1}M_{21}$$

Similarly, an upper LFT:

$$\mathcal{F}_u(M, \Delta_u) = M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12}$$




$$\begin{aligned} \begin{bmatrix} z_1 \\ y_1 \end{bmatrix} &= M \begin{bmatrix} w_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ u_1 \end{bmatrix}, \\ u_1 &= \Delta\, y_1 \end{aligned}$$

## Properties

- $\mathcal{F}_\ell(M,\Delta)$ is well-posed if $(I - M_{22}\Delta)$ is invertible.
- $(\mathcal{F}_u(M,\Delta))^{-1} = \mathcal{F}_u(N,\Delta)$ with $N$ given by

$$N = \begin{bmatrix} M_{11} - M_{12}M_{22}^{-1}M_{21} & -M_{12}M_{22}^{-1} \\ M_{22}^{-1}M_{21} & M_{22}^{-1} \end{bmatrix}.$$

- Suppose $C$ is invertible. Then

$$\begin{aligned} (A+BQ)(C+DQ)^{-1} &= \mathcal{F}_\ell(M,Q) \\ (C+QD)^{-1}(A+QB) &= \mathcal{F}_\ell(N,Q) \end{aligned}$$

  with

$$M = \begin{bmatrix} AC^{-1} & B - AC^{-1}D \\ C^{-1} & -C^{-1}D \end{bmatrix},$$

$$N = \begin{bmatrix} C^{-1}A & C^{-1} \\ B - DC^{-1}A & -DC^{-1} \end{bmatrix}.$$

- if $M_{12}$ is invertible, then

$$\mathcal{F}_\ell(M,Q) = (C+QD)^{-1}(A+QB)$$

  with $A = M_{12}^{-1}M_{11}$, $B = M_{21} - M_{22}M_{12}^{-1}M_{11}$, $C = M_{12}^{-1}$ and $D = -M_{22}M_{12}^{-1}$.

- if $M_{21}$ is invertible, then

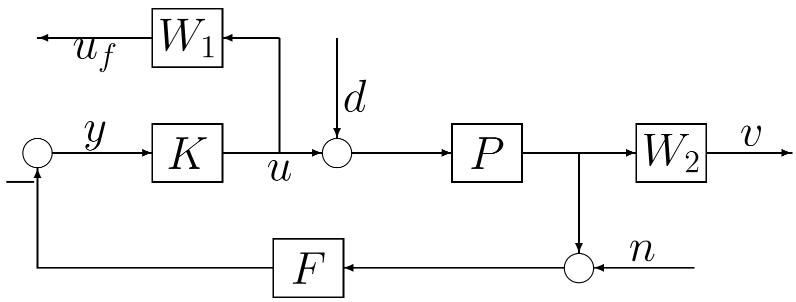
$$\mathcal{F}_\ell(M,Q) = (A+BQ)(C+DQ)^{-1}$$

  with $A = M_{11}M_{21}^{-1}$, $B = M_{12} - M_{11}M_{21}^{-1}M_{22}$, $C = M_{21}^{-1}$ and $D = -M_{21}^{-1}M_{22}$.

# Example

The following diagram can be rearranged as an LFT $z = \mathcal{F}_\ell(G, K)w$ with

$$w = \begin{pmatrix} d \\ n \end{pmatrix} \qquad z = \begin{pmatrix} v \\ u_f \end{pmatrix} \quad G = \left[\begin{array}{cc|c} W_2P & 0 & W_2P \\ 0 & 0 & W_1 \\ \hline -FP & -F & -FP \end{array}\right].$$




$$P = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & 0 \end{array}\right], \ F = \left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & D_f \end{array}\right], \ W_1 = \left[\begin{array}{c|c} A_u & B_u \\ \hline C_u & D_u \end{array}\right], \ W_2 = \left[\begin{array}{c|c} A_v & B_v \\ \hline C_v & D_v \end{array}\right].$$

That is,

$$\dot{x}_p = A_p x_p + B_p(d + u), \quad y_p = C_p x_p,$$
$$\dot{x}_f = A_f x_f + B_f(y_p + n), \quad -y = C_f x_f + D_f(y_p + n),$$
$$\dot{x}_u = A_u x_u + B_u u, \quad u_f = C_u x_u + D_u u,$$
$$\dot{x}_v = A_v x_v + B_v y_p, \quad v = C_v x_v + D_v y_p.$$

Now define a new state vector

$$x = \begin{bmatrix} x_p \\ x_f \\ x_u \\ x_v \end{bmatrix}$$

and eliminate the variable $y_p$ to get a realization of $G$ as

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned}$$

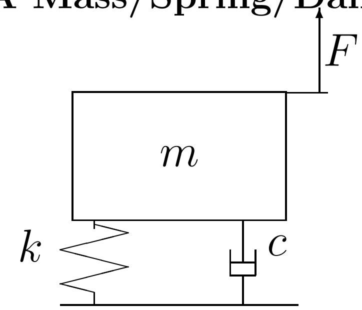
with

$$A = \begin{bmatrix} A_p & 0 & 0 & 0 \\ B_f C_p & A_f & 0 & 0 \\ 0 & 0 & A_u & 0 \\ B_v C_p & 0 & 0 & A_v \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_p & 0 \\ 0 & B_f \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_p \\ 0 \\ B_u \\ 0 \end{bmatrix}$$

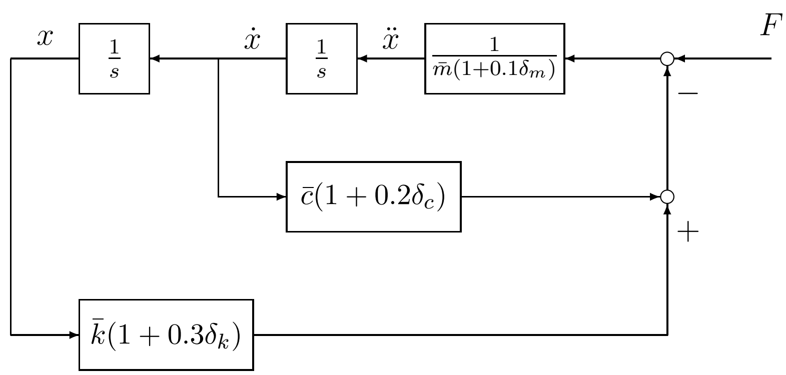
$$C_1 = \begin{bmatrix} D_v C_p & 0 & 0 & C_v \\ 0 & 0 & C_u & 0 \end{bmatrix}, \quad D_{11} = 0, \quad D_{12} = \begin{bmatrix} 0 \\ D_u \end{bmatrix}$$

$$C_2 = \begin{bmatrix} -D_f C_p & -C_f & 0 & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & -D_f \end{bmatrix}, \quad D_{22} = 0.$$

**Parametric Uncertainty: A Mass/Spring/Damper System**




$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F}{m}.$$




Then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \mathcal{F}_\ell(M, \Delta) \begin{bmatrix} x_1 \\ x_2 \\ F \end{bmatrix}$$

where

$$M = \left[\begin{array}{ccc:ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\bar{k}}{\bar{m}} & -\frac{\bar{c}}{\bar{m}} & \frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{0.1}{\bar{m}} \\ \hdashline 0.3\bar{k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2\bar{c} & 0 & 0 & 0 & 0 \\ -\bar{k} & -\bar{c} & 1 & -1 & -1 & -0.1 \end{array}\right], \quad \Delta = \begin{bmatrix} \delta_k & 0 & 0 \\ 0 & \delta_c & 0 \\ 0 & 0 & \delta_m \end{bmatrix}.$$

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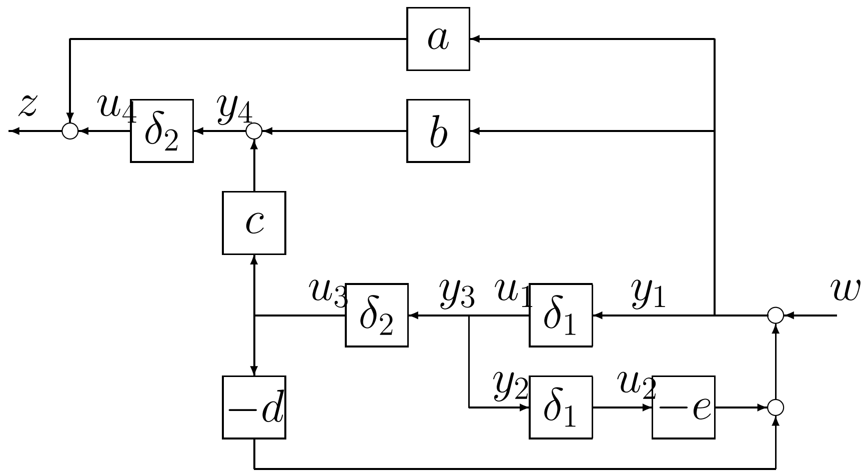



$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ z \end{bmatrix} = M \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ w \end{bmatrix}$$

where

$$M = \left[\begin{array}{cccc:c} 0 & -e & -d & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -be & -bd+c & 0 & b \\ \hdashline 0 & -ae & -ad & 1 & a \end{array}\right].$$

Then

$$z = \mathcal{F}_u(M, \Delta)w, \quad \Delta = \begin{bmatrix} \delta_1 I_2 & 0 \\ 0 & \delta_2 I_2 \end{bmatrix}.$$

# HIMAT Example

$$W_{\mathrm{del}} = \begin{bmatrix} \dfrac{50(s+100)}{s+10000} & 0 \\ 0 & \dfrac{50(s+100)}{s+10000} \end{bmatrix}, \quad W_p = \begin{bmatrix} \dfrac{0.5(s+3)}{s+0.03} & 0 \\ 0 & \dfrac{0.5(s+3)}{s+0.03} \end{bmatrix},$$

$$W_n = \begin{bmatrix} \dfrac{2(s+1.28)}{s+320} & 0 \\ 0 & \dfrac{2(s+1.28)}{s+320} \end{bmatrix},$$

$$P_0 = \left[\begin{array}{cccc|cc} -0.0226 & -36.6 & -18.9 & -32.1 & 0 & 0 \\ 0 & -1.9 & 0.983 & 0 & -0.414 & 0 \\ 0.0123 & -11.7 & -2.63 & 0 & -77.8 & 22.4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 57.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 57.3 & 0 & 0 \end{array}\right]$$

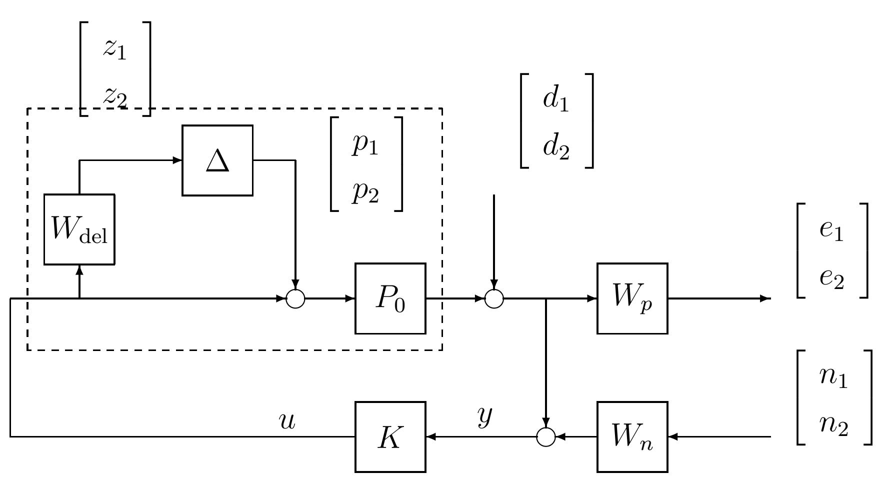



Figure 0.19: HIMAT closed-loop interconnection

The open-loop interconnection is

$$\begin{bmatrix} z_1 \\ z_2 \\ e_1 \\ e_2 \\ y_1 \\ y_2 \end{bmatrix} = \hat{G}(s) \begin{bmatrix} p_1 \\ p_2 \\ d_1 \\ d_2 \\ n_1 \\ n_2 \\ u_1 \\ u_2 \end{bmatrix}.$$

The SIMULINK block diagram:

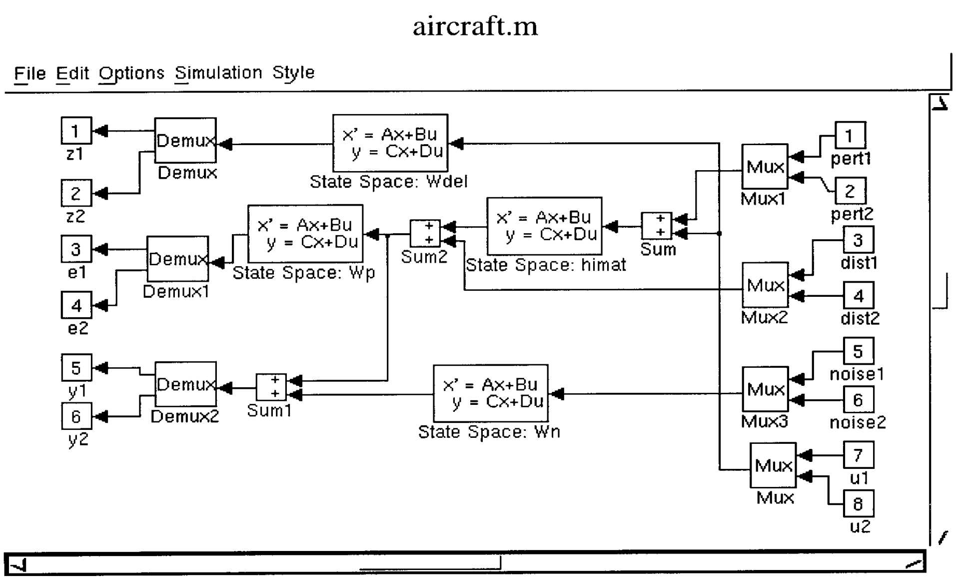



Figure 0.20: SIMULINK block diagram for HIMAT (aircraft.m)

The $\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ can be computed by

$$\gg\ [\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}] = \mathbf{linmod}('\mathbf{aircraft}')$$

which gives

$$A = \begin{bmatrix} -10000I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0226 & -36.6 & -18.9 & -32.1 & 0 & 0 & 0 \\ 0 & 0 & -1.9 & 0.983 & 0 & 0 & 0 & 0 \\ 0 & 0.0123 & -11.7 & -2.63 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -54.087 & 0 & 0 & -0.018 & 0 & 0 \\ 0 & 0 & 0 & 0 & -54.087 & 0 & -0.018 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -320I_2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & -703.5624 & 0 \\ 0 & 0 & 0 & 0 & 0 & -703.5624 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.4140 & 0 & 0 & 0 & -0.4140 & 0 \\ -77.8 & 22.4 & 0 & 0 & -77.8 & 22.4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.9439I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -25.2476I_2 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 703.5624I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 28.65 & 0 & 0 & -0.9439 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 28.65 & 0 & -0.9439 & 0 & 0 \\ 0 & 0 & 57.3 & 0 & 0 & 0 & 0 & 25.2476 & 0 \\ 0 & 0 & 0 & 0 & 57.3 & 0 & 0 & 0 & 25.2476 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 50 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \end{bmatrix}.$$

# Redheffer Star Products

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$$P \star K := \begin{bmatrix} \mathcal{F}_l(P, K_{11}) & P_{12}(I - K_{11}P_{22})^{-1} K_{12} \\ K_{21}(I - P_{22}K_{11})^{-1} P_{21} & \mathcal{F}_u(K, P_{22}) \end{bmatrix}.$$

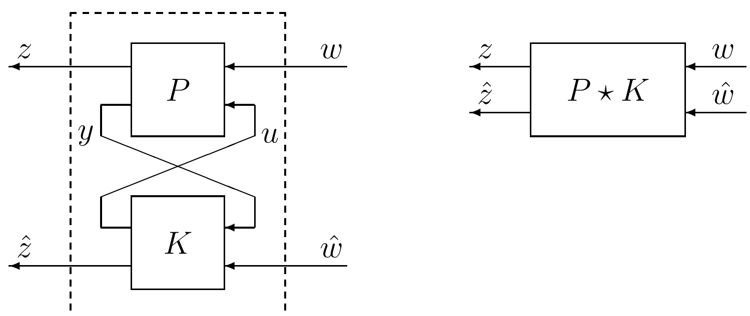



Figure 0.21: Interconnection of LFTs

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \qquad K = \left[\begin{array}{c|cc} A_K & B_{K1} & B_{K2} \\ \hline C_{K1} & D_{K11} & D_{K12} \\ C_{K2} & D_{K21} & D_{K22} \end{array}\right].$$

Then the transfer matrix

$$P \star K : \begin{bmatrix} w \\ \hat{w} \end{bmatrix} \mapsto \begin{bmatrix} z \\ \hat{z} \end{bmatrix}$$

has a representation

$$P \star K = \left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & \bar{D}_{11} & \bar{D}_{12} \\ \bar{C}_2 & \bar{D}_{21} & \bar{D}_{22} \end{array}\right] = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$$

where

$$\begin{aligned}
\bar{A} &= \begin{bmatrix} A + B_2\tilde{R}^{-1}D_{K11}C_2 & B_2\tilde{R}^{-1}C_{K1} \\ B_{K1}R^{-1}C_2 & A_K + B_{K1}R^{-1}D_{22}C_{K1} \end{bmatrix} \\
\bar{B} &= \begin{bmatrix} B_1 + B_2\tilde{R}^{-1}D_{K11}D_{21} & B_2\tilde{R}^{-1}D_{K12} \\ B_{K1}R^{-1}D_{21} & B_{K2} + B_{K1}R^{-1}D_{22}D_{K12} \end{bmatrix} \\
\bar{C} &= \begin{bmatrix} C_1 + D_{12}D_{K11}R^{-1}C_2 & D_{12}\tilde{R}^{-1}C_{K1} \\ D_{K21}R^{-1}C_2 & C_{K2} + D_{K21}R^{-1}D_{22}C_{K1} \end{bmatrix} \\
\bar{D} &= \begin{bmatrix} D_{11} + D_{12}D_{K11}R^{-1}D_{21} & D_{12}\tilde{R}^{-1}D_{K12} \\ D_{K21}R^{-1}D_{21} & D_{K22} + D_{K21}R^{-1}D_{22}D_{K12} \end{bmatrix}
\end{aligned}$$

$$R = I - D_{22}D_{K11}, \quad \tilde{R} = I - D_{K11}D_{22}.$$

$$\begin{aligned}
\bar{A} &= \begin{bmatrix} A & B_2 \\ C_2 & D_{22} \end{bmatrix} \star \begin{bmatrix} D_{K11} & C_{K1} \\ B_{K1} & A_K \end{bmatrix}, \\
\bar{B} &= \begin{bmatrix} B_1 & B_2 \\ D_{21} & D_{22} \end{bmatrix} \star \begin{bmatrix} D_{K11} & D_{K12} \\ B_{K1} & B_{K2} \end{bmatrix}, \\
\bar{C} &= \begin{bmatrix} C_1 & D_{12} \\ C_2 & D_{22} \end{bmatrix} \star \begin{bmatrix} D_{K11} & C_{K1} \\ D_{K21} & C_{K2} \end{bmatrix}, \\
\bar{D} &= \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \star \begin{bmatrix} D_{K11} & D_{K12} \\ D_{K21} & D_{K22} \end{bmatrix}.
\end{aligned}$$

$$\gg \ \mathbf{P} \star \mathbf{K} = \mathbf{starp}(\mathbf{P}, \mathbf{K}, \mathbf{dimy}, \mathbf{dimu})$$

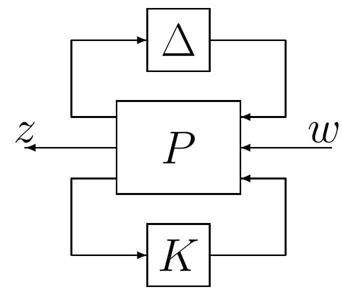
$$\gg \ \mathcal{F}_\ell(\mathbf{P}, \mathbf{K}) = \mathbf{starp}(\mathbf{P}, \mathbf{K})$$

# Chapter 10: $\mu$ and $\mu$ Synthesis

- general framework
- analysis and synthesis methods for unstructured uncertainty
- stability with structured uncertainties
- structured singular value
- structured robust stability
- robust performance
- extension to nonlinear time varying uncertainties
- skewed problem
- overview on $\mu$ synthesis

# General Framework

General Framework:



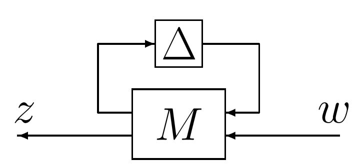

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) & P_{13}(s) \\ P_{21}(s) & P_{22}(s) & P_{23}(s) \\ P_{31}(s) & P_{32}(s) & P_{33}(s) \end{bmatrix}$$

$$\begin{aligned} z &= \mathcal{F}_u\left(\mathcal{F}_\ell(P, K), \Delta\right) w \\ &= \mathcal{F}_\ell\left(\mathcal{F}_u(P, \Delta), K\right) w. \end{aligned}$$

# Analysis Framework

$$M(s) = \mathcal{F}_\ell\left(P(s), K(s)\right) = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix},$$

$$z = \mathcal{F}_u(M, \Delta) w = \left[ M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1} M_{12} \right] w.$$



# Analysis and Synthesis Methods for Unstructured Uncertainty

<table>
<tr><th>Input Assumptions</th><th>Performance Specifications</th><th>Perturbation Assumptions</th><th>Analysis Tests</th><th>Synthesis Methods</th></tr>
<tr><td>$E(w(t)w(\tau)^*)$<br>$= \delta(t-\tau)I$</td><td>$E(z(t)^*z(t)) \leq 1$</td><td rowspan="2">$\Delta = 0$</td><td rowspan="2">$\|M_{22}\|_2 \leq 1$</td><td rowspan="2">LQG<br>Wiener-Hopf<br>$\mathcal{H}_2$</td></tr>
<tr><td>$w = U_0\delta(t)$<br>$E(U_0U_0^*) = I$</td><td>$E(\|z\|_2^2) \leq 1$</td></tr>
<tr><td>$\|w\|_2 \leq 1$</td><td>$\|z\|_2 \leq 1$</td><td>$\Delta = 0$</td><td>$\|M_{22}\|_\infty \leq 1$</td><td rowspan="2">Singular Value<br>Loop Shaping<br>$\mathcal{H}_\infty$</td></tr>
<tr><td>$\|w\|_2 \leq 1$</td><td>Internal Stability</td><td>$\|\Delta\|_\infty < 1$</td><td>$\|M_{11}\|_\infty \leq 1$</td></tr>
</table>

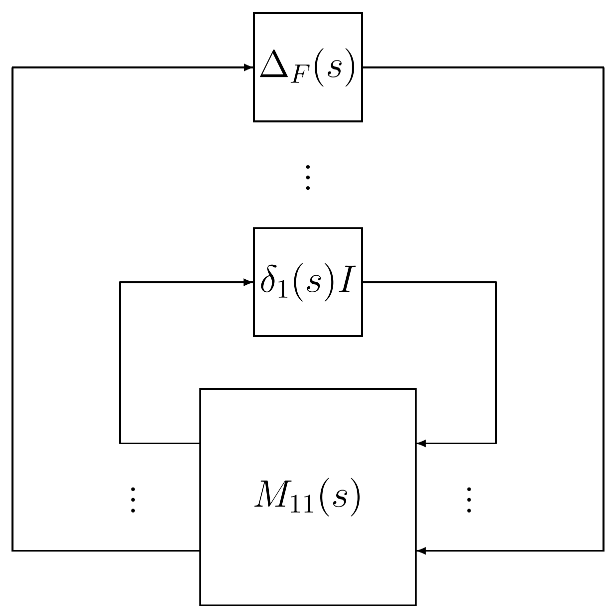
# Stability with Structured Uncertainties

Assume

$$\Delta(s) = \text{diag}\ [\delta_1 I_{r_1}, \ldots, \delta_s I_{r_S}, \Delta_1, \ldots, \Delta_F] \in \mathcal{RH}_\infty$$

with $\|\delta_i\|_\infty < 1$ and $\|\Delta_j\|_\infty < 1$.

Robust Stability $\iff$ The following interconnection is stable.




Stability Conditions:

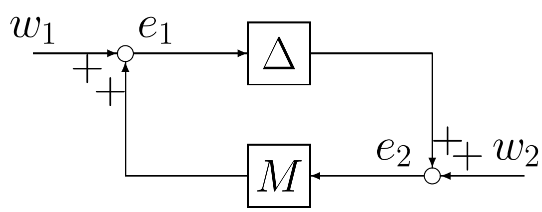
(1) (sufficient conditions) $\|M_{11}\|_\infty \leq 1$.

Conservative, ignoring structure of the uncertainties.

(2) (necessary conditions) Test for each $\delta_i$ ($\Delta_j$) individually (assuming no uncertainty in other channels): $\|(M_{11})_{ii}\|_\infty \leq 1$.

Optimistic because it ignores interaction between the $\delta_i$ ($\Delta_j$).

## Structured Singular Value




## Unstructured $\Delta$

Problem: Given $M \in \mathbb{C}^{p\times q}$, find a smallest $\Delta \in \mathbb{C}^{q\times p}$ in the sense of $\overline{\sigma}(\Delta)$ such that

$$\det(I - M\Delta) = 0.$$

It is easy to see that

$$\alpha_{\min} := \inf\left\{\overline{\sigma}(\Delta):\ \det(I - M\Delta) = 0,\ \Delta \in \mathbb{C}^{q\times p}\right\}$$

$$= \inf\left\{\alpha:\ \det(I - \alpha M\Delta) = 0,\ \overline{\sigma}(\Delta) \le 1,\ \Delta \in \mathbb{C}^{q\times p}\right\}$$

and

$$\max_{\overline{\sigma}(\Delta)\le 1} \rho(M\Delta) = \alpha_{\min}^{-1} = \overline{\sigma}(M)$$

with a smallest "destabilizing" $\Delta$:

$$\Delta_{\text{des}} = \frac{1}{\overline{\sigma}(M)} v_1 u_1^*, \quad \det(I - M\Delta_{\text{des}}) = 0$$

where $M = \overline{\sigma}(M) u_1 v_1^* + \sigma_2 u_2 v_2^* + \cdots$

So $\overline{\sigma}(M)$ can be defined as

$$\overline{\sigma}(M) := \frac{1}{\inf\{\overline{\sigma}(\Delta):\ \det(I - M\Delta) = 0,\ \Delta \in \mathbb{C}^{q\times p}\}}$$

## Structured $\Delta$

$$\mathbf{\Delta} = \left\{ \text{diag} \left[ \delta_1 I_{r_1}, \ldots, \delta_s I_{r_S}, \Delta_1, \ldots, \Delta_F \right] : \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j \times m_j} \right\}.$$

$$\alpha_{\min} := \inf \left\{ \overline{\sigma}(\Delta) : \ \det(I - M\Delta) = 0, \ \Delta \in \mathbf{\Delta} \right\}$$
$$= \inf \left\{ \alpha : \ \det(I - \alpha M\Delta) = 0, \ \overline{\sigma}(\Delta) \le 1, \ \Delta \in \mathbf{\Delta} \right\}$$

and

$$\max_{\overline{\sigma}(\Delta) \le 1} \rho(M\Delta) = \alpha_{\min}^{-1} \le \overline{\sigma}(M)$$

## Definition of SSV

*For $M \in \mathbb{C}^{n \times n}$, $\mu_{\mathbf{\Delta}}(M)$ is defined as*

$$\mu_{\mathbf{\Delta}}(M) := \frac{1}{\min \left\{ \overline{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \det(I - M\Delta) = 0 \right\}} \tag{0.6}$$

*unless no $\Delta \in \mathbf{\Delta}$ makes $I - M\Delta$ singular, in which case $\mu_{\mathbf{\Delta}}(M) := 0$.*

- If $\mathbf{\Delta} = \{\delta I : \delta \in \mathbb{C}\}$ $(S=1, F=0, r_1=n)$, then $\mu_{\mathbf{\Delta}}(M) = \rho(M)$, the spectral radius of $M$.
- If $\mathbf{\Delta} = \mathbb{C}^{n \times n}$ $(S=0, F=1, m_1=n)$, then $\mu_{\mathbf{\Delta}}(M) = \overline{\sigma}(M)$.

$$\rho(M) \le \mu_{\mathbf{\Delta}}(M) \le \overline{\sigma}(M).$$

(1) $M = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$ for any $\beta > 0$. Then $\rho(M) = 0$ and $\overline{\sigma}(M) = \beta$. But $\mu(M) = 0$ since $\det(I - M\Delta) = 1$ for all admissible $\Delta$.

(2) $M = \begin{bmatrix} -1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$. Then $\rho(M) = 0$ and $\overline{\sigma}(M) = 1$. Since

$$\det(I - M\Delta) = 1 + \frac{\delta_1 - \delta_2}{2} = 0$$

if $\delta_1 = -\delta_2 = -1$. so $\mu(M) = 1$.

Thus neither $\rho$ nor $\overline{\sigma}$ provide useful bounds even in these simple cases.

$$\mathcal{U} = \{U \in \mathbf{\Delta} : \; UU^* = I_n\}$$

$$\mathcal{D} = \left\{ \begin{array}{c} \text{diag}\left[D_1, \ldots, D_S, d_1 I_{m_1}, \ldots, d_{F-1} I_{m_{F-1}}, I_{m_F}\right] : \\ D_i \in \mathbb{C}^{r_i \times r_i}, D_i = D_i^* > 0, d_j \in \mathbb{R}, d_j > 0 \end{array} \right\}.$$

Note that for any $\Delta \in \mathbf{\Delta}, U \in \mathcal{U}$, and $D \in \mathcal{D}$,

$$U^* \in \mathcal{U} \quad U\Delta \in \mathbf{\Delta} \quad \Delta U \in \mathbf{\Delta} \quad \overline{\sigma}(U\Delta) = \overline{\sigma}(\Delta U) = \overline{\sigma}(\Delta)$$

$$D\Delta = \Delta D.$$

---

*For all $U \in \mathcal{U}$ and $D \in \mathcal{D}$*

$$\mu_{\mathbf{\Delta}}(MU) = \mu_{\mathbf{\Delta}}(UM) = \mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(DMD^{-1}).$$

---

$$\max_{U \in \mathcal{U}} \rho(UM) \le \max_{\Delta \in \mathbf{B\Delta}} \rho(\Delta M) = \mu_{\mathbf{\Delta}}(M) \le \inf_{D \in \mathcal{D}} \overline{\sigma}\left(DMD^{-1}\right)$$

$$\max_{U \in \mathcal{U}} \rho(UM) \le \mu_{\mathbf{\Delta}}(M) \le \inf_{D \in \mathcal{D}} \overline{\sigma}\left(DMD^{-1}\right).$$

---

*[Doyle, 1982]* $\max_{U \in \mathcal{U}} \rho(MU) = \mu_{\mathbf{\Delta}}(M)$. *Not Convex.*

---

$\mu_{\mathbf{\Delta}}(M) = \inf_{D \in \mathcal{D}} \overline{\sigma}(DMD^{-1})$ *if* $2S + F \le 3$

| F= <br> S= | *0* | *1* | *2* | *3* | *4* |
|---|---|---|---|---|---|
| *0* | | *yes* | *yes* | *yes* | *no* |
| *1* | *yes* | *yes* | *no* | *no* | *no* |
| *2* | *no* | *no* | *no* | *no* | *no* |

$\gg$ **[bounds,rowd] = mu(M,blk)**

$$\Delta = \begin{bmatrix} \delta_1 I_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_5 I_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_6 \end{bmatrix}$$

$$\delta_1, \delta_2, \delta_5, \in \mathbb{C},\ \ \Delta_3 \in \mathbb{C}^{2\times 3}, \Delta_4 \in \mathbb{C}^{3\times 3}, \Delta_6 \in \mathbb{C}^{2\times 1}$$

*can be specified by*

$$\mathbf{blk} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 2 & 3 \\ 3 & 3 \\ 3 & 0 \\ 2 & 1 \end{bmatrix}.$$

$\gg [\mathbf{D}_\ell, \mathbf{D_r}] = \mathbf{unwrapd}(\mathbf{rowd}, \mathbf{blk})$

## Structured Robust Stability

*How large* $\Delta$ *(in the sense of* $\|\Delta\|_\infty$*) can be without destabilizing the feedback system?*

*Since the closed-loop poles are given by* $\det(I - M\Delta) = 0$*, the feedback system becomes unstable if* $\det(I - M(s)\Delta(s)) = 0$ *for some* $s \in \overline{\mathbb{C}}_+$*. Now let* $\alpha > 0$ *be a sufficiently small number such that the closed-loop system is stable for all stable* $\|\Delta\|_\infty < \alpha$*. Next increase* $\alpha$ *until* $\alpha_{max}$ *so that the closed-loop system becomes unstable. So* $\alpha_{max}$ *is the robust stability margin.*

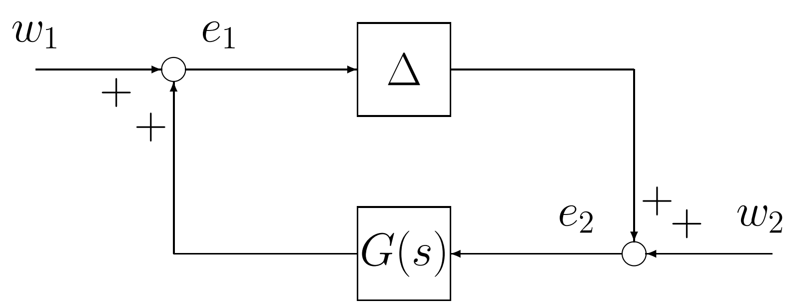
*Define*

$$\mathbf{\Delta} := \{\Delta(\cdot) \in \mathcal{RH}_\infty : \Delta(s_o) \in \mathbf{\Delta} \text{ for all } s_o \in \overline{\mathbb{C}}_+\}$$

---

*Let* $\beta > 0$*. The system is well-posed and internally stable for all* $\Delta(\cdot) \in \mathbf{\Delta}$ *with* $\|\Delta\|_\infty < \frac{1}{\beta}$ *if and only if*

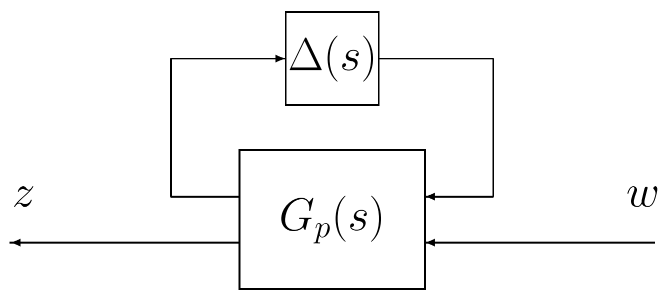
$$\sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta}}(G(j\omega)) \leq \beta$$



# Robust Performance

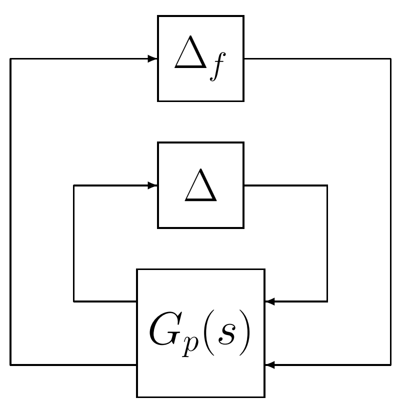
$$G_p(s) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

$$\mathbf{\Delta}_P := \left\{ \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_f \end{bmatrix} : \Delta \in \mathbf{\Delta}, \Delta_f \in \mathbb{C}^{q_2 \times p_2} \right\}.$$




*Let $\beta > 0$. For all $\Delta(s) \in \mathbf{\Delta}$ with $\|\Delta\|_\infty < \frac{1}{\beta}$, the system is well-posed, internally stable, and $\|F_u(G_p, \Delta)\|_\infty \leq \beta$ if and only if*

$$\sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta}_P}(G_p(j\omega)) \leq \beta.$$

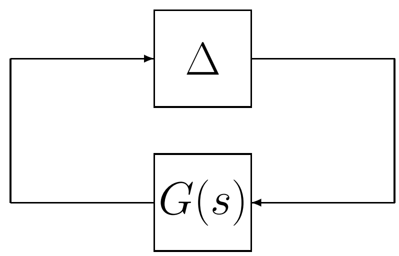


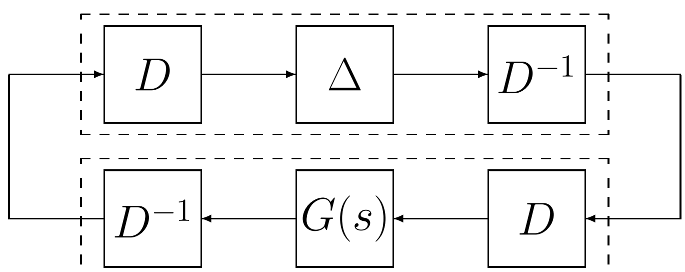
# Extension to Nonlinear Time Varying Uncertainty

*Suppose* $\Delta \in \mathbf{\Delta}_N$ *is a structured Nonlinear (Time-varying) Uncertainty and suppose* $D$ *is constant scaling matrix such that* $D\Delta D^{-1} \in \mathbf{\Delta}_N$.

*Then a sufficient condition for stability is (by small gain theorem)*

$$\left\| D^{-1}G(s)D \right\|_\infty \leq 1$$



$\gg [\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}] = \mathbf{linmod}('\mathbf{aircraft}')$

$\gg \hat{\mathbf{G}} = \mathbf{pck}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D});$

$\gg [\mathbf{K}, \mathbf{G_p}, \gamma] = \mathbf{hinfsyn}(\hat{\mathbf{G}}, \mathbf{2}, \mathbf{2}, \mathbf{0}, \mathbf{10}, \mathbf{0.001}, \mathbf{2});$

*which gives* $\gamma = 1.8612 = \|G_p\|_\infty$, *a stabilizing controller* $K$, *and a closed loop transfer matrix* $G_p$*:*

$$\begin{bmatrix} z_1 \\ z_2 \\ \hdashline e_1 \\ e_2 \end{bmatrix} = G_p(s) \begin{bmatrix} p_1 \\ p_2 \\ \hdashline d_1 \\ d_2 \\ n_1 \\ n_2 \end{bmatrix}, \quad G_p(s) = \begin{bmatrix} G_{p11} & G_{p12} \\ G_{p21} & G_{p22} \end{bmatrix}.$$

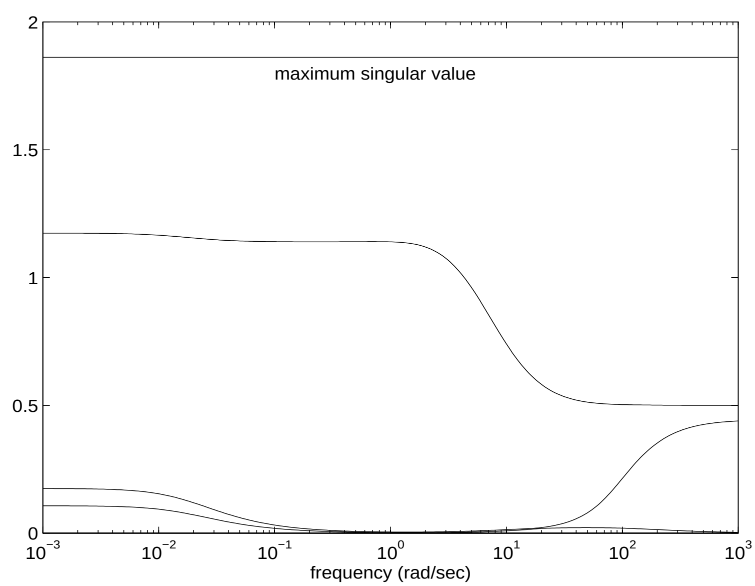



Figure 0.22: Singular values of $G_p(j\omega)$

*Now generate the singular value frequency responses of* $G_p$*:*

$\gg$ **w=logspace(-3,3,300);**

$\gg \mathbf{Gpf} = \mathbf{frsp}(\mathbf{G_p}, \mathbf{w})$; *% Gpf is the frequency response of $G_p$;*

$\gg [\mathbf{u}, \mathbf{s}, \mathbf{v}] = \mathbf{vsvd}(\mathbf{Gpf})$;

$\gg \mathbf{vplot}('\mathbf{liv}, \mathbf{m}', \mathbf{s})$

*The singular value frequency responses of $G_p$ are shown in Figure 0.22. To test the robust stability, we need to compute $\|G_{p11}\|_\infty$:*

$\gg \mathbf{G_{p11}} = \mathbf{sel}(\mathbf{G_p}, \mathbf{1}:\mathbf{2}, \mathbf{1}:\mathbf{2})$;

$\gg \mathbf{norm\_of\_G_{p11}} = \mathbf{hinfnorm}(\mathbf{G_{p11}}, \mathbf{0.001})$;

*which gives $\|G_{P11}\|_\infty = 0.933 < 1$. So the system is robustly stable. To check the robust performance, we shall compute the $\mu_{\Delta_P}(G_p(j\omega))$ for each frequency with*

$$\Delta_P = \begin{bmatrix} \Delta & \\ & \Delta_f \end{bmatrix}, \ \Delta \in \mathbb{C}^{2\times 2}, \quad \Delta_f \in \mathbb{C}^{4\times 2}.$$

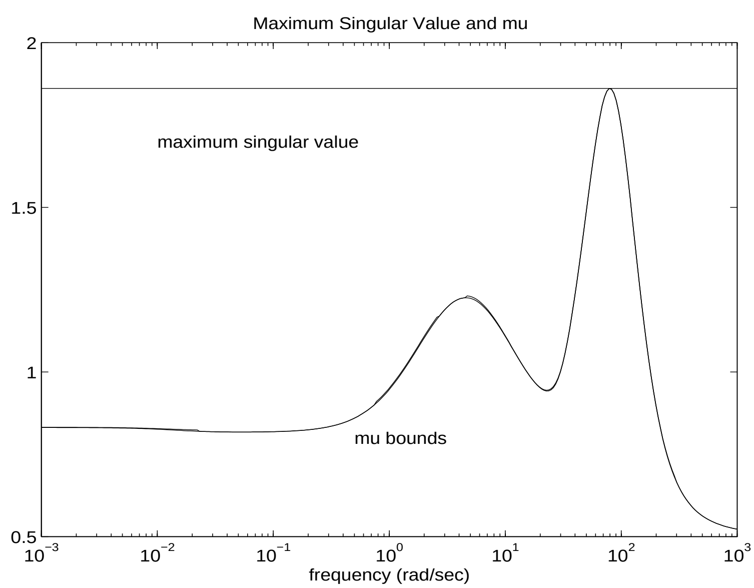



Figure 0.23: $\mu_{\Delta_P}(G_p(j\omega))$ and $\overline{\sigma}(G_p(j\omega))$

$\gg$ **blk=[2,2;4,2];**

$\gg$ **[bnds,dvec,sens,pvec]=mu(Gpf,blk);**

$\gg$ $\mathbf{vplot('liv, m', vnorm(Gpf), bnds)}$

$\gg$ $\mathbf{title('Maximum\ Singular\ Value\ and\ mu')}$

$\gg$ $\mathbf{xlabel('frequency(rad/sec)')}$

$\gg$ $\mathbf{text(0.01, 1.7,' maximum\ singular\ value')}$

$\gg$ $\mathbf{text(0.5, 0.8,' mu\ bounds')}$

*The structured singular value* $\mu_{\Delta_P}(G_p(j\omega))$ *and* $\overline{\sigma}(G_p(j\omega))$ *are shown in Figure 0.23. It is clear that the robust performance is not satisfied. Note that*

$$\max_{\|\Delta\|_\infty \le 1} \|\mathcal{F}_u(G_p, \Delta)\|_\infty \le \gamma \Longleftrightarrow \sup_\omega \mu_{\Delta_P}\left(\begin{bmatrix} G_{p11} & G_{p12} \\ G_{p21}/\gamma & G_{p22}/\gamma \end{bmatrix}\right) \le 1.$$

*Using a bisection algorithm, we can also find the worst performance:*

$$\max_{\|\Delta\|_\infty \le 1} \|\mathcal{F}_u(G_p, \Delta)\|_\infty = 12.7824.$$

## Skewed Problem

$$G = \begin{bmatrix} -W_2 T_i W_1 & -W_2 K S_o W_d \\ W_e S_o P W_1 & W_e S_o W_d \end{bmatrix}.$$

robust performance condition:

$$\mu_{\mathbf{\Delta}}(G(j\omega)) = \inf_{d_\omega \in \mathbb{R}_+} \overline{\sigma} \left( \begin{bmatrix} -W_2 T_i W_1 & -d_\omega W_2 K S_o W_d \\ \frac{1}{d_\omega} W_e S_o P W_1 & W_e S_o W_d \end{bmatrix} \right) \leq 1$$

for all $\omega \geq 0$. An upper bound:

$$\mu_{\mathbf{\Delta}}(G(j\omega)) \leq \sqrt{\kappa(W_d^{-1} P W_1)}(\|W_2 T_i W_1\| + \|W_e S_o W_d\|).$$

$\mu$ is proportional to the square root of the plant condition number.

Assumptions:

$$W_e = w_s I,\ W_d = I,\ W_1 = I,\ W_2 = w_t I,$$

and $P$ is stable and has a stable inverse (i.e., minimum phase) and

$$K(s) = P^{-1}(s) l(s)$$

such that $K(s)$ is proper and the closed-loop is stable. Then

$$S_o = S_i = \frac{1}{1 + l(s)} I = \varepsilon(s) I, \quad T_o = T_i = \frac{l(s)}{1 + l(s)} I = \tau(s) I$$

$$G = \begin{bmatrix} -w_t \tau I & -w_t \tau P^{-1} \\ w_s \varepsilon P & w_s \varepsilon I \end{bmatrix}.$$

Then

$$\mu_{\mathbf{\Delta}}(G(j\omega)) = \inf_{d \in \mathbb{R}_+} \overline{\sigma} \left( \begin{bmatrix} -w_t \tau I & -w_t \tau (dP)^{-1} \\ w_s \varepsilon dP & w_s \varepsilon I \end{bmatrix} \right).$$

Let the SVD of $P(j\omega)$ be

$$P(j\omega) = U\Sigma V^*, \quad \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m)$$

with $\sigma_1 = \overline{\sigma}$ and $\sigma_m = \underline{\sigma}$ where $m$ is the dimension of $P$. Then

$$\mu_{\mathbf{\Delta}}(G(j\omega)) = \inf_{d\in\mathbb{R}_+} \overline{\sigma}\left(\begin{bmatrix} -w_t\tau I & -w_t\tau(d\Sigma)^{-1} \\ w_s\varepsilon d\Sigma & w_s\varepsilon I \end{bmatrix}\right)$$

$$\begin{bmatrix} -w_t\tau I & -w_t\tau(d\Sigma)^{-1} \\ w_s\varepsilon d\Sigma & w_s\varepsilon I \end{bmatrix} = P_1 \mathrm{diag}(M_1, M_2, \ldots, M_m) P_2$$

where $P_1$ and $P_2$ are permutation matrices and where

$$M_i = \begin{bmatrix} -w_t\tau & -w_t\tau(d\sigma_i)^{-1} \\ w_s\varepsilon d\sigma_i & w_s\varepsilon \end{bmatrix}.$$

Hence

$$\begin{aligned}
\mu_{\mathbf{\Delta}}(G(j\omega)) &= \inf_{d\in\mathbb{R}_+} \max_i \overline{\sigma}\left(\begin{bmatrix} -w_t\tau & -w_t\tau(d\sigma_i)^{-1} \\ w_s\varepsilon d\sigma_i & w_s\varepsilon \end{bmatrix}\right) \\
&= \inf_{d\in\mathbb{R}_+} \max_i \overline{\sigma}\left(\begin{bmatrix} -w_t\tau \\ w_s\varepsilon d\sigma_i \end{bmatrix} \begin{bmatrix} 1 & (d\sigma_i)^{-1} \end{bmatrix}\right) \\
&= \inf_{d\in\mathbb{R}_+} \max_i \sqrt{(1+|d\sigma_i|^{-2})(|w_s\varepsilon d\sigma_i|^2 + |w_t\tau|^2)} \\
&= \inf_{d\in\mathbb{R}_+} \max_i \sqrt{|w_s\varepsilon|^2 + |w_t\tau|^2 + |w_s\varepsilon d\sigma_i|^2 + \left|\frac{w_t\tau}{d\sigma_i}\right|^2}.
\end{aligned}$$

The maximum is achieved at

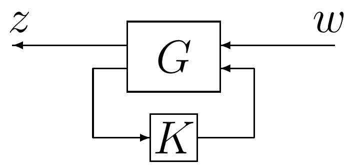
$$d^2 = \frac{|w_t\tau|}{|w_s\varepsilon|\underline{\sigma}\overline{\sigma}},$$

and

$$\mu_{\mathbf{\Delta}}(G(j\omega)) = \sqrt{|w_s\varepsilon|^2 + |w_t\tau|^2 + |w_s\varepsilon||w_t\tau|[\kappa(P) + \frac{1}{\kappa(P)}]}.$$

$$\mu_{\mathbf{\Delta}}(G(j\omega)) \approx \sqrt{|w_s\varepsilon||w_t\tau|\kappa(P)}\ .$$

## Overview on $\mu$ Synthesis




$$\mathcal{F}_\ell(G,K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}.$$

$$\min_K \|\mathcal{F}_\ell(G,K)\|_\mu$$

The $\mu$-synthesis is not yet fully solved. But a reasonable approach is to "solve"

$$\min_K \inf_{D,D^{-1}\in\mathcal{H}_\infty} \left\| D\mathcal{F}_\ell(G,K)D^{-1} \right\|_\infty$$

by iteratively solving for $K$ and $D$, i.e., first minimizing over $K$ with $D$ fixed, then minimizing pointwise over $D$ with $K$ fixed, then again over $K$, and again over $D$, etc. This is the so-called *D-K Iteration*.

- Fix $D$

  $$\min_K \left\| D\mathcal{F}_\ell(G,K)D^{-1} \right\|_\infty$$

  is a standard $\mathcal{H}_\infty$ optimization problem.

- Fix $K$

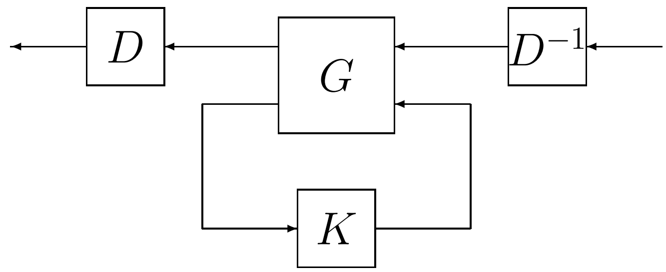
  $$\inf_{D,D^{-1}\in\mathcal{H}_\infty} \left\| D\mathcal{F}_\ell(G,K)D^{-1} \right\|_\infty$$

  is a standard convex optimization problem and it can be solved pointwise in the frequency domain:

  $$\sup_\omega \inf_{D_\omega\in\mathcal{D}} \overline{\sigma}\left[ D_\omega\mathcal{F}_\ell(G,K)(j\omega)D_\omega^{-1} \right].$$

  Note that when $S = 0$, (no scalar blocks)

  $$D_\omega = \operatorname{diag}(d_1^\omega I, \ldots, d_{F-1}^\omega I, I) \in \mathcal{D},$$




D-K Iterations:

(i) Fix an initial estimate of the scaling matrix $D_\omega \in \mathcal{D}$ pointwise across frequency.

(ii) Find scalar transfer functions $d_i(s), d_i^{-1}(s) \in \mathcal{RH}_\infty$ for $i = 1, \ldots, (F-1)$ such that $|d_i(j\omega)| \approx d_i^\omega$.

(iii) Let

$$D(s) = \operatorname{diag}(d_1(s)I, \ldots, d_{F-1}(s)I, I).$$

Construct a state space model for system

$$\hat{G}(s) = \begin{bmatrix} D(s) & \\ & I \end{bmatrix} G(s) \begin{bmatrix} D^{-1}(s) & \\ & I \end{bmatrix}.$$

(iv) Solve an $\mathcal{H}_\infty$-optimization problem to minimize

$$\left\| \mathcal{F}_\ell(\hat{G}, K) \right\|_\infty$$

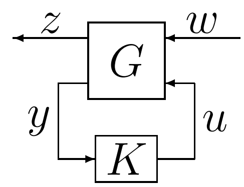
over all stabilizing $K$'s. Denote the minimizing controller by $\hat{K}$.

(v) Minimize $\overline{\sigma}[D_\omega \mathcal{F}_\ell(G, \hat{K}) D_\omega^{-1}]$ over $D_\omega$, pointwise across frequency. The minimization itself produces a new scaling function.

(vi) Compare $\hat{D}_\omega$ with the previous estimate $D_\omega$. Stop if they are close, otherwise, replace $D_\omega$ with $\hat{D}_\omega$ and return to step (ii).

The joint optimization of $D$ and $K$ is not convex and the global convergence is not guaranteed, many designs have shown that this approach works very well.

# Chapter 11: Controller Parameterization




$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}.$$

Suppose $(A, B_2)$ is stabilizable and $(C_2, A)$ is detectable.

**Youla parameterization:**

*all controllers $K$ that internally stabilize $G$.*

- Suppose $G \in \mathcal{RH}_\infty$. Then

  $$K = Q(I + G_{22}Q)^{-1}, \quad Q \in \mathcal{RH}_\infty$$

  and $I + D_{22}Q(\infty)$ nonsingular.

  $K$ stabilizes a stable plant $G_{22}$ iff $K(I - G_{22}K)^{-1}$ is stable. So let $Q = K(I - G_{22}K)^{-1}$.

- General Case: Let $F$ and $L$ be such that $A + LC_2$ and $A + B_2F$ are stable. Then $K = \mathcal{F}_\ell(J, Q)$:

  $$J = \left[\begin{array}{c|cc} A + B_2F + LC_2 + LD_{22}F & -L & B_2 + LD_{22} \\ \hline F & 0 & I \\ -(C_2 + D_{22}F) & I & -D_{22} \end{array}\right]$$

  with any $Q \in \mathcal{RH}_\infty$ and $I + D_{22}Q(\infty)$ nonsingular.

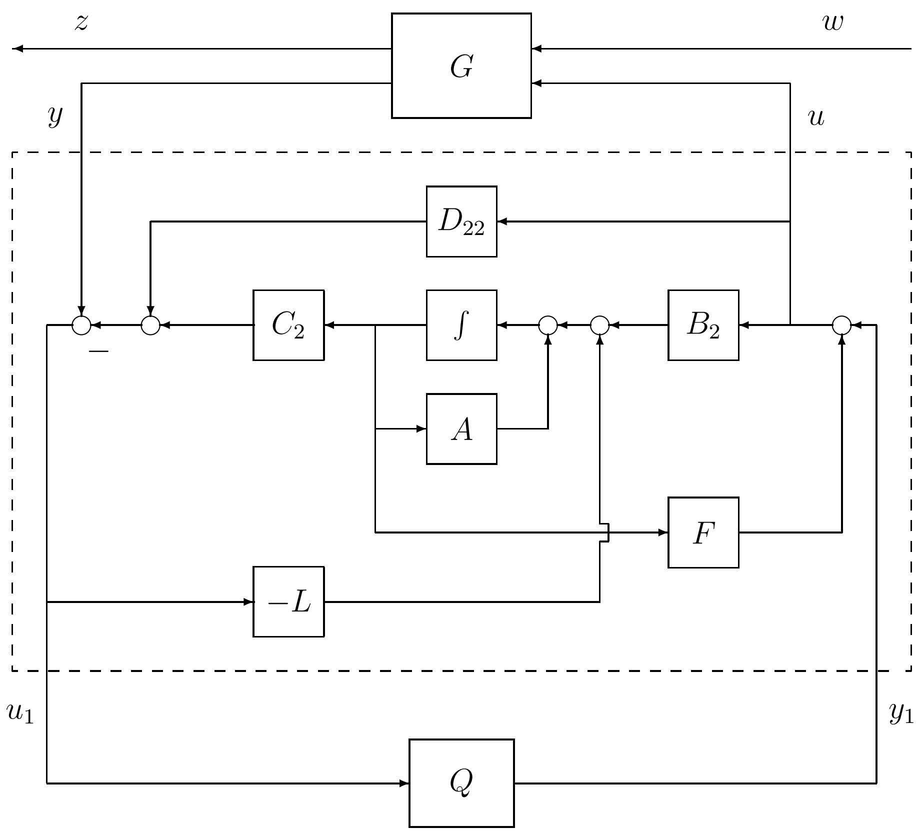



Figure 0.24: Structure of Stabilizing Controllers

- Closed-loop Matrix:

$$\mathcal{F}_\ell(G, K) = \mathcal{F}_\ell(G, \mathcal{F}_\ell(J, Q)) = \mathcal{F}_\ell(T, Q).$$

$$= \{T_{11} + T_{12}QT_{21} : \ Q \in \mathcal{RH}_\infty, \ I + D_{22}Q(\infty) \text{ invertible}\}$$

where $T$ is given by

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} A + B_2F & -B_2F & B_1 & B_2 \\ 0 & A + LC_2 & B_1 + LD_{21} & 0 \\ \hline C_1 + D_{12}F & -D_{12}F & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right].$$

- Coprime factorization approach: Let $G_{22} = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ be rcf and lcf of $G_{22}$ over $\mathcal{RH}_\infty$, respectively. And let $U_0, V_0, \tilde{U}_0, \tilde{V}_0 \in \mathcal{RH}_\infty$ satisfy the Bezout identity:

$$\begin{bmatrix} \tilde{V}_0 & -\tilde{U}_0 \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U_0 \\ N & V_0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Then

$$\begin{aligned} K &= (U_0 + MQ_y)(V_0 + NQ_y)^{-1} \\ &= (\tilde{V}_0 + Q_y\tilde{N})^{-1}(\tilde{U}_0 + Q_y\tilde{M}) \\ &= \mathcal{F}_\ell(J_y, Q_y), \quad Q_y \in \mathcal{RH}_\infty \end{aligned}$$

where

$$J_y := \begin{bmatrix} U_0V_0^{-1} & \tilde{V}_0^{-1} \\ V_0^{-1} & -V_0^{-1}N \end{bmatrix}$$

and $(I + V_0^{-1}NQ_y)(\infty)$ is invertible

# Chapter 12: Algebraic Riccati Equations

$$A^*X + XA + XRX + Q = 0, \quad R = R^*, \;\; Q = Q^*$$

The associated Hamiltonian matrix:

$$H := \begin{bmatrix} A & R \\ -Q & -A^* \end{bmatrix}.$$

Then

$$J^{-1}HJ = -JHJ = -H^*, \quad J := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

so $H$ and $-H^*$ are similar. Thus $\lambda$ is an eigenvalue iff $-\bar{\lambda}$ is.

---

$eig(H) \neq j\omega \Leftrightarrow H$ has $n$ eigenvalues in Re $s < 0$ and $n$ in Re $s > 0$.

Let $\mathcal{X}_-(H)$ be the $n$-dimensional spectral subspace corresponding to eigenvalues in Re $s < 0$.

$$\mathcal{X}_-(H) = \operatorname{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where $X_1, X_2 \in \mathbb{C}^{n\times n}$. ($X_1$ and $X_2$ can be chosen to be real matrices.)

If $X_1$ is nonsingular, define

$$X := Ric(H) = X_2 X_1^{-1} : \; dom(Ric) \subset \mathbb{R}^{2n\times 2n} \longmapsto \mathbb{R}^{n\times n}.$$

where $dom(Ric)$ consists of all $H$ matrices such that

- $H$ has no eigenvalues on the imaginary axis
- $\mathcal{X}_-(H), \quad \operatorname{Im} \begin{bmatrix} 0 \\ I \end{bmatrix}$ are complementary (or $X_1$ is nonsingular.)

**Theorem:** Suppose $H \in dom(Ric)$ and $X = Ric(H)$. Then

(i) $X$ is real symmetric;

(ii) $X$ satisfies the algebraic Riccati equation

$$A^* X + XA + XRX + Q = 0;$$

(iii) $A + RX$ is stable .

---

**Proof**. (i) Let $\mathcal{X}_-(H) = \operatorname{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$. We show $X_1^* X_2$ is symmetric. Note that there exists a stable matrix $H_-$ in $\mathbb{R}^{n\times n}$ such that

$$H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_- \quad .$$

Pre-multiply this equation by

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^* J$$

to get

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^* JH \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^* J \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_-$$

Since $JH$ is symmetric $\Rightarrow$:

$$(-X_1^* X_2 + X_2^* X_1) H_- = H_-^* (-X_1^* X_2 + X_2^* X_1)^*$$

$$= -H_-^* (-X_1^* X_2 + X_2^* X_1).$$

This is a Lyapunov equation. Since $H_-$ is stable, the unique solution is

$$-X_1^* X_2 + X_2^* X_1 = 0.$$

i.e., $X_1^* X_2$ is symmetric. $\Rightarrow X = (X_1^{-1})^*(X_1^* X_2)X_1^{-1}$ is symmetric.

(ii) Start with the equation

$$H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_-$$

and post-multiply by $X_1^{-1}$ to get

$$H \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} X_1 H_- X_1^{-1}.$$

Now pre-multiply by $[X \quad -I]$:

$$[X \quad -I] H \begin{bmatrix} I \\ X \end{bmatrix} = 0.$$

This is precisely the Riccati equation.

(iii) $[I \quad 0] \left( H \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} X_1 H_- X_1^{-1} \right) \Rightarrow$

$$A + RX = X_1 H_- X_1^{-1}.$$

Thus $A + RX$ is stable because $H_-$ is. □

$\gg$ $[\mathbf{X_1}, \mathbf{X_2}] = \mathbf{ric\_schr}(\mathbf{H})$, $\mathbf{X} = \mathbf{X_2}/\mathbf{X_1}$

**Theorem:** Suppose $eig(H) \neq j\omega$ and $R$ is semi-definite ($\geq 0$ or $\leq 0$). Then $H \in dom(Ric) \Leftrightarrow (A, R)$ is stabilizable.

---

**Proof.** ($\Leftarrow$) Note that $\mathcal{X}_-(H) = \text{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad H\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_-$

We need to show that $X_1$ is nonsingular, i.e., Ker $X_1 = 0$.

<u>Claim: Ker $X_1$ is $H_-$-invariant.</u>

Let $x \in \text{Ker}\, X_1$ and note that $X_2^* X_1$ is symmetric and

$$AX_1 + RX_2 = X_1 H_- \quad .$$

Pre-multiply by $x^* X_2^*$, post-multiply by $x$ to get

$$x^* X_2^* R X_2 x = 0 \Rightarrow R X_2 x = 0 \Rightarrow X_1 H_- x = 0$$

i.e. $H_- x \in \text{Ker}\, X_1$.

Suppose Ker $X_1 \neq 0$. Then $H_-|_{\text{Ker}\, X_1}$ has an eigenvalue, $\lambda$, and a corresponding eigenvector, $x$:

$$H_- x = \lambda x, \qquad \text{Re}\, \lambda < 0, \quad 0 \neq x \in \text{Ker}\, X_1.$$

Note that

$$-QX_1 - A^* X_2 = X_2 H_- \quad .$$

Post-multiply the above equation by $x$:

$$(A^* + \lambda I) X_2 x = 0.$$

Recall that $RX_2 x = 0$, we have

$$x^* X_2^* [A + \overline{\lambda} I \quad R] = 0.$$

$(A, R)$ stabilizable $\Rightarrow X_2 x = 0 \Rightarrow \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} x = 0 \Rightarrow x = 0$ since $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ has full column rank, which is a contradiction.

($\Rightarrow$) $H \in dom(Ric) \Rightarrow A + RX$ stable $\Rightarrow (A, R)$ stabilizable.

$\square$

**Bounded Real Lemma:** Let $\gamma > 0$, $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{RH}_\infty$ and

$$H := \begin{bmatrix} A + BR^{-1}D^*C & BR^{-1}B^* \\ -C^*(I + DR^{-1}D^*)C & -(A + BR^{-1}D^*C)^* \end{bmatrix}$$

where $R = \gamma^2 I - D^*D$. Then the following conditions are equivalent:

(i) $\|G\|_\infty < \gamma$.

(ii) $\bar{\sigma}(D) < \gamma$ and $H$ has no eigenvalues on the imaginary axis.

(iii) $\bar{\sigma}(D) < \gamma$ and $H \in dom(Ric)$ .

(iv) $\bar{\sigma}(D) < \gamma$ and $H \in dom(Ric)$ and $Ric(H) \geq 0$ ($Ric(H) > 0$ if $(C, A)$ is observable).

(v) $\bar{\sigma}(D) < \gamma$ and there exists an $X \geq 0$ such that

$$X(A+BR^{-1}D^*C)+(A+BR^{-1}D^*C)^*X+XBR^{-1}B^*X+C^*(I+DR^{-1}D^*)C = 0$$

and $A+BR^{-1}D^*C+BR^{-1}B^*X$ has no eigenvalues on the imaginary axis.

(vi) $\bar{\sigma}(D) < \gamma$ and there exists an $X > 0$ such that

$$X(A+BR^{-1}D^*C)+(A+BR^{-1}D^*C)^*X+XBR^{-1}B^*X+C^*(I+DR^{-1}D^*)C < 0.$$

(vii) there exists an $X > 0$ such that

$$\begin{bmatrix} XA + A^*X & XB & C^* \\ B^*X & -\gamma I & D^* \\ C & D & -\gamma I \end{bmatrix} < 0.$$

**Proof.** $(v) \to (i)$: Assume $D = 0$ for simplicity. Then there is an $X \geq 0$

$$XA + A^*X + XBB^*X/\gamma^2 + C^*C = 0$$

and $A + BB^*X/\gamma^2$ has no $j\omega$-axis eigenvalue. Hence

$$W(s) := \left[\begin{array}{c|c} A & -B \\ \hline B^*X/\gamma & \gamma I \end{array}\right]$$

has no zeros on the imaginary axis since

$$W^{-1}(s) = \left[\begin{array}{c|c} A + BB^*X/\gamma^2 & B/\gamma \\ \hline B^*X/\gamma^2 & I/\gamma \end{array}\right]$$

has no poles on the imaginary axis. Next, note that

$$-X(j\omega I - A) - (j\omega I - A)^*X + XBB^*X/\gamma^2 + C^*C = 0.$$

Multiply $B^*\{(j\omega I - A)^*\}^{-1}$ on the left and $(j\omega I - A)^{-1}B$ on the right of the above equation to get

$$\begin{aligned} -B^*\{(j\omega I - A)^*\}^{-1}XB - B^*X(j\omega I - A)^{-1}B \\ +B^*\{(j\omega I - A)^*\}^{-1}XBB^*X(j\omega I - A)^{-1}B/\gamma^2 \\ +B^*\{(j\omega I - A)^*\}^{-1}C^*C(j\omega I - A)^{-1}B = 0. \end{aligned}$$

Completing square, we have

$$G^*(j\omega)G(j\omega) = \gamma^2 I - W^*(j\omega)W(j\omega).$$

Since $W(s)$ has no $j\omega$-axis zeros , we conclude that $\|G\|_\infty < \gamma$.

$(vi) \Rightarrow (vii)$ follows from Schur complement.

$(vi) \Rightarrow (i)$ by following the similar procedure as above.

$(i) \Rightarrow (vi)$: let

$$\hat{G} = \left[\begin{array}{c|c} A & B \\ \hline C & D \\ \epsilon I & 0 \end{array}\right].$$

Then there exists an $\epsilon > 0$ such that $\left\|\hat{G}\right\|_\infty < \gamma$. Now $(vi)$ follows by applying part $(v)$ to $\hat{G}$. $\square$

**Theorem:** Suppose $H$ has the form

$$H = \begin{bmatrix} A & -BB^* \\ -C^*C & -A^* \end{bmatrix}.$$

Then $H \in dom(Ric)$ iff $(A, B)$ is stabilizable and $(C, A)$ has no unobservable modes on the imaginary axis. Furthermore, $X = Ric(H) \geq 0$. And $X > 0$ if and only if $(C, A)$ has no stable unobservable modes.

---

**Proof.** Only need to show that, assuming $(A, B)$ is stabilizable, $H$ has no imaginary eigenvalues iff $(C, A)$ has no unobservable modes on the imaginary axis. Suppose that $j\omega$ is an eigenvalue and $0 \neq \begin{bmatrix} x \\ z \end{bmatrix}$ is a corresponding eigenvector. Then

$$Ax - BB^*z = j\omega x, \qquad -C^*Cx - A^*z = j\omega z.$$

Re-arrange:

$$(A - j\omega I)x = BB^*z, \qquad -(A - j\omega I)^*z = C^*Cx.$$

Thus

$$\langle z, (A - j\omega I)x \rangle = \langle z, BB^*z \rangle = \|B^*z\|^2$$
$$-\langle x, (A - j\omega I)^*z \rangle = \langle x, C^*Cx \rangle = \|Cx\|^2$$

so $\langle x, (A - j\omega I)^*z \rangle$ is real and

$$-\|Cx\|^2 = \langle (A - j\omega I)x, z \rangle = \overline{\langle z, (A - j\omega I)x \rangle} = \|B^*z\|^2.$$

Therefore $B^*z = 0$ and $Cx = 0$. So

$$(A - j\omega I)x = 0, \qquad (A - j\omega I)^*z = 0.$$

Combine the last four equations to get

$$z^*[A - j\omega I \quad B] = 0, \qquad \begin{bmatrix} A - j\omega I \\ C \end{bmatrix} x = 0.$$

The stabilizability of $(A, B)$ gives $z = 0$. Now it is clear that $j\omega$ is an eigenvalue of $H$ iff $j\omega$ is an unobservable mode of $(C, A)$.

$$(A - BB^*X)^*X + X(A - BB^*X) + XBB^*X + C^*C = 0.$$

$X \geq 0$ since $A - BB^*X$ is stable. □

---

**Corollary:** Suppose $(A, B)$ is stabilizable and $(C, A)$ is detectable. Then

$$A^*X + XA - XBB^*X + C^*C = 0$$

has a unique positive semidefinite solution. Moreover, it is stabilizing.

---

**Corollary:** Suppose $D$ has full column rank and denote $R = D^*D > 0$. Let $H$ have the form

$$\begin{aligned} H &= \begin{bmatrix} A & 0 \\ -C^*C & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -C^*D \end{bmatrix} R^{-1} \begin{bmatrix} D^*C & B^* \end{bmatrix} \\ &= \begin{bmatrix} A - BR^{-1}D^*C & -BR^{-1}B^* \\ -C^*(I - DR^{-1}D^*)C & -(A - BR^{-1}D^*C)^* \end{bmatrix}. \end{aligned}$$

Then $H \in dom(Ric)$ iff $(A, B)$ is stabilizable and $\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$ has full column rank for all $\omega$. Furthermore, $X = Ric(H) \geq 0$ if $H \in dom(Ric)$, and $\mathrm{Ker}(X) = 0$ if and only if $(D_\perp^* C, A - BR^{-1}D^*C)$ has no stable unobservable modes.

---

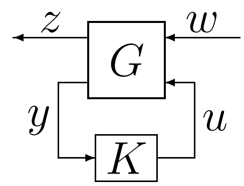
This is because $\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$ has full column rank for all $\omega \Longleftrightarrow$

$((I - DR^{-1}D^*)C,\ A - BR^{-1}D^*C)$ has no unobservable modes on $j\omega$-axis.

# Chapter 13: $\mathcal{H}_2$ Optimal Control

- $\mathcal{H}_2$ optimal control
- stability margins of $\mathcal{H}_2$ controllers

# $\mathcal{H}_2$ Optimal Control




$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$

Assumptions:

(i) $(A, B_2)$ is stabilizable and $(C_2, A)$ is detectable;

(ii) $D_{12}$ has full column rank with $\begin{bmatrix} D_{12} & D_\perp \end{bmatrix}$ unitary, and $D_{21}$ has full row rank with $\begin{bmatrix} D_{21} \\ \tilde{D}_\perp \end{bmatrix}$ unitary;

(iii) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega$;

(iv) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all $\omega$.

**$\mathcal{H}_2$ Problem:** *find a stabilizing controller $K$ that minimizes* $\|T_{zw}\|_2$ .

$$X_2(A - B_2 D_{12}^* C_1) + (A - B_2 D_{12}^* C_1)^* X_2 - X_2 B_2 B_2^* X_2 + C_1^* D_\perp D_\perp^* C_1 = 0$$

$$Y_2(A - B_1 D_{21}^* C_2)^* + (A - B_1 D_{21}^* C_2) Y_2 - Y_2 C_2^* C_2 Y_2 + B_1 \tilde{D}_\perp^* \tilde{D}_\perp B_1^* = 0$$

Define

$$F_2 := -(B_2^* X_2 + D_{12}^* C_1), \quad L_2 := -(Y_2 C_2^* + B_1 D_{21}^*)$$

$$G_c(s) := \left[\begin{array}{c|c} A + B_2 F_2 & I \\ \hline C_1 + D_{12} F_2 & 0 \end{array}\right], \qquad G_f(s) := \left[\begin{array}{c|c} A + L_2 C_2 & B_1 + L_2 D_{21} \\ \hline I & 0 \end{array}\right].$$

---

There exists a unique optimal controller

$$K_{opt}(s) := \left[\begin{array}{c|c} A + B_2 F_2 + L_2 C_2 & -L_2 \\ \hline F_2 & 0 \end{array}\right].$$

Moreover, $\min \|T_{zw}\|_2^2 = \|G_c B_1\|_2^2 + \|F_2 G_f\|_2^2 = \|G_c L_2\|_2^2 + \|C_1 G_f\|_2^2.$

---

- $U := \left[\begin{array}{c|c} A + B_2 F_2 & B_2 \\ \hline C_1 + D_{12} F_2 & D_{12} \end{array}\right] \in \mathcal{RH}_\infty$ is inner and $U^\sim G_c \in \mathcal{RH}_2^\perp$.

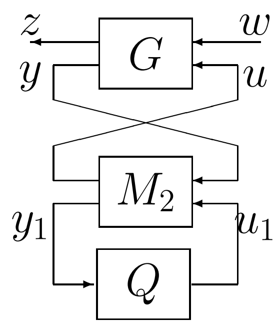
- $V := \left[\begin{array}{c|c} A + L_2 C_2 & B_1 + L_2 D_{21} \\ \hline C_2 & D_{21} \end{array}\right] \in \mathcal{RH}_\infty$ is co-inner and $G_f V^\sim \in \mathcal{RH}_2^\perp$.

- all stabilizing controllers $K(s) = \mathcal{F}_\ell(M_2, Q), \quad Q \in \mathcal{RH}_\infty$ with

$$M_2(s) = \left[\begin{array}{c|cc} A + B_2 F_2 + L_2 C_2 & -L_2 & B_2 \\ \hline F_2 & 0 & I \\ -C_2 & I & 0 \end{array}\right].$$

- Closed-loop with $K$

$$T_{zw} = G_c B_1 - U F_2 G_f + U Q V.$$




$$\|T_{zw}\|_2^2 = \|G_cB_1\|_2^2 + \|F_2G_f - QV\|_2^2 = \|G_cB_1\|_2^2 + \|F_2G_f\|_2^2 + \|Q\|_2^2$$

and $Q = 0$ gives the unique optimal control: $K = \mathcal{F}_\ell(M_2, 0)$.

## Stability Margins of $\mathcal{H}_2$ Controllers

- LQR margin: $\geq 60^o$ phase margin and $\geq 6dB$ gain margin.
- LQG or $\mathcal{H}_2$ Controller: No guaranteed margin:

$$G(s) = \left[\begin{array}{c|cc} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} \sqrt{\sigma} & 0 \\ \sqrt{\sigma} & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \hline \begin{bmatrix} \sqrt{q} & \sqrt{q} \\ 0 & 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \end{bmatrix} & 0 \end{array}\right].$$

Then

$$X_2 = \begin{bmatrix} 2\alpha & \alpha \\ \alpha & \alpha \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 2\beta & \beta \\ \beta & \beta \end{bmatrix}$$

and

$$F_2 = -\alpha \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad L_2 = -\beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where

$$\alpha = 2 + \sqrt{4+q}\,, \quad \beta = 2 + \sqrt{4+\sigma}.$$

$$K_{opt} = \left[\begin{array}{cc|c} 1-\beta & 1 & \beta \\ -(\alpha+\beta) & 1-\alpha & \beta \\ \hline -\alpha & -\alpha & 0 \end{array}\right].$$

Suppose the controller implemented in the system (or plant $G_{22}$) is actually

$$K = kK_{opt},$$

with a nominal value $k = 1$. Then the closed-loop system $A$-matrix becomes

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -k\alpha & -k\alpha \\ \beta & 0 & 1-\beta & 1 \\ \beta & 0 & -\alpha-\beta & 1-\alpha \end{bmatrix}.$$

The characteristic polynomial has the form

$$\det(sI - \tilde{A}) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

with

$$a_1 = \alpha + \beta - 4 + 2(k-1)\alpha\beta, \quad a_0 = 1 + (1-k)\alpha\beta.$$

– necessary for stability: $a_0 > 0$ and $a_1 > 0$.

– $\alpha \gg 1$ and $\beta \gg 1$ and $k \neq 1 \Rightarrow a_0 \approx (1-k)\alpha\beta$ and $a_1 \approx 2(k-1)\alpha\beta$.

– $\alpha \gg 1$ and $\beta \gg 1$ (or $q$ and $\sigma$), the system is unstable for arbitrarily small perturbations in $k$ in either direction. Thus, by choice of $q$ and $\sigma$, the gain margins may be made arbitrarily small.

It is interesting to note that the margins deteriorate as control weight $(1/q)$ gets small (large $q$) and/or system driving noise gets large (large $\sigma$). In modern control folklore, these have often been considered ad hoc means of improving sensitivity.

- $\mathcal{H}_2$ (LQG) controllers have no global system-independent guaranteed robustness properties.

- Improve the robustness of a given design by relaxing the optimality of the filter (or FC controller) with respect to error properties. LQG loop transfer recovery (LQG/LTR) design technique. The idea is to design a filtering gain (or FC control law) in such way so that the LQG (or $\mathcal{H}_2$) control law will approximate the loop properties of the regular LQR control.

# Chapter 14a: Understanding $H_\infty$ Control

Objective: Derivation of $H_\infty$ controller

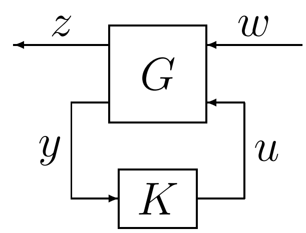
Methods: Intuition and handwaving

Background: State Feedback and Observer

- Problem Formulation and Solutions
- An intuitive Derivation

## Problem Formulation and Solutions

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$




(i) $(A, B_1)$ is stabilizable and $(C_1, A)$ is detectable

(ii) $(A, B_2)$ is stabilizable and $(C_2, A)$ is detectable

(iii) $D_{12}^* \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$

(iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^* = \begin{bmatrix} 0 \\ I \end{bmatrix}$

---

(i) Together with (ii) guarantees that the two AREs have nonnegative definite stabilizing solutions.

(ii) Necessary and sufficient for $G$ to be internally stabilizable.

(iii) The penalty on $z = C_1 x + D_{12} u$ includes a nonsingular, normalized penalty on the control $u$. In the conventional $\mathcal{H}_2$ setting this means that there is no cross weighting between the state and control and that the control weight matrix is the identity.

(iv) $w$ includes both plant disturbance and sensor noise, these are orthogonal, and the sensor noise weighting is normalized and nonsingular.

---

These assumptions can be relaxed.

# Output Feedback $\mathcal{H}_\infty$ Control

$\exists K$ such that $\|T_{zw}\|_\infty < \gamma$ if and only if

(i) $\exists\ X_\infty \geq 0$

$$X_\infty A + A^* X_\infty + X_\infty (B_1 B_1^*/\gamma^2 - B_2 B_2^*) X_\infty + C_1^* C_1 = 0$$

(ii) $\exists\ Y_\infty \geq 0$

$$A Y_\infty + Y_\infty A^* + Y_\infty (C_1^* C_1/\gamma^2 - C_2^* C_2) Y_\infty + B_1 B_1^* = 0$$

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$ .

$$K_{sub}(s) := \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

where

$$\hat{A}_\infty := A + \gamma^{-2} B_1 B_1^* X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2$$
$$F_\infty := -B_2^* X_\infty, \quad L_\infty := -Y_\infty C_2^*$$
$$Z_\infty := (I - \gamma^{-2} Y_\infty X_\infty)^{-1}.$$

## Bounded Real Lemma

$$z = G(s)w, \quad G(s) = C(sI - A)^{-1}B \in \mathcal{H}_\infty$$

$$\|G\|_\infty = \sup_w \frac{\|z\|_2}{\|w\|_2} := \sup_w \frac{\sqrt{\int_0^\infty \|z\|^2 \, dt}}{\sqrt{\int_0^\infty \|w\|^2 \, dt}}$$

$$\|G\|_\infty < \gamma$$

$$\Updownarrow$$

$$\int_0^\infty \left( \|z\|^2 - \gamma^2 \|w\|^2 \right) dt < 0, \quad \forall w \neq 0$$

$$\Updownarrow$$

$$\boxed{\begin{array}{c} \exists\ X = X^* \geq 0 \text{ such that} \\ XA + A^*X + XBB^*X/\gamma^2 + C^*C = 0 \\ \text{and } A + BB^*X/\gamma^2 \text{ is stable} \end{array}}$$

$$\Updownarrow$$

$$\boxed{\begin{array}{c} \exists\ Y = Y^* \geq 0 \text{ such that} \\ YA^* + AY + YC^*CY/\gamma^2 + BB^* = 0 \\ \text{and } A + YC^*C/\gamma^2 \text{ is stable} \end{array}}$$

Let $\Phi(s) = \gamma^2 I - G^{\sim}(s)G(s)$. Then

$$\|G\|_\infty < \gamma \Longleftrightarrow \Phi(j\omega) > 0,\ \forall \omega \in \mathbb{R} \quad \Longleftrightarrow \det \Phi(j\omega) \neq 0$$

since $\Phi(\infty) = \gamma^2 I > 0$ and $\Phi(j\omega)$ is continuous

$\Longleftrightarrow \Phi(s)$ has no imaginary axis zero.

$\Longleftrightarrow \Phi^{-1}(s)$ has no imaginary axis pole.

$$\Phi(s) = \left[\begin{array}{cc|c} A & 0 & -B \\ -C^*C & -A^* & 0 \\ \hline 0 & B^* & \gamma^2 I \end{array}\right]$$

$$\Phi^{-1} = \left[\begin{array}{cc|c} A & BB^*/\gamma^2 & B/\gamma^2 \\ -C^*C & -A^* & 0 \\ \hline 0 & B^*/\gamma^2 & \gamma^{-2} I \end{array}\right]$$

$$\Longleftrightarrow \begin{bmatrix} A & BB^*/\gamma^2 \\ -C^*C & -A^* \end{bmatrix} \text{ has no } j\omega \text{ axis eigenvalues}$$

Apply the following similarity transformation to $\Phi^{-1}$

$$T = \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}$$

$$\Phi^{-1} = \left[\begin{array}{cc|c} A + BB^*X/\gamma^2 & BB^*/\gamma^2 & B/\gamma^2 \\ M(X) & -A^* - XBB^*/\gamma^2 & -XB/\gamma^2 \\ \hline B^*X/\gamma^2 & B^*/\gamma^2 & \gamma^{-2} I \end{array}\right]$$

$$M(X) := -XA - A^*X - XBB^*X/\gamma^2 - C^*C$$

If $M(X) = 0$, we have

$$\Phi^{-1} = \gamma^2 \left[\begin{array}{c|c} A + BB^*X/\gamma^2 & B/\gamma^2 \\ \hline B^*X/\gamma^2 & I/\gamma^2 \end{array}\right] \left[\begin{array}{c|c} -(A + BB^*X/\gamma^2)^* & -XB/\gamma^2 \\ \hline B/\gamma^2 & I/\gamma^2 \end{array}\right]$$

$\Phi(j\omega) > 0$ if $A + BB^*X/\gamma^2$ has no $j\omega$ eigenvalue

System Equations:

$$
\begin{aligned}
\dot{x} &= Ax + B_1 w + B_2 u \\
z &= C_1 x + D_{12} u \\
y &= C_2 x + D_{21} w
\end{aligned}
$$

$\boxed{\text{State feedback}}$ $u = Fx$:

$$
\begin{aligned}
\dot{x} &= (A + B_2 F)x + B_1 w \\
z &= (C_1 + D_{12} F)x
\end{aligned}
$$

By Bounded Real Lemma, $\|T_{zw}\|_\infty < \gamma$

$$\Updownarrow$$

$$
\boxed{
\begin{array}{c}
\exists\ X = X^* \geq 0 \text{ such that} \\
X(A + B_2F) + (A + B_2F)^*X + XB_1B_1^*X/\gamma^2 + (C_1 + D_{12}F)^*(C_1 + D_{12}F) = 0 \\
\text{and } A + B_2F + B_1B_1^*X/\gamma^2 \text{ is stable}
\end{array}
}
$$

$$\text{complete} \Updownarrow \text{ square}$$

$$
\boxed{
\begin{array}{c}
\exists\ X = X^* \geq 0 \text{ such that} \\
XA + A^*X + XB_1B_1^*X/\gamma^2 - XB_2B_2^*X + C_1^*C_1 + (F + B_2^*X)^*(F + B_2^*X) = 0 \\
\text{and } A + B_2F + B_1B_1^*X/\gamma^2 \text{ is stable}
\end{array}
}
$$

$$\boxed{\text{Intuition} \Longrightarrow \quad F = -B_2^*X}$$

$$\Updownarrow$$

$$
\boxed{
\begin{array}{c}
\exists\ X = X^* \geq 0 \text{ such that} \\
XA + A^*X + XB_1B_1^*X/\gamma^2 - XB_2B_2^*X + C_1^*C_1 = 0 \\
\text{and } A + B_1B_1^*X/\gamma^2 - B_2B_2^*X \text{ is stable}
\end{array}
}
$$

$$\Longrightarrow \quad F = F_\infty, \quad X = X_\infty$$

$\boxed{\text{Output Feedback}}$: Converting to State Estimation
Suppose $\exists$ a $K$ such that

$$\|T_{zw}\|_\infty < \gamma$$

Then $x(\infty) = 0$ by stability (note also $x(0) = 0$)

$$\int_0^\infty \left( \|z\|^2 - \gamma^2 \|w\|^2 \right) dt$$

$$= \int_0^\infty \left( \|z\|^2 - \gamma^2 \|w\|^2 + \frac{d}{dt}(x^* X_\infty x) \right) dt$$

$$= \int_0^\infty \left( \|z\|^2 - \gamma^2 \|w\|^2 + \dot{x}^* X_\infty x + x^* X_\infty \dot{x} \right) dt$$

Substituting $\dot{x} = Ax + B_1 w + B_2 u$ and $z = C_1 x + D_{12} u$

$$= \int_0^\infty \left( \|C_1 x\|^2 + \|u\|^2 - \gamma^2 \|w\|^2 \right.$$

$$\left. +2x^* X_\infty A x + 2x^* X_\infty B_1 w + 2x^* X_\infty B_2 u \right) dt$$

$$= \int_0^\infty \left( x^*(C_1^* C_1 + X_\infty A + A^* X_\infty) x + \|u\|^2 \right.$$

$$\left. -\gamma^2 \|w\|^2 + 2x^* X_\infty B_1 w + 2x^* X_\infty B_2 u \right) dt$$

using $X_\infty$ equation

$$= \int_0^\infty \left( x^*(-X_\infty B_1 B_1^* X_\infty/\gamma^2 + X_\infty B_2 B_2^* X_\infty) x + \|u\|^2 \right.$$

$$\left. -\gamma^2 \|w\|^2 + 2x^* X_\infty B_1 w + 2x^* X_\infty B_2 u \right) dt$$

$$= \int_0^\infty \left( - \|B_1^* X_\infty x/\gamma\|^2 - \gamma^2 \|w\|^2 + 2x^* X_\infty B_1 w \right.$$

$$\left. + \|B_2^* X_\infty x\|^2 + \|u\|^2 + 2x^* X_\infty B_2 u \right) dt$$

completing the squares with respect to $u$ and $w$

$$= \int_0^\infty \left( \|u + B_2^* X_\infty x\|^2 - \gamma^2 \left\|w - \gamma^{-2} B_1^* X_\infty x\right\|^2 \right) dt$$

Summary:

$$\int_0^\infty \left(\|z\|^2 - \gamma^2 \|w\|^2\right) dt = \int_0^\infty \left(\|v\|^2 - \gamma^2 \|r\|^2\right) dt$$

$$v = u + B_2^* X_\infty x = u - F_\infty x, \quad r = w - \gamma^{-2} B_1^* X_\infty x$$

Rewrite the system equation with: $w = r + \gamma^{-2} B_1^* X_\infty x$

$$\begin{aligned} \dot{x} &= (A + B_1 B_1^* X_\infty / \gamma^2) x + B_1 r + B_2 u \\ v &= -F_\infty x + u \\ y &= C_2 x + D_{21} r \end{aligned}$$

$$\|T_{zw}\|_\infty < \gamma \Longleftrightarrow \|T_{vr}\|_\infty < \gamma$$

$$\Longleftrightarrow \quad \int_0^\infty \left(\|u - F_\infty x\|^2 - \gamma^2 \|r\|^2\right) dt < 0$$

---

If state is available: $u = F_\infty x$

worst disturbance: $w_* = \gamma^{-2} B_1^* X_\infty x$

State is not available: using estimated state

$$u = F_\infty \hat{x}$$

A standard observer:

$$\dot{\hat{x}} = (A + B_1 B_1^* X_\infty / \gamma^2) \hat{x} + B_2 u + L(C_2 \hat{x} - y)$$

where $L$ is the observer gain to be determined.

Let $e := x - \hat{x}$. Then

$$\begin{aligned}\dot{e} &= (A + B_1 B_1^* X_\infty/\gamma^2 + LC_2)e + (B_1 + LD_{21})r \\ v &= -F_\infty e\end{aligned}$$

$\|T_{vr}\|_\infty < \gamma \Longrightarrow \exists$ a $Y \geq 0$ by bounded real lemma

$$Y(A+B_1B_1^*X_\infty/\gamma^2+LC_2)^*+(A+B_1B_1^*X_\infty/\gamma^2+LC_2)Y+YF_\infty^*F_\infty Y/\gamma^2$$
$$+(B_1 + LD_{21})(B_1 + LD_{21})^* = 0$$

Complete square w.r.t. $L$

$$(A+B_1B_1^*X_\infty/\gamma^2)^*+(A+B_1B_1^*X_\infty/\gamma^2)Y+YF_\infty^*F_\infty Y/\gamma^2+B_1B_1^*-YC_2^*C_2Y$$
$$+(L + YC_2^*)(L + YC_2^*)^* = 0$$

Again, intuition suggests that we can take

$$L = -YC_2^*$$

which gives

$$Y(A + B_1B_1^*X_\infty/\gamma^2)^* + (A + B_1B_1^*X_\infty/\gamma^2)Y$$
$$+YF_\infty^*F_\infty Y/\gamma^2 - YC_2^*C_2Y + B_1B_1^* = 0$$

It is easy to verify that

$$Y = Y_\infty(I - \gamma^{-2}X_\infty Y_\infty)^{-1}$$

Since $Y \geq 0$, we must have

$$\rho(X_\infty Y_\infty) < \gamma^2$$

Hence $L = Z_\infty L_\infty$ and the controller is give by

$$\begin{aligned}\dot{\hat{x}} &= (A + B_1B_1^*X_\infty/\gamma^2)\hat{x} + B_2u + Z_\infty L_\infty(C_2\hat{x} - y) \\ u &= F_\infty \hat{x}\end{aligned}$$

# Chapter 14: $\mathcal{H}_\infty$ Control

- $\mathcal{H}_\infty$ background
- $\mathcal{H}_\infty$: 1984 workshop approach
- Assumptions
- output feedback $\mathcal{H}_\infty$ control
- a matrix fact
- inequality characterization
- connection between ARE and ARI (LMI)
- proof for necessity
- proof for sufficiency
- comments
- optimality and dependence on $\gamma$
- $\mathcal{H}_\infty$ controller structure
- example
- an optimal controller
- $\mathcal{H}_\infty$ control: general case
- relaxing assumptions
- $\mathcal{H}_2$ and $\mathcal{H}_\infty$ integral control
- $\mathcal{H}_\infty$ filtering

## $\mathcal{H}_\infty$ Background

- Initial theory was SISO (Zames, Helton, Tannenbaum)
- Nevanlinna-Pick interpolation
- Operator-theoretic methods (Sarason, Adamjan *et al*, Ball-Helton)
- Initial work handled restricted problems
  ( “1-block” and “2-block” )
- Solution to “$2 \times 2$-block” problem
  (1984 Honeywell-ONR Workshop)

# $\mathcal{H}_\infty$: 1984 H/ONR Workshop Approach

Solution approach:

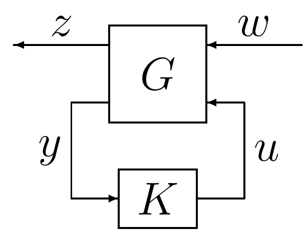
- Parameterize all stabilizing controllers via [Youla *et al*]
- Obtain realizations of the closed-loop transfer matrix
- Transform to "$2 \times 2$-block" general distance problem
- Reduce to the Nehari problem and solve via Glover

Properties of the solution:

- State-space using standard operations
- Computationally intensive (many Ric. eqns.)
- Potentially high-order controllers
- Find solution $< \gamma$, iterate for optimal

# Assumptions

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$




(i) $(A, B_1)$ is Controllable and $(C_1, A)$ is observable

(ii) $(A, B_2)$ is stabilizable and $(C_2, A)$ is detectable

(iii) $D_{12}^* \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$

(iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^* = \begin{bmatrix} 0 \\ I \end{bmatrix}$

---

(i) Together with (ii) guarantees that the two AREs have positive definite stabilizing solution.

(ii) Necessary and sufficient for $G$ to be internally stabilizable.

(iii) The penalty on $z = C_1 x + D_{12} u$ includes a nonsingular, normalized penalty on the control $u$. In the conventional $\mathcal{H}_2$ setting this means that there is no cross weighting between the state and control input, and that the control weight matrix is the identity.

(iv) $w$ includes both plant disturbance and sensor noise, these are orthogonal, and the sensor noise weighting is normalized and nonsingular.

---

These assumptions simplify the theorem statements and proofs, and can be relaxed.

## Output Feedback $\mathcal{H}_\infty$ Control

$\exists K$ such that $\|T_{zw}\|_\infty < \gamma$ iff

(i) $\exists\ X_\infty > 0$

$$X_\infty A + A^* X_\infty + X_\infty (B_1 B_1^*/\gamma^2 - B_2 B_2^*) X_\infty + C_1^* C_1 = 0$$

(ii) $\exists\ Y_\infty > 0$

$$A Y_\infty + Y_\infty A^* + Y_\infty (C_1^* C_1/\gamma^2 - C_2^* C_2) Y_\infty + B_1 B_1^* = 0$$

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$ .

---

$$K_{sub}(s) := \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

where

$$\hat{A}_\infty := A + \gamma^{-2} B_1 B_1^* X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2$$

$$F_\infty := -B_2^* X_\infty, \quad L_\infty := -Y_\infty C_2^*$$

$$Z_\infty := (I - \gamma^{-2} Y_\infty X_\infty)^{-1}.$$

## A Matrix Fact

[Packard, 1994] Suppose $X, Y \in \mathbb{R}^{n\times n}$ and $X = X^* > 0, Y = Y^* > 0$. Let $r$ be a positive integer. Then there exists matrices $X_{12} \in \mathbb{R}^{n\times r}$, $X_2 \in \mathbb{R}^{r\times r}$ such that $X_2 = X_2^*$, and

$$\begin{bmatrix} X & X_{12} \\ X_{12}^* & X_2 \end{bmatrix} > 0 \qquad \& \qquad \begin{bmatrix} X & X_{12} \\ X_{12}^* & X_2 \end{bmatrix}^{-1} = \begin{bmatrix} Y & \star \\ \star & \star \end{bmatrix}$$

if and only if

$$\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \geq 0 \qquad \& \qquad \operatorname{rank}\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \leq n + r.$$

**Proof.** ($\Leftarrow$) By assumption, there is a matrix $X_{12} \in \mathbb{R}^{n\times r}$ such that $X - Y^{-1} = X_{12}X_{12}^*$. Defining $X_2 := I_r$ completes the construction.

($\Rightarrow$) Using Schur complements,

$$Y = X^{-1} + X^{-1}X_{12}(X_2 - X_{12}^*X^{-1}X_{12})^{-1}X_{12}^*X^{-1}.$$

Inverting, using the matrix inversion lemma, gives

$$Y^{-1} = X - X_{12}X_2^{-1}X_{12}^*.$$

Hence, $X - Y^{-1} = X_{12}X_2^{-1}X_{12}^* \geq 0$, and indeed,

rank$(X - Y^{-1})$ =rank$(X_{12}X_2^{-1}X_{12}^*) \leq r$. $\square$

## Inequality Characterization

**Lemma IC:** $\exists$ $r$-th order $K$ such that $\|T_{zw}\|_\infty < \gamma$ only if

(i) $\exists$ $Y_1 > 0$

$$AY_1 + Y_1A^* + Y_1C_1^*C_1Y_1/\gamma^2 + B_1B_1^* - \gamma^2 B_2B_2^* < 0$$

(ii) $\exists X_1 > 0$

$$X_1A + A^*X_1 + X_1B_1B_1^*X_1/\gamma^2 + C_1^*C_1 - \gamma^2 C_2^*C_2 < 0$$

(iii) $\begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix} \geq 0$ and rank $\begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix} \leq n + r.$

*Proof.* Suppose that there exists an $r$-th order controller $K(s)$ such that $\|T_{zw}\|_\infty < \gamma$. Let $K(s)$ have a state space realization

$$K(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$$

then

$$T_{zw} = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] := \left[\begin{array}{cc|c} A + B_2\hat{D}C_2 & B_2\hat{C} & B_1 + B_2\hat{D}D_{21} \\ \hat{B}C_2 & \hat{A} & \hat{B}D_{21} \\ \hline C_1 + D_{12}\hat{D}C_2 & D_{12}\hat{C} & D_{12}\hat{D}D_{21} \end{array}\right].$$

Denote

$$R = \gamma^2 I - D_c^*D_c, \qquad \tilde{R} = \gamma^2 I - D_cD_c^*.$$

By Bounded Real Lemma, $\exists \tilde{X} = \begin{bmatrix} X_1 & X_{12} \\ X_{12}^* & X_2 \end{bmatrix} > 0$ such that

$$\tilde{X}(A_c + B_cR^{-1}D_c^*C_c) + (A_c + B_cR^{-1}D_c^*C_c)^*\tilde{X}$$

$$+\tilde{X}B_cR^{-1}B_c^*\tilde{X}+C_c^*\tilde{R}^{-1}C_c<0$$

This gives after much algebraic manipulation

$$X_1A+A^*X_1+X_1B_1B_1^*X_1/\gamma^2+C_1^*C_1-\gamma^2C_2^*C_2$$

$$+(X_1B_1\hat{D}+X_{12}\hat{B}+\gamma^2C_2^*)(\gamma^2I-\hat{D}^*\hat{D})^{-1}(X_1B_1\hat{D}+X_{12}\hat{B}+\gamma^2C_2^*)^*<0$$

which implies that

$$X_1A+A^*X_1+X_1B_1B_1^*X_1/\gamma^2+C_1^*C_1-\gamma^2C_2^*C_2<0.$$

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Let $\tilde{Y}=\gamma^2\tilde{X}^{-1}$ and partition $\tilde{Y}$ as $\tilde{Y}=\begin{bmatrix} Y_1 & Y_{12} \\ Y_{12}^* & Y_2 \end{bmatrix}>0$ then

$$(A_c+B_cR^{-1}D_c^*C_c)\tilde{Y}+\tilde{Y}(A_c+B_cR^{-1}D_c^*C_c)^*$$

$$+\tilde{Y}C_c^*\tilde{R}^{-1}C_c\tilde{Y}+B_cR^{-1}B_c^*<0$$

This gives

$$AY_1+Y_1A^*+B_1B_1^*-\gamma^2B_2B_2^*+Y_1C_1^*C_1Y_1/\gamma^2$$

$$+(Y_1C_1^*\hat{D}^*+Y_{12}\hat{C}^*+\gamma^2B_2)(\gamma^2I-\hat{D}\hat{D}^*)^{-1}(Y_1C_1^*\hat{D}^*+Y_{12}\hat{C}^*+\gamma^2B_2)^*<0$$

which implies that

$$AY_1+Y_1A^*+B_1B_1^*-\gamma^2B_2B_2^*+Y_1C_1^*C_1Y_1/\gamma^2<0.$$

By the matrix fact, given $X_1>0$ and $Y_1>0$, there exists $X_{12}$ and $X_2$ such that $\tilde{Y}=\gamma^2\tilde{X}^{-1}$ or $\tilde{Y}/\gamma=(\tilde{X}/\gamma)^{-1}$:

$$\begin{bmatrix} X_1/\gamma & X_{12}/\gamma \\ X_{12}^*/\gamma & X_2/\gamma \end{bmatrix}^{-1}=\begin{bmatrix} Y_1/\gamma & \star \\ \star & \star \end{bmatrix}$$

$$\Longleftrightarrow \begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix}\geq 0 \text{ and rank}\begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix}\leq n+r. \qquad \square$$

## Connection between ARE and ARI (LMI)

**Lemma ARE:** [Ran and Vreugdenhil, 1988] Suppose $(A, B)$ is controllable and there is an $X = X^*$ such that

$$\mathcal{Q}(X) := XA + A^*X + XBB^*X + Q < 0.$$

Then there exists a solution $X_+ > X$ to the Riccati equation

$$XA + A^*X + XBB^*X + Q = 0 \tag{0.7}$$

such that $A + BB^*X_+$ is antistable.

*Proof.* Let $X$ be such that $\mathcal{Q}(X) < 0$.
Choose $F_0$ such that $A_0 := A - BF_0$ is antistable.
Let $X_0 = X_0^*$ solve

$$X_0A_0 + A_0^*X_0 - F_0^*F_0 + Q = 0.$$

Define $\hat{F}_0 := F_0 + B^*X$. Then

$$(X_0 - X)A_0 + A_0^*(X_0 - X) = \hat{F}_0^*\hat{F}_0 - \mathcal{Q}(X) > 0.$$

and $X_0 > X$ (by anti-stability of $A_0$)
Define a non-increasing sequence of hermitian matrices $\{X_i\}$:

$$X_0 \geq X_1 \geq \cdots \geq X_{n-1} > X,$$

$$A_i = A - BF_i, \text{ is antistable, } i = 0, \ldots, n-1;$$

$$F_i = -B^*X_{i-1}, \ i = 1, \ldots, n-1;$$

$$X_iA_i + A_i^*X_i = F_i^*F_i - Q, \ i = 0, 1, \ldots, n-1. \tag{0.8}$$

By Induction: We show this sequence can indeed be defined:
Introduce

$$F_n = -B^*X_{n-1}, \quad A_n = A - BF_n.$$

We show that $A_n$ is antistable. Using (0.8), with $i = n-1$, we get

$$X_{n-1}A_n + A_n^* X_{n-1} + Q - F_n^* F_n - (F_n - F_{n-1})^*(F_n - F_{n-1}) = 0.$$

Let $\hat{F}_n := F_n + B^* X$; then

$$(X_{n-1} - X)A_n + A_n^*(X_{n-1} - X) = -\mathcal{Q}(X)$$
$$+\hat{F}_n^* \hat{F}_n + (F_n - F_{n-1})^*(F_n - F_{n-1}) > 0$$

$\Rightarrow A_n$ is antistable by Lyapunov theorem since $X_{n-1} - X > 0$.

Let $X_n$ be the unique solution of

$$X_n A_n + A_n^* X_n = F_n^* F_n - Q. \tag{0.9}$$

Then $X_n$ is hermitian. Next, we have

$$(X_n - X)A_n + A_n^*(X_n - X) = -\mathcal{Q}(X) + \hat{F}_n^* \hat{F}_n > 0,$$

$$(X_{n-1} - X_n)A_n + A_n^*(X_{n-1} - X_n) = (F_n - F_{n-1})^*(F_n - F_{n-1}) \geq 0.$$

Since $A_n$ is antistable, we have $X_{n-1} \geq X_n > X$.

We have a non-increasing sequence $\{X_i\}$.

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Since the sequence is bounded below by $X_i > X$. Hence the limit

$$X_+ := \lim_{n\to\infty} X_n$$

exists and is hermitian, and we have $X_+ \geq X$. Passing the limit $n \to \infty$ in (0.9), we get $\mathcal{Q}(X_+) = 0$. So $X_+$ is a solution of (0.7).

Note that $X_+ - X \geq 0$ and

$$(X_+ - X)A_+ + A_+^*(X_+ - X) = -\mathcal{Q}(X)$$
$$+(X_+ - X)BB^*(X_+ - X) > 0 \tag{0.10}$$

hence, $X_+ - X > 0$ and $A_+ = A + BB^* X_+$ is antistable.

$\square$

## Proof for Necessary

There exists a controller such that $\|T_{zw}\|_\infty < \gamma$ only if the following three conditions hold:

(i) there exists a stabilizing solution $X_\infty > 0$ to

$$X_\infty A + A^* X_\infty + X_\infty (B_1 B_1^*/\gamma^2 - B_2 B_2^*) X_\infty + C_1^* C_1 = 0$$

(ii) there exists a stabilizing solution $Y_\infty > 0$ to

$$A Y_\infty + Y_\infty A^* + Y_\infty (C_1^* C_1/\gamma^2 - C_2^* C_2) Y_\infty + B_1 B_1^* = 0$$

(iii)

$$\begin{bmatrix} \gamma Y_\infty^{-1} & I_n \\ I_n & \gamma X_\infty^{-1} \end{bmatrix} > 0 \quad \text{or} \quad \rho(X_\infty Y_\infty) < \gamma^2.$$

*Proof.* Applying Lemma ARE to part (i) of Lemma IC, we conclude that there exists a $Y > Y_1 > 0$ such that

$$AY + YA^* + YC_1^* C_1 Y/\gamma^2 + B_1 B_1^* - \gamma^2 B_2 B_2^* = 0$$

and $A + C_1^* C_1 Y/\gamma^2$ is antistable. Let $X_\infty := \gamma^2 Y^{-1}$, we have

$$X_\infty A + A^* X_\infty + X_\infty (B_1 B_1^*/\gamma^2 - B_2 B_2^*) X_\infty + C_1^* C_1 = 0$$

and

$$\begin{aligned} A + (B_1 B_1^*/\gamma^2 - B_2 B_2^*) X_\infty &= -X_\infty^{-1} (A + C_1^* C_1 X_\infty^{-1}) X_\infty \\ &= -X_\infty^{-1} (A + C_1^* C_1 Y/\gamma^2) X_\infty \end{aligned}$$

is stable.

Similarly, applying Lemma ARE to part (ii) of Lemma IC, we conclude that there exists an $X > X_1 > 0$ such that

$$XA + A^* X + XB_1 B_1^* X/\gamma^2 + C_1^* C_1 - \gamma^2 C_2^* C_2 = 0$$

and $A + B_1 B_1^* X/\gamma^2$ is antistable. Let $Y_\infty := \gamma^2 X^{-1}$, we have

$$AY_\infty + Y_\infty A^* + Y_\infty (C_1^* C_1/\gamma^2 - C_2^* C_2) Y_\infty + B_1 B_1^* = 0 \tag{0.11}$$

and $A + (C_1^* C_1/\gamma^2 - C_2^* C_2) Y_\infty$ is stable.

Finally, note that the rank condition in part (iii) of Lemma IC is automatically satisfied by $r \geq n$, and

$$\begin{aligned} \begin{bmatrix} \gamma Y_\infty^{-1} & I_n \\ I_n & \gamma X_\infty^{-1} \end{bmatrix} &= \begin{bmatrix} X/\gamma & I_n \\ I_n & Y/\gamma \end{bmatrix} \\ &> \begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix} \geq 0. \end{aligned}$$

or $\rho(X_\infty Y_\infty) < \gamma^2$. □

## Proof for Sufficiency

Show $K_{sub}$ renders $\|T_{zw}\|_\infty < \gamma$.

The closed-loop transfer function with $K_{sub}$:

$$T_{zw} = \left[\begin{array}{cc|c} A & B_2F_\infty & B_1 \\ -Z_\infty L_\infty C_2 & \hat{A}_\infty & -Z_\infty L_\infty D_{21} \\ \hline C_1 & D_{12}F_\infty & 0 \end{array}\right] =: \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right]$$

Define

$$P = \begin{bmatrix} \gamma^2 Y_\infty^{-1} & -\gamma^2 Y_\infty^{-1} Z_\infty^{-1} \\ -\gamma^2 (Z_\infty^*)^{-1} Y_\infty^{-1} & \gamma^2 Y_\infty^{-1} Z_\infty^{-1} \end{bmatrix}$$

Then $P > 0$ and

$$PA_c + A_c^* P + PB_c B_c^* P/\gamma^2 + C_c^* C_c = 0.$$

Moreover

$$A_c + B_c B_c^* P/\gamma^2 = \begin{bmatrix} A + B_1 B_1^* Y_\infty^{-1} & B_2 F_\infty - B_1 B_1^* Y_\infty^{-1} Z_\infty^{-1} \\ 0 & A + B_1 B_1^* X_\infty/\gamma^2 + B_2 F_\infty \end{bmatrix}$$

has no eigenvalues on the imaginary axis since

$$A + B_1 B_1^* Y_\infty^{-1} \quad \text{is antistable}$$

and

$$A + B_1 B_1^* X_\infty/\gamma^2 + B_2 F_\infty \quad \text{is stable}$$

By Bounded Real Lemma, $\|T_{zw}\|_\infty < \gamma$.

## Comments

The conditions in Lemma IC are in fact necessary and sufficient.

But the three conditions have to be checked simultaneously. This is because if one finds an $X_1 > 0$ and a $Y_1 > 0$ satisfying conditions $(i)$ and $(ii)$ but not condition $(iii)$, this does not imply that there is no admissible $\mathcal{H}_\infty$ controller since there might be other $X_1 > 0$ and $Y_1 > 0$ that satisfy all three conditions.

For example, consider $\gamma = 1$ and

$$G(s) = \left[\begin{array}{c|cc} -1 & \begin{bmatrix} 1 & 0 \end{bmatrix} & 1 \\ \hline \begin{bmatrix} 1 \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 & \begin{bmatrix} 0 & 1 \end{bmatrix} & 0 \end{array}\right].$$

It is easy to check that $X_1 = Y_1 = 0.5$ satisfy $(i)$ and $(ii)$ but not $(iii)$. Nevertheless, we can show that $\gamma_{opt} = 0.7321$ and thus a suboptimal controller exists for $\gamma = 1$. In fact, we can check that $1 < X_1 < 2,\ 1 < Y_1 < 2$ also satisfy $(i), (ii)$ and $(iii)$. For this reason, Riccati equation approach is usually preferred over the Riccati inequality and LMI approaches whenever possible.

## Example

Consider the feedback system shown in Figure 0.4 with

$$P = \frac{50(s+1.4)}{(s+1)(s+2)}, \quad W_e = \frac{2}{s+0.2}, \quad W_u = \frac{s+1}{s+10}.$$

Design a $K$ to minimize the $\mathcal{H}_\infty$ norm from $w = \begin{bmatrix} d \\ d_i \end{bmatrix}$ to $z = \begin{bmatrix} e \\ \tilde{u} \end{bmatrix}$:

$$\begin{bmatrix} e \\ \tilde{u} \end{bmatrix} = \begin{bmatrix} W_e(I+PK)^{-1} & W_e(I+PK)^{-1}P \\ -W_uK(I+PK)^{-1} & -W_uK(I+PK)^{-1}P \end{bmatrix} \begin{bmatrix} d \\ d_i \end{bmatrix} =: T_{zw} \begin{bmatrix} d \\ d_i \end{bmatrix}.$$

LFT framework:

$$G(s) = \left[\begin{array}{cc:c} W_e & W_eP & -W_eP \\ 0 & 0 & -W_u \\ \hdashline I & P & -P \end{array}\right] = \left[\begin{array}{cccc|cc:c} -0.2 & 2 & 2 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 20 & -20 \\ 0 & 0 & -2 & 0 & 0 & 30 & -30 \\ 0 & 0 & 0 & -10 & 0 & 0 & -3 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & -1 \\ \hdashline 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right].$$

$$\gg [\mathbf{K}, \mathbf{T_{zw}}, \gamma_{\mathbf{subopt}}] = \mathbf{hinfsyn}(\mathbf{G}, \mathbf{n_y}, \mathbf{n_u}, \gamma_{\min}, \gamma_{\max}, \mathbf{tol})$$

where $n_y$ = dimensions of $y$, $n_u$ = dimensions of $u$; $\gamma_{\min}$ = a lower bound, $\gamma_{\max}$ = an upper bound for $\gamma_{\text{opt}}$; and tol is a tolerance to the optimal value. Set $n_y = 1, n_u = 1$, $\gamma_{\min} = 0$, $\gamma_{\max} = 10$, tol $= 0.0001$; we get $\gamma_{\text{subopt}} = 0.7849$ and a suboptimal controller

$$K = \frac{12.82(s/10+1)(s/7.27+1)(s/1.4+1)}{(s/32449447.67+1)(s/22.19+1)(s/1.4+1)(s/0.2+1)}.$$

If we set tol $= 0.01$, we would get $\gamma_{\text{subopt}} = 0.7875$ and a suboptimal controller

$$\tilde{K} = \frac{12.78(s/10+1)(s/7.27+1)(s/1.4+1)}{(s/2335.59+1)(s/21.97+1)(s/1.4+1)(s/0.2+1)}.$$

The only significant difference between $K$ and $\tilde{K}$ is the exact location of the far-away stable controller pole. Figure 0.25 shows the closed-loop frequency response of $\overline{\sigma}\,(T_{zw})$ and Figure 0.26 shows the frequency responses of $S, T, KS$, and $SP$.

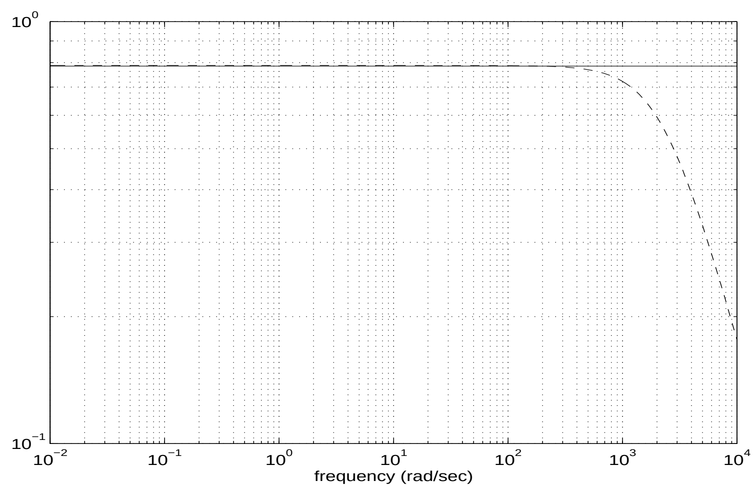


Figure 0.25: The closed-loop frequency responses of $\overline{\sigma}(T_{zw})$ with $K$ (solid line) and $\tilde{K}$ (dashed line)

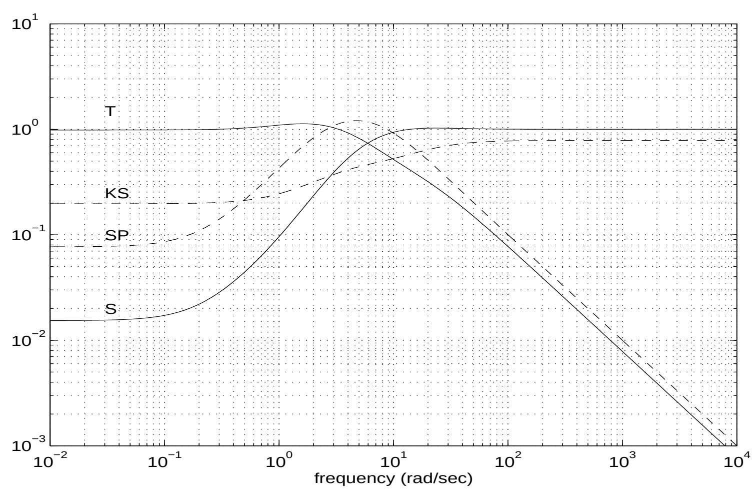


Figure 0.26: The frequency responses of $S, T, KS$, and $SP$ with $K$

Consider again the two-mass/spring/damper system shown in Figure 0.1. Assume that $F_1$ is the control force, $F_2$ is the disturbance force, and the measurements of $x_1$ and $x_2$ are corrupted by measurement noise:

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + W_n \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, \quad W_n = \begin{bmatrix} \frac{0.01(s+10)}{s+100} & 0 \\ 0 & \frac{0.01(s+10)}{s+100} \end{bmatrix}.$$

Our objective is to design a control law so that the effect of the disturbance force $F_2$ on the positions of the two masses, $x_1$ and $x_2$, are reduced in a frequency range $0 \le \omega \le 2$.

The problem can be set up as shown in Figure 0.27, where $W_e = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}$ is the performance weight and $W_u$ is the control weight. In order to limit the control force, we shall choose

$$W_u = \frac{s+5}{s+50}.$$

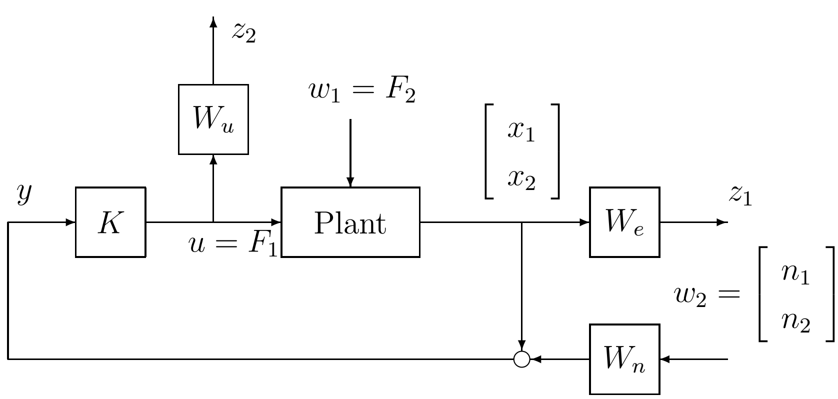


Figure 0.27: Rejecting the disturbance force $F_2$ by a feedback control

Let $u = F_1$, $w = \begin{bmatrix} F_2 \\ n_1 \\ n_2 \end{bmatrix}$:

$$G(s) = \begin{bmatrix} \begin{bmatrix} W_e P_1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} W_e P_2 \\ W_u \end{bmatrix} \\ \begin{bmatrix} P_1 & W_n \end{bmatrix} & P_2 \end{bmatrix}$$

where $P_1$ and $P_2$ denote the transfer matrices from $F_1$ and $F_2$ to $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, respectively.

- $W_1 = \frac{5}{s/2+1}$, $W_2 = 0$: only reject the effect of the disturbance force $F_2$ on the position $x_1$.

  $\|\mathcal{F}_\ell(G, K_2)\|_2 = 2.6584$

  $\|\mathcal{F}_\ell(G, K_2)\|_\infty = 2.6079$

  $\|\mathcal{F}_\ell(G, K_\infty)\|_\infty = 1.6101.$

  This means that the effect of the disturbance force $F_2$ in the desired frequency rang $0 \leq \omega \leq 2$ will be effectively reduced with the $\mathcal{H}_\infty$ controller $K_\infty$ by $5/1.6101 = 3.1054$ times at $x_1$.

- $W_1 = 0$, $W_2 = \frac{5}{s/2+1}$: only reject the effect of the disturbance force $F_2$ on the position $x_2$.

  $\|\mathcal{F}_\ell(G, K_2)\|_2 = 0.1659$

  $\|\mathcal{F}_\ell(G, K_2)\|_\infty = 0.5202$

  $\|\mathcal{F}_\ell(G, K_\infty)\|_\infty = 0.5189.$

  This means that the effect of the disturbance force $F_2$ in the desired frequency rang $0 \leq \omega \leq 2$ will be effectively reduced with the $\mathcal{H}_\infty$ controller $K_\infty$ by $5/0.5189 = 9.6358$ times at $x_2$.

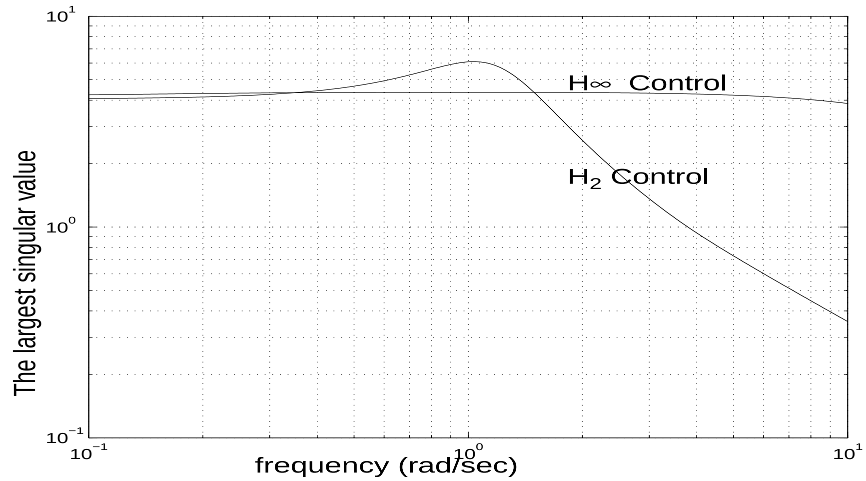


Figure 0.28: The largest singular value plot of the closed-loop system $T_{zw}$ with an $\mathcal{H}_2$ controller and an $\mathcal{H}_\infty$ controller

- $W_1 = W_2 = \frac{5}{s/2+1}$: want to reject the effect of the disturbance force $F_2$ on both $x_1$ and $x_2$.

  $\|\mathcal{F}_\ell(G, K_2)\|_2 = 4.087$

  $\|\mathcal{F}_\ell(G, K_2)\|_\infty = 6.0921$

  $\|\mathcal{F}_\ell(G, K_\infty)\|_\infty = 4.3611.$

  This means that the effect of the disturbance force $F_2$ in the desired frequency rang $0 \leq \omega \leq 2$ will only be effectively reduced with the $\mathcal{H}_\infty$ controller $K_\infty$ by $5/4.3611 = 1.1465$ times at both $x_1$ and $x_2$.

  This result shows clearly that it is very hard to reject the disturbance effect on both positions at the same time. The largest singular value Bode plots of the closed-loop system are shown in Figure 0.28. We note that the $\mathcal{H}_\infty$ controller typically gives a relatively flat frequency response since it tries to minimize the peak of the frequency response. On the other hand, the $\mathcal{H}_2$ controller would typically produce a frequency response that rolls off fast in the high-frequency range but with a large peak in the low-frequency range.

## Optimality and dependence on $\gamma$

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There exists an admissible controller such that $\|T_{zw}\|_\infty < \gamma$ iff the following three conditions hold:

(i) $\exists$ a stabilizing $X_\infty > 0$

(ii) $\exists$ a stabilizing $Y_\infty > 0$

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$

---

- Denote by $\gamma_o$ the infimum over all $\gamma$ such that conditions (i)-(iii) are satisfied.
- Descriptor formulae can be obtained for $\gamma = \gamma_o$.
- As $\gamma \to \infty$, $H_\infty \to H_2$, $X_\infty \to X_2$, etc., and $K_{sub} \to K_2$.
- If $\gamma_2 \geq \gamma_1 > \gamma_0$ then $X_\infty(\gamma_1) \geq X_\infty(\gamma_2)$ and $Y_\infty(\gamma_1) \geq Y_\infty(\gamma_2)$.
- Thus $X_\infty$ and $Y_\infty$ are decreasing functions of $\gamma$, as is $\rho(X_\infty Y_\infty)$.
- At $\gamma = \gamma_o$, any one of the 3 conditions can fail.
- It is most likely that condition (iii) will fail first.
- To understand this, consider (i) and let $\gamma_1$ be the largest $\gamma$ for which $H_\infty$ fails to be in $dom(Ric)$, because it fails to have either the stability property or the complementarity property. The same remarks will apply to (ii) by duality.
- If the stability property fails at $\gamma = \gamma_1$, then $H_\infty \notin dom(Ric)$ but $Ric$ can be extended to obtain $X_\infty$ and the controller $u = -B_2^* X_\infty x$ is stabilizing and makes $\|T_{zw}\|_\infty = \gamma_1$. The stability property will also not hold for any $\gamma \leq \gamma_1$, and no controller whatsoever exists which makes $\|T_{zw}\|_\infty < \gamma_1$.

- In other words, if stability breaks down first then the infimum over stabilizing controllers equals the infimum over all controllers, stabilizing or otherwise.
- In view of this, we would expect that typically complementarity would fail first.

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- Complementarity failing at $\gamma = \gamma_1$ means $\rho(X_\infty) \to \infty$ as $\gamma \to \gamma_1$ so condition (iii) would fail at even larger values of $\gamma$, unless the eigenvectors associated with $\rho(X_\infty)$ as $\gamma \to \gamma_1$ are in the null space of $Y_\infty$.
- Thus condition (iii) is the most likely of all to fail first.

# $\mathcal{H}_\infty$ Controller Structure

$$K_{sub}(s) := \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

$$\hat{A}_\infty := A + \gamma^{-2} B_1 B_1^* X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2$$

$$F_\infty := -B_2^* X_\infty, \quad L_\infty := -Y_\infty C_2^*, \quad Z_\infty := (I - \gamma^{-2} Y_\infty X_\infty)^{-1}$$

---

$$\dot{\hat{x}} = A\hat{x} + B_1 \hat{w}_{worst} + B_2 u + Z_\infty L_\infty (C_2 \hat{x} - y)$$

$$u = F_\infty \hat{x}, \qquad \hat{w}_{worst} = \gamma^{-2} B_1^* X_\infty \hat{x}$$

---

1) $\hat{w}_{worst}$ is the estimate of $w_{worst}$

2) $Z_\infty L_\infty$ is the filter gain for the OE problem of estimating $F_\infty x$ in the presence of the "worst-case" $w$, $w_{worst}$

3) The $\mathcal{H}_\infty$ controller has a separation interpretation

---

Optimal Controller:

$$(I - \gamma_{opt}^{-2} Y_\infty X_\infty)\dot{\hat{x}} = A_s \hat{x} - L_\infty y \tag{0.12}$$

$$u = F_\infty \hat{x} \tag{0.13}$$

$$A_s := A + B_2 F_\infty + L_\infty C_2$$
$$+ \gamma_{opt}^{-2} Y_\infty A^* X_\infty + \gamma_{opt}^{-2} B_1 B_1^* X_\infty + \gamma_{opt}^{-2} Y_\infty C_1^* C_1$$

See the example below.

## Example

$$G(s) = \left[\begin{array}{c|cc} a & \begin{bmatrix} 1 & 0 \end{bmatrix} & 1 \\ \hline \begin{bmatrix} 1 \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 & \begin{bmatrix} 0 & 1 \end{bmatrix} & 0 \end{array}\right].$$

Then all assumptions for output feedback problem are satisfied and

$$H_\infty = \begin{bmatrix} a & \frac{1-\gamma^2}{\gamma^2} \\ -1 & -a \end{bmatrix}, \quad J_\infty = \begin{bmatrix} a & \frac{1-\gamma^2}{\gamma^2} \\ -1 & -a \end{bmatrix}.$$

The eigenvalues of $H_\infty$ and $J_\infty$ are given, respectively, by

$$\sigma(H_\infty) = \left\{ \pm \frac{\sqrt{(a^2+1)\gamma^2 - 1}}{\gamma} \right\}, \quad \sigma(J_\infty) = \left\{ \pm \frac{\sqrt{(a^2+1)\gamma^2 - 1}}{\gamma} \right\}.$$

If $\gamma > \dfrac{1}{\sqrt{a^2+1}}$, then $\mathcal{X}_-(H_\infty)$ and $\mathcal{X}_-(J_\infty)$ exist and

$$\mathcal{X}_-(H_\infty) = \text{Im} \begin{bmatrix} \frac{\sqrt{(a^2+1)\gamma^2-1} - a\gamma}{\gamma} \\ 1 \end{bmatrix}$$

$$\mathcal{X}_-(J_\infty) = \text{Im} \begin{bmatrix} \frac{\sqrt{(a^2+1)\gamma^2-1} - a\gamma}{\gamma} \\ 1 \end{bmatrix}.$$

Note that if $\gamma > 1$, then $H_\infty \in dom(Ric)$, $J_\infty \in dom(Ric)$, and

$$X_\infty = \frac{\gamma}{\sqrt{(a^2+1)\gamma^2 - 1} - a\gamma} > 0$$

$$Y_\infty = \frac{\gamma}{\sqrt{(a^2+1)\gamma^2 - 1} - a\gamma} > 0.$$

It can be shown that

$$\rho(X_\infty Y_\infty) = \frac{\gamma^2}{(\sqrt{(a^2+1)\gamma^2 - 1} - a\gamma)^2} < \gamma^2$$

is satisfied if and only if

$$\gamma > \sqrt{a^2+2} + a.$$

So condition (iii) will fail before either (i) or (ii) fails.

The optimal $\gamma$ for the output feedback is given by

$$\gamma_{opt} = \sqrt{a^2+2} + a$$

and the optimal controller given by the descriptor formula in equations (0.12) and (0.13) is a constant. In fact,

$$u_{opt} = -\frac{\gamma_{opt}}{\sqrt{(a^2+1)\gamma_{opt}^2 - 1} - a\gamma_{opt}}\, y.$$

For instance, let $a = -1$ then $\gamma_{opt} = \sqrt{3} - 1 = 0.7321$ and $u_{opt} = -0.7321\, y$. Further,

$$T_{zw} = \left[\begin{array}{c|cc} -1.7321 & 1 & -0.7321 \\ \hline 1 & 0 & 0 \\ -0.7321 & 0 & -0.7321 \end{array}\right].$$

It is easy to check that $\|T_{zw}\|_\infty = 0.7321$.

## An Optimal Controller

There exists an admissible controller such that $\|T_{zw}\|_\infty \le \gamma$ iff the following three conditions hold:

(i) there exists a full column rank matrix $\begin{bmatrix} X_{\infty 1} \\ X_{\infty 2} \end{bmatrix} \in \mathbb{R}^{2n\times n}$ such that

$$H_\infty \begin{bmatrix} X_{\infty 1} \\ X_{\infty 2} \end{bmatrix} = \begin{bmatrix} X_{\infty 1} \\ X_{\infty 2} \end{bmatrix} T_X, \quad \operatorname{Re}\lambda_i(T_X) \le 0 \ \forall i$$

and

$$X_{\infty 1}^* X_{\infty 2} = X_{\infty 2}^* X_{\infty 1};$$

(ii) there exists a full column rank matrix $\begin{bmatrix} Y_{\infty 1} \\ Y_{\infty 2} \end{bmatrix} \in \mathbb{R}^{2n\times n}$ such that

$$J_\infty \begin{bmatrix} Y_{\infty 1} \\ Y_{\infty 2} \end{bmatrix} = \begin{bmatrix} Y_{\infty 1} \\ Y_{\infty 2} \end{bmatrix} T_Y, \quad \operatorname{Re}\lambda_i(T_Y) \le 0 \ \forall i$$

and

$$Y_{\infty 1}^* Y_{\infty 2} = Y_{\infty 2}^* Y_{\infty 1};$$

(iii) $\begin{bmatrix} X_{\infty 2}^* X_{\infty 1} & \gamma^{-1} X_{\infty 2}^* Y_{\infty 2} \\ \gamma^{-1} Y_{\infty 2}^* X_{\infty 2} & Y_{\infty 2}^* Y_{\infty 1} \end{bmatrix} \ge 0.$

Moreover, when these conditions hold, one such controller is

$$K_{opt}(s) := C_K(sE_K - A_K)^+ B_K$$

where

$$\begin{aligned} E_K &:= Y_{\infty 1}^* X_{\infty 1} - \gamma^{-2} Y_{\infty 2}^* X_{\infty 2} \\ B_K &:= Y_{\infty 2}^* C_2^* \\ C_K &:= -B_2^* X_{\infty 2} \\ A_K &:= E_K T_X - B_K C_2 X_{\infty 1} = T_Y^* E_K + Y_{\infty 1}^* B_2 C_K. \end{aligned}$$

# $\mathcal{H}_\infty$ Control: General Case

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Assumptions:

(A1) $(A, B_2)$ is stabilizable and $(C_2, A)$ is detectable;

(A2) $D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}$;

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega$;

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all $\omega$.

$$\begin{aligned}
R &:= D_{1\bullet}^* D_{1\bullet} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad D_{1\bullet} := [D_{11} \quad D_{12}] \\
\tilde{R} &:= D_{\bullet 1} D_{\bullet 1}^* - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad D_{\bullet 1} := \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} \\
H_\infty &:= \begin{bmatrix} A & 0 \\ -C_1^* C_1 & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -C_1^* D_{1\bullet} \end{bmatrix} R^{-1} \begin{bmatrix} D_{1\bullet}^* C_1 & B^* \end{bmatrix} \\
J_\infty &:= \begin{bmatrix} A^* & 0 \\ -B_1 B_1^* & -A \end{bmatrix} - \begin{bmatrix} C^* \\ -B_1 D_{\bullet 1}^* \end{bmatrix} \tilde{R}^{-1} \begin{bmatrix} D_{\bullet 1} B_1^* & C \end{bmatrix}
\end{aligned}$$

$$X_\infty := Ric(H_\infty) \qquad Y_\infty := Ric(J_\infty)$$

$$
\begin{aligned}
F &:= \begin{bmatrix} F_{1\infty} \\ F_{2\infty} \end{bmatrix} := -R^{-1}\left[D_{1\bullet}^* C_1 + B^* X_\infty\right] \\
L &:= \begin{bmatrix} L_{1\infty} & L_{2\infty} \end{bmatrix} := -[B_1 D_{\bullet 1}^* + Y_\infty C^*]\tilde{R}^{-1}
\end{aligned}
$$

$D$, $F_{1\infty}$, and $L_{1\infty}$ are Partitioned as follows:

$$
\left[\begin{array}{c|c} & F' \\ \hline L' & D \end{array}\right] = \left[\begin{array}{c|ccc} & F_{11\infty}^* & F_{12\infty}^* & F_{2\infty}^* \\ \hline L_{11\infty}^* & D_{1111} & D_{1112} & 0 \\ L_{12\infty}^* & D_{1121} & D_{1122} & I \\ L_{2\infty}^* & 0 & I & 0 \end{array}\right].
$$

---

There exists a stabilizing controller $K(s)$ such that

$$
\|\mathcal{F}_\ell(G, K)\|_\infty < \gamma
$$

if and only if

(i) $\gamma > max(\overline{\sigma}[D_{1111}, D_{1112}, ], \overline{\sigma}[D_{1111}^*, D_{1121}^*]);$

(ii) $H_\infty \in dom(Ric)$ with $X_\infty = Ric(H_\infty) \geq 0;$

(iii) $J_\infty \in dom(Ric)$ with $Y_\infty = Ric(J_\infty) \geq 0;$

(iv) $\rho(X_\infty Y_\infty) < \gamma^2.$

$$
K = \mathcal{F}_\ell(M_\infty, Q), \quad Q \in \mathcal{RH}_\infty, \quad \|Q\|_\infty < \gamma
$$

where

$$
M_\infty = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array}\right]
$$

$$
\hat{D}_{11} = -D_{1121} D_{1111}^* (\gamma^2 I - D_{1111} D_{1111}^*)^{-1} D_{1112} - D_{1122},
$$

$\hat{D}_{12} \in \mathbb{R}^{m_2 \times m_2}$ and $\hat{D}_{21} \in \mathbb{R}^{p_2 \times p_2}$ are any matrices satisfying

$$\begin{aligned}
\hat{D}_{12}\hat{D}_{12}^* &= I - D_{1121}(\gamma^2 I - D_{1111}^* D_{1111})^{-1} D_{1121}^*, \\
\hat{D}_{21}^*\hat{D}_{21} &= I - D_{1112}^*(\gamma^2 I - D_{1111} D_{1111}^*)^{-1} D_{1112},
\end{aligned}$$

and

$$\begin{aligned}
\hat{B}_2 &= Z_\infty (B_2 + L_{12\infty})\hat{D}_{12}, \\
\hat{C}_2 &= -\hat{D}_{21}(C_2 + F_{12\infty}), \\
\hat{B}_1 &= -Z_\infty L_{2\infty} + \hat{B}_2 \hat{D}_{12}^{-1} \hat{D}_{11}, \\
\hat{C}_1 &= F_{2\infty} + \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2, \\
\hat{A} &= A + BF + \hat{B}_1 \hat{D}_{21}^{-1}\hat{C}_2
\end{aligned}$$

where

$$Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1}.$$

**Some Special Cases:**

- $D_{12} = I$. Then (i) becomes $\gamma > \overline{\sigma}(D_{1121})$ and

$$\hat{D}_{11} = -D_{1122}, \quad \hat{D}_{12}\hat{D}_{12}^* = I - \gamma^{-2} D_{1121} D_{1121}^*, \quad \hat{D}_{21}^*\hat{D}_{21} = I.$$

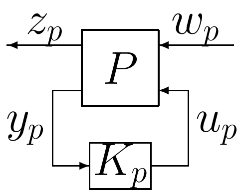
- $D_{21} = I$. Then (i) becomes $\gamma > \overline{\sigma}(D_{1112})$ and

$$\hat{D}_{11} = -D_{1122}, \quad \hat{D}_{12}\hat{D}_{12}^* = I, \quad \hat{D}_{21}^*\hat{D}_{21} = I - \gamma^{-2} D_{1112}^* D_{1112}.$$

- $D_{12} = I$ & $D_{21} = I$. Then (i) drops out and

$$\hat{D}_{11} = -D_{1122}, \quad \hat{D}_{12}\hat{D}_{12}^* = I, \quad \hat{D}_{21}^*\hat{D}_{21} = I.$$

# Relaxing Assumptions




$$P(s) = \left[\begin{array}{c|cc} A_p & B_{p1} & B_{p2} \\ \hline C_{p1} & D_{p11} & D_{p12} \\ C_{p2} & D_{p21} & D_{p22} \end{array}\right]$$

Assume $D_{p12}$ *has full column rank and* $D_{p21}$ *has full row rank:*

**Normalize $D_{12}$ and $D_{21}$**

Perform SVD

$$D_{p12} = U_p \begin{bmatrix} 0 \\ I \end{bmatrix} R_p, \quad D_{p21} = \tilde{R}_p \begin{bmatrix} 0 & I \end{bmatrix} \tilde{U}_p$$

such that $U_p$ and $\tilde{U}_p$ are square and unitary. Now let

$$z_p = U_p z, \quad w_p = \tilde{U}_p^* w, \quad y_p = \tilde{R}_p y, \quad u_p = R_p u$$

$$K(s) = R_p K_p(s) \tilde{R}_p$$

$$\begin{aligned} G(s) &= \begin{bmatrix} U_p^* & 0 \\ 0 & \tilde{R}_p^{-1} \end{bmatrix} P(s) \begin{bmatrix} \tilde{U}_p^* & 0 \\ 0 & R_p^{-1} \end{bmatrix} \\ &= \left[\begin{array}{c|cc} A_p & B_{p1}\tilde{U}_p^* & B_{p2}R_p^{-1} \\ \hline U_p^* C_{p1} & U_p^* D_{p11} \tilde{U}_p^* & U_p^* D_{p12} R_p^{-1} \\ \tilde{R}_p^{-1} C_{p2} & \tilde{R}_p^{-1} D_{p21} \tilde{U}_p^* & \tilde{R}_p^{-1} D_{p22} R_p^{-1} \end{array}\right] \\ &=: \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \end{aligned}$$

Then

$$D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad D_{21} = \begin{bmatrix} 0 & I \end{bmatrix},$$

$$\|\mathcal{F}_\ell(P, K_p)\|_\infty = \|\mathcal{F}_\ell(G, K)\|_\infty$$

**Remove the Assumption $D_{22} = 0$**

Suppose $K(s)$ is a controller for $G$ with $D_{22}$ set to zero. Then the controller for $D_{22} \neq 0$ is $K(I + D_{22}K)^{-1}$.

**Relaxing A3 and A4**

Complicated. Suppose that

$$G = \left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right]$$

which violates both A3 and A4 and corresponds to the robust stabilization of an integrator. If the controller $u = -\epsilon x$ where $\epsilon > 0$ is used, then

$$T_{zw} = \frac{-\epsilon s}{s + \epsilon}, \quad \text{with } \|T_{zw}\|_\infty = \epsilon.$$

Hence the norm can be made arbitrarily small as $\epsilon \to 0$, but $\epsilon = 0$ is not stabilizing.

**Relaxing A1**

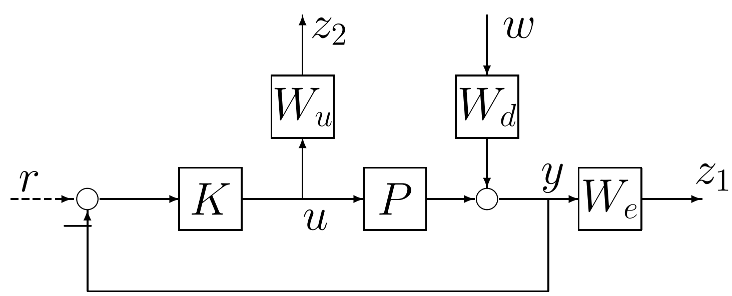
Complicated.

**Relaxing A2**

Singular Problem: reduced ARE or LMI, . . .

# $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Integral Control

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ design frameworks do not in general produce integral control.




Ways to achieve the integral control:

1. introduce an integral in the performance weight $W_e$:

$$z_1 = W_e(I + PK)^{-1}W_d w.$$

Now if the norm (2-norm or $\infty$-norm) between $w$ and $z_1$ is finite, then $K$ must have a pole at $s = 0$ which is the zero of the sensitivity function.

The standard $\mathcal{H}_2$ (or $\mathcal{H}_\infty$) control theory can not be applied to this problem formulation directly because the pole $s = 0$ of $W_e$ becomes an uncontrollable pole of the feedback system (A1 is violated).

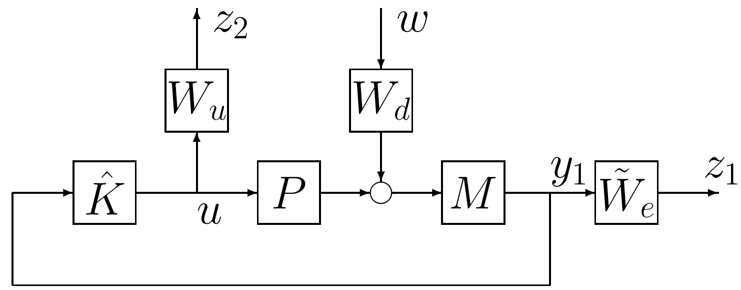
Suppose $W_e$ can be factorized as follows

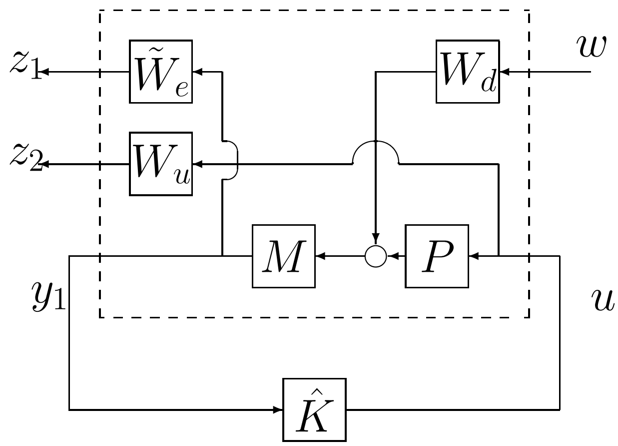
$$W_e = \tilde{W}_e(s)M(s)$$

where $M(s)$ is proper, containing all the imaginary axis poles of $W_e$, and $M^{-1}(s) \in \mathcal{RH}_\infty$, $\tilde{W}_e(s)$ is stable and minimum phase. Now suppose there exists a controller $K(s)$ which contains the same imaginary axis poles that achieves the performance. Then without loss of generality, $K$ can be factorized as

$$K(s) = -\hat{K}(s)M(s)$$

Now the problem can be reformulated as




A simple numerical example:

$$P = \frac{s-2}{(s+1)(s-3)} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 3 & 2 & 1 \\ \hline -2 & 1 & 0 \end{array}\right], \quad W_d = 1,$$

$$W_u = \frac{s+10}{s+100} = \left[\begin{array}{c|c} -100 & -90 \\ \hline 1 & 1 \end{array}\right], \quad W_e = \frac{1}{s}.$$

Then we can choose without loss of generality that

$$M = \frac{s+\alpha}{s}, \quad \tilde{W}_e = \frac{1}{s+\alpha}, \quad \alpha > 0.$$

This gives the following generalized system

$$G(s) = \left[\begin{array}{ccccc|cc} -\alpha & 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & -100 & 0 & 0 & 0 & 0 & -90 \\ 0 & 0 & 0 & -2\alpha & \alpha & \alpha & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 1 & 0 \end{array}\right]$$

suboptimal $\mathcal{H}_\infty$ controller $\hat{K}_\infty$:

$$\hat{K}_\infty = \frac{-2060381.4(s+1)(s+\alpha)(s+100)(s-0.1557)}{(s+\alpha)^2(s+32.17)(s+262343)(s-19.89)}$$

which gives the closed-loop $\infty$ norm 7.854.

$$K_\infty = -\hat{K}_\infty(s)M(s) = \frac{2060381.4(s+1)(s+100)(s-0.1557)}{s(s+32.17)(s+262343)(s-19.89)}$$

$$\approx \frac{7.85(s+1)(s+100)(s-0.1557)}{s(s+32.17)(s-19.89)}$$

An optimal $\mathcal{H}_2$ controller

$$\hat{K}_2 = \frac{-43.487(s+1)(s+\alpha)(s+100)(s-0.069)}{(s+\alpha)^2(s^2+30.94s+411.81)(s-7.964)}$$

and

$$K_2(s) = -\hat{K}_2(s)M(s) = \frac{43.487(s+1)(s+100)(s-0.069)}{s(s^2+30.94s+411.81)(s-7.964)}.$$

---

2. An approximate integral control:

$$W_e = \tilde{W}_e = \frac{1}{s+\epsilon}, \quad M(s) = 1$$

for a sufficiently small $\epsilon > 0$. For example, a controller for $\epsilon = 0.001$ is given by

$$K_\infty = \frac{316880(s+1)(s+100)(s-0.1545)}{(s+0.001)(s+32)(s+40370)(s-20)}$$
$$\approx \frac{7.85(s+1)(s+100)(s-0.1545)}{s(s+32)(s-20)}$$

which gives the closed-loop $\mathcal{H}_\infty$ norm of 7.85.

$$K_2 = \frac{43.47(s+1)(s+100)(s-0.0679)}{(s+0.001)(s^2+30.93s+411.7)(s-7.9718)}.$$

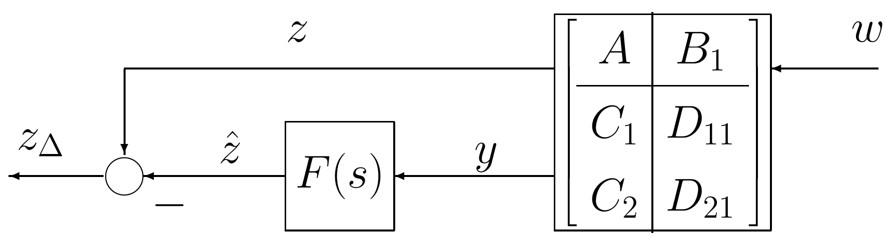
# $\mathcal{H}_\infty$ Filtering

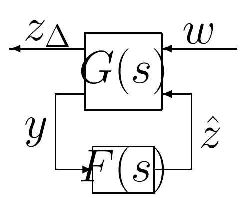
$$
\begin{aligned}
\dot{x} &= Ax + B_1 w(t), \qquad x(0) = 0 \\
y &= C_2 x + D_{21} w(t) \\
z &= C_1 x, \quad B_1 D_{21}^* = 0, \ D_{21} D_{21}^* = I
\end{aligned}
$$

**$\mathcal{H}_\infty$ Filtering**: Given a $\gamma > 0$, find a causal filter $F(s) \in \mathcal{RH}_\infty$ if it exists such that

$$
J := \sup_{w \in \mathcal{L}_2[0,\infty)} \frac{\|z - \hat{z}\|_2^2}{\|w\|_2^2} < \gamma^2
$$

with $\hat{z} = F(s)y$.




$$
G(s) = \left[\begin{array}{c|cc} A & B_1 & 0 \\ \hline C_1 & D_{11} & -I \\ C_2 & D_{21} & 0 \end{array}\right]
$$

This can be regarded as a $\mathcal{H}_\infty$ problem without internal stability.

---

There exists a causal filter $F(s) \in \mathcal{RH}_\infty$ such that $J < \gamma^2$ if and only if $J_\infty \in dom(Ric)$ and $Y_\infty = Ric(J_\infty) \geq 0$

$$
\hat{z} = F(s)y = \left[\begin{array}{c|c} A - Y_\infty C_2^* C_2 & Y_\infty C_2^* \\ \hline C_1 & 0 \end{array}\right] y
$$

where $Y_\infty$ is the stabilizing solution to

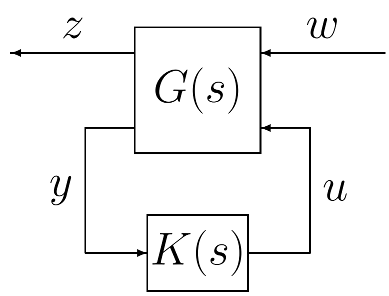
$$
Y_\infty A^* + A Y_\infty + Y_\infty(\gamma^{-2} C_1^* C_1 - C_2^* C_2) Y_\infty + B_1 B_1^* = 0.
$$

# Chapter 15: $\mathcal{H}_\infty$ Controller Reduction

- problem formulation
- additive reduction
- coprime factor reduction

## Problem Formulation

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$




All stabilizing controllers satisfying $\|T_{zw}\|_\infty < \gamma$:

$$K = \mathcal{F}_\ell(M_\infty, Q), \quad Q \in \mathcal{RH}_\infty, \quad \|Q\|_\infty < \gamma$$

where $M_\infty$ is of the form

$$M_\infty = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right]$$

such that $\hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1$ and $\hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2$ are both stable, i.e., $M_{12}^{-1}$ and $M_{21}^{-1}$ are both stable.

Find a controller $\hat{K}$ with a minimal order such that $\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma$.

$$\Updownarrow$$

Find a stable $Q$ such that $K = \mathcal{F}_\ell(M_\infty, Q)$ has minimal order and $\|Q\|_\infty < \gamma$.

## Additive Reduction

Consider the class of (reduced order) controllers:

$$\hat{K} = K_0 + W_2 \Delta W_1, \quad \Delta \in \mathcal{RH}_\infty$$

$$W_1, W_1^{-1}, W_2, W_2^{-1} \in \mathcal{RH}_\infty$$

such that $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$

$\hat{K}$ and $K_0$ have the same right half plane poles.

Then

$$\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma$$

$$\Updownarrow$$

$$\exists\ Q \in \mathcal{RH}_\infty \text{ with } \|Q\|_\infty < \gamma \text{ such that } \hat{K} = \mathcal{F}_\ell(M_\infty, Q).$$

$$\Downarrow$$

$$Q = \mathcal{F}_\ell(\bar{K}_a^{-1}, \hat{K}), \quad \bar{K}_a^{-1} := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} M_\infty^{-1} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

$$\begin{aligned} \|Q\|_\infty < \gamma \iff & \left\|\mathcal{F}_\ell(\bar{K}_a^{-1}, \hat{K})\right\|_\infty < \gamma \\ \iff & \left\|\mathcal{F}_\ell(\bar{K}_a^{-1}, K_0 + W_2 \Delta W_1)\right\|_\infty < \gamma \\ \iff & \left\|\mathcal{F}_\ell(\tilde{R}, \Delta)\right\|_\infty < 1 \end{aligned}$$

where

$$\tilde{R} = \begin{bmatrix} \gamma^{-1/2} I & 0 \\ 0 & W_1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} \gamma^{-1/2} I & 0 \\ 0 & W_2 \end{bmatrix}$$

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \mathcal{S}(\bar{K}_a^{-1}, \begin{bmatrix} K_o & I \\ I & 0 \end{bmatrix}).$$

---

Redheffer's Lemma: $\left\|\tilde{R}\right\|_\infty \le 1$ and $\|\Delta\|_\infty < 1 \Rightarrow \left\|\mathcal{F}_\ell(\tilde{R}, \Delta)\right\|_\infty < 1.$

---

Suppose $W_1$ and $W_2$ are stable, minimum phase and invertible transfer matrices such that $\tilde{R}$ is a contraction. Let $K_0$ be a stabilizing controller such that $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$. Then $\hat{K}$ is also a stabilizing controller such that $\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma$ if

$$\|\Delta\|_\infty = \left\|W_2^{-1}(\hat{K} - K_0)W_1^{-1}\right\|_\infty < 1.$$

---

$\tilde{R}$ can always be made contractive for sufficiently small $W_1$ and $W_2$. We would like to select the "largest" $W_1$ and $W_2$.

Assume $\|R_{22}\|_\infty < \gamma$ and define

$$L = \begin{bmatrix} L_1 & L_2 \\ L_2^\sim & L_3 \end{bmatrix} = \mathcal{F}_\ell(\left[\begin{array}{cc:cc} 0 & -R_{11} & 0 & R_{12} \\ -R_{11}^\sim & 0 & R_{21}^\sim & 0 \\ \hdashline 0 & R_{21} & 0 & -R_{22} \\ R_{12}^\sim & 0 & -R_{22}^\sim & 0 \end{array}\right], \gamma^{-1} I).$$

Then $\tilde{R}$ is a contraction if $W_1$ and $W_2$ satisfy

$$\begin{bmatrix} (W_1^\sim W_1)^{-1} & 0 \\ 0 & (W_2 W_2^\sim)^{-1} \end{bmatrix} \geq \begin{bmatrix} L_1 & L_2 \\ L_2^\sim & L_3 \end{bmatrix}.$$

An algorithm that maximizes $\det(W_1^\sim W_1)\det(W_2 W_2^\sim)$ has been developed by Goddard and Glover [1993].

## Coprime Factor Reduction

All controllers such that $\|T_{zw}\|_\infty < \gamma$ can also be written as

$$\begin{aligned} K(s) = \mathcal{F}_\ell(M_\infty, Q) &= (\Theta_{11}Q + \Theta_{12})(\Theta_{21}Q + \Theta_{22})^{-1} := UV^{-1} \\ &= (Q\tilde{\Theta}_{12} + \tilde{\Theta}_{22})^{-1}(Q\tilde{\Theta}_{11} + \tilde{\Theta}_{21}) := \tilde{V}^{-1}\tilde{U} \end{aligned}$$

where $Q \in \mathcal{RH}_\infty$, $\|Q\|_\infty < \gamma$, and $UV^{-1}$ and $\tilde{V}^{-1}\tilde{U}$ are respectively right and left coprime factorizations over $\mathcal{RH}_\infty$, and

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 & \hat{B}_2 - \hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22} & \hat{B}_1\hat{D}_{21}^{-1} \\ \hline \hat{C}_1 - \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2 & \hat{D}_{12} - \hat{D}_{11}\hat{D}_{21}^{-1}\hat{D}_{22} & \hat{D}_{11}\hat{D}_{21}^{-1} \\ -\hat{D}_{21}^{-1}\hat{C}_2 & -\hat{D}_{21}^{-1}\hat{D}_{22} & \hat{D}_{21}^{-1} \end{array}\right]$$

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_{11} & \tilde{\Theta}_{12} \\ \tilde{\Theta}_{21} & \tilde{\Theta}_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1 & \hat{B}_1 - \hat{B}_2\hat{D}_{12}^{-1}\hat{D}_{11} & -\hat{B}_2\hat{D}_{12}^{-1} \\ \hline \hat{C}_2 - \hat{D}_{22}\hat{D}_{12}^{-1}\hat{C}_1 & \hat{D}_{21} - \hat{D}_{22}\hat{D}_{12}^{-1}\hat{D}_{11} & -\hat{D}_{22}\hat{D}_{12}^{-1} \\ \hat{D}_{12}^{-1}\hat{C}_1 & \hat{D}_{12}^{-1}\hat{D}_{11} & \hat{D}_{12}^{-1} \end{array}\right]$$

$$\Theta^{-1} = \left[\begin{array}{c|cc} \hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1 & \hat{B}_2\hat{D}_{12}^{-1} & \hat{B}_1 - \hat{B}_2\hat{D}_{12}^{-1}\hat{D}_{11} \\ \hline -\hat{D}_{12}^{-1}\hat{C}_1 & \hat{D}_{12}^{-1} & -\hat{D}_{12}^{-1}\hat{D}_{11} \\ \hat{C}_2 - \hat{D}_{22}\hat{D}_{12}^{-1}\hat{C}_1 & \hat{D}_{22}\hat{D}_{12}^{-1} & \hat{D}_{21} - \hat{D}_{22}\hat{D}_{12}^{-1}\hat{D}_{11} \end{array}\right]$$

$$\tilde{\Theta}^{-1} = \left[\begin{array}{c|cc} \hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 & -\hat{B}_1\hat{D}_{21}^{-1} & \hat{B}_2 - \hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22} \\ \hline \hat{D}_{21}^{-1}\hat{C}_2 & \hat{D}_{21}^{-1} & \hat{D}_{21}^{-1}\hat{D}_{22} \\ \hat{C}_1 - \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2 & -\hat{D}_{11}\hat{D}_{21}^{-1} & \hat{D}_{12} - \hat{D}_{11}\hat{D}_{21}^{-1}\hat{D}_{22} \end{array}\right].$$

Let $K_0 = \Theta_{12}\Theta_{22}^{-1}$ be the central $\mathcal{H}_\infty$ controller: $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$
Let $\hat{U}, \hat{V} \in \mathcal{RH}_\infty$ with $\det \hat{V}(\infty) \neq 0$ be such that

$$\left\| \begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix} \Theta^{-1} \left( \begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right) \right\|_\infty < 1/\sqrt{2}.$$

Then $\hat{K} = \hat{U}\hat{V}^{-1}$ is also a stabilizing controller and $\|\mathcal{F}_\ell(G, \hat{K})\|_\infty < \gamma$.

Note that $K$ is a stabilizing controller such that $\|T_{zw}\|_\infty < \gamma$ if and only if there exists a $Q \in \mathcal{RH}_\infty$ with $\|Q\|_\infty < \gamma$ such that

$$\begin{bmatrix} U \\ V \end{bmatrix} := \begin{bmatrix} \Theta_{11}Q + \Theta_{12} \\ \Theta_{21}Q + \Theta_{22} \end{bmatrix} = \Theta \begin{bmatrix} Q \\ I \end{bmatrix} \tag{0.14}$$

and

$$K = UV^{-1}.$$

Define

$$\Delta := \begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix} \Theta^{-1} \left( \begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right)$$

and partition $\Delta$ as

$$\Delta := \begin{bmatrix} \Delta_U \\ \Delta_V \end{bmatrix}.$$

Then

$$\begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} = \begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \Theta \begin{bmatrix} \gamma I & 0 \\ 0 & I \end{bmatrix} \Delta = \Theta \begin{bmatrix} -\gamma\Delta_U \\ I - \Delta_V \end{bmatrix}$$

and

$$\begin{bmatrix} \hat{U}(I - \Delta_V)^{-1} \\ \hat{V}(I - \Delta_V)^{-1} \end{bmatrix} = \Theta \begin{bmatrix} -\gamma\Delta_U(I - \Delta_V)^{-1} \\ I \end{bmatrix}.$$

Define

$$U := \hat{U}(I - \Delta_V)^{-1}, \quad V := \hat{V}(I - \Delta_V)^{-1}$$

$$Q := -\gamma\Delta_U(I-\Delta_V)^{-1}$$

Then $\hat{K} = \hat{U}\hat{V}^{-1} = UV^{-1}$ and

$$\begin{aligned} Q := -\gamma\Delta_U(I-\Delta_V)^{-1} &= -\gamma \begin{bmatrix} I & 0 \end{bmatrix} \Delta \left( I - \begin{bmatrix} 0 & I \end{bmatrix} \Delta \right)^{-1} \\ &= -\gamma \mathcal{F}_\ell \left( \begin{bmatrix} 0 & \begin{bmatrix} I & 0 \end{bmatrix} \\ I/\sqrt{2} & \begin{bmatrix} 0 & I/\sqrt{2} \end{bmatrix} \end{bmatrix}, \sqrt{2}\Delta \right) \end{aligned}$$

Again by Redheffer's Lemma, $\|\Delta_U(I-\Delta_V)^{-1}\|_\infty < 1$ since

$$\begin{bmatrix} 0 & \begin{bmatrix} I & 0 \end{bmatrix} \\ I/\sqrt{2} & \begin{bmatrix} 0 & I/\sqrt{2} \end{bmatrix} \end{bmatrix}$$

is a contraction and $\left\|\sqrt{2}\Delta\right\|_\infty < 1$.

$$\Longrightarrow \|Q\|_\infty = \left\|\gamma\Delta_U(I-\Delta_V)^{-1}\right\|_\infty < \gamma$$

Therefore $\|\mathcal{F}_\ell(G,\hat{K})\|_\infty < \gamma$.

---

Let $K_0 = \tilde{\Theta}_{22}^{-1}\tilde{\Theta}_{21}$ be the central $\mathcal{H}_\infty$ controller: $\|\mathcal{F}_\ell(G,K_0)\|_\infty < \gamma$

Let $\hat{\hat{U}}, \hat{\hat{V}} \in \mathcal{RH}_\infty$ with $\det \hat{\hat{V}}(\infty) \neq 0$ be such that

$$\left\| \left( \begin{bmatrix} \tilde{\Theta}_{21} & \tilde{\Theta}_{22} \end{bmatrix} - \begin{bmatrix} \hat{\hat{U}} & \hat{\hat{V}} \end{bmatrix} \right) \tilde{\Theta}^{-1} \begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix} \right\|_\infty < 1/\sqrt{2}.$$

Then $\hat{K} = \hat{\hat{V}}^{-1}\hat{\hat{U}}$ is also a stabilizing controller and $\|\mathcal{F}_\ell(G,\hat{K})\|_\infty < \gamma$.

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sufficient conditions:

$\mathcal{H}_\infty$ controller reduction $\Longrightarrow$ frequency weighted $\mathcal{H}_\infty$ model reduction.

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# Chapter 16: $\mathcal{H}_\infty$ Loop Shaping

- Robust Stabilization of Coprime factors
- Robust Stabilization of Normalized Coprime Factors
- $\mathcal{H}_\infty$ Loop Shaping Design
- Weighted $\mathcal{H}_\infty$ Control Interpretation
- Further Guidelines for Loop Shaping

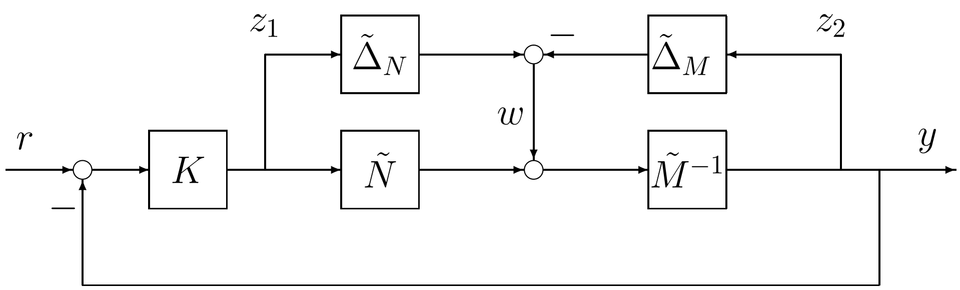
# Robust Stabilization of Coprime Factors

## Robust Stabilization Condition:

Let $P = \tilde{M}^{-1}\tilde{N}$ be the nominal model and

$$P_\Delta = (\tilde{M} + \tilde{\Delta}_M)^{-1}(\tilde{N} + \tilde{\Delta}_N)$$

with $\tilde{M}, \tilde{N}, \tilde{\Delta}_M, \tilde{\Delta}_N \in \mathcal{RH}_\infty$ and $\left\| \begin{bmatrix} \tilde{\Delta}_N & \tilde{\Delta}_M \end{bmatrix} \right\|_\infty < \epsilon$.




The perturbed system is robustly stable iff

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\|_\infty \le 1/\epsilon.$$

## State Space Coprime Factorization:

Let

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

and let $L$ be such that $A + LC$ is stable. Then

$$P = \tilde{M}^{-1}\tilde{N}, \quad \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & B + LD & L \\ \hline C & D & I \end{array}\right].$$

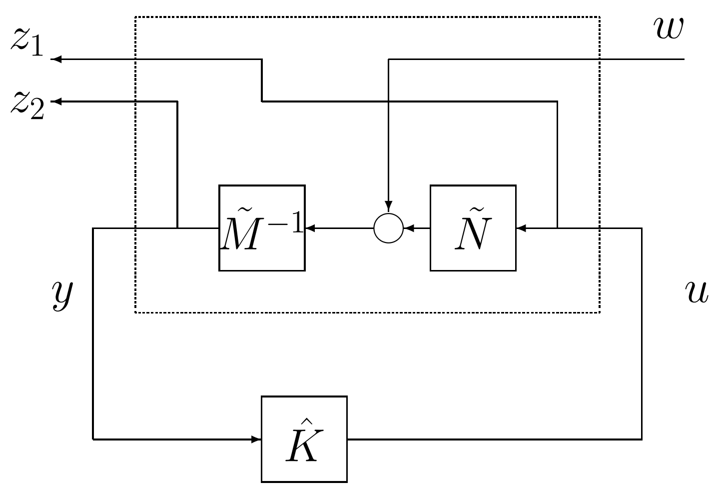
Denote

$$\hat{K} = -K$$

**LFT framework:**

$$G(s) = \begin{bmatrix} \begin{bmatrix} 0 \\ \tilde{M}^{-1} \end{bmatrix} & \begin{bmatrix} I \\ P \end{bmatrix} \\ \tilde{M}^{-1} & P \end{bmatrix} = \left[\begin{array}{c|cc} A & -L & B \\ \hline \begin{bmatrix} 0 \\ C \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} I \\ D \end{bmatrix} \\ C & I & D \end{array}\right]$$

$$=: \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

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**Controller for a Special Case:** $D = 0$.

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\|_\infty < \gamma$$

iff $\gamma > 1$ and there exists a stabilizing solution $X_\infty \geq 0$ solving

$$X_\infty(A-\frac{LC}{\gamma^2-1})+(A-\frac{LC}{\gamma^2-1})^*X_\infty-X_\infty(BB^*-\frac{LL^*}{\gamma^2-1})X_\infty+\frac{\gamma^2C^*C}{\gamma^2-1}=0.$$

a central controller:

$$K = \left[\begin{array}{c|c} A - BB^*X_\infty + LC & L \\ \hline -B^*X_\infty & 0 \end{array}\right].$$

## Robust Stabilization of Normalized Coprime Factors

Suppose $\tilde{M}$ and $\tilde{N}$ are normalized coprime factors

$$\tilde{M}(j\omega)\tilde{M}^*(j\omega) + \tilde{N}(j\omega)\tilde{N}^*(j\omega) = I$$

Then $\tilde{M}$ and $\tilde{N}$ can be obtained as

$$\begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A - YC^*C & B & -YC^* \\ \hline C & 0 & I \end{array}\right]$$

where $L = -YC^*$ and $Y \geq 0$ is the stabilizing solution to

$$AY + YA^* - YC^*CY + BB^* = 0$$

---

$$\gamma_{\min} := \inf_{K \ stabilizing} \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\|_\infty = \frac{1}{\sqrt{1 - \lambda_{max}(YQ)}}$$

$$\lambda_{\max}(YQ) = \left\| \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} \right\|_H^2$$

where $Q$ is the solution to

$$Q(A - YC^*C) + (A - YC^*C)^*Q + C^*C = 0.$$

Moreover, for any $\gamma > \gamma_{min}$ a controller achieving

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\|_\infty < \gamma$$

is given by

$$K(s) = \left[\begin{array}{c|c} A - BB^*X_\infty - YC^*C & -YC^* \\ \hline -B^*X_\infty & 0 \end{array}\right]$$

where

$$X_\infty = \frac{\gamma^2}{\gamma^2 - 1} Q \left( I - \frac{\gamma^2}{\gamma^2 - 1} YQ \right)^{-1}.$$

- Let $P = \tilde{M}^{-1}\tilde{N}$ be a normalized left coprime factorization and
$$P_\Delta = (\tilde{M} + \tilde{\Delta}_M)^{-1}(\tilde{N} + \tilde{\Delta}_N)$$
with
$$\left\| \begin{bmatrix} \tilde{\Delta}_N & \tilde{\Delta}_M \end{bmatrix} \right\|_\infty < \epsilon.$$
Then there is a robustly stabilizing controller for $P_\Delta$ if and only if
$$\epsilon \le \sqrt{1 - \lambda_{max}(YQ)} = \sqrt{1 - \left\| \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} \right\|_H^2}.$$

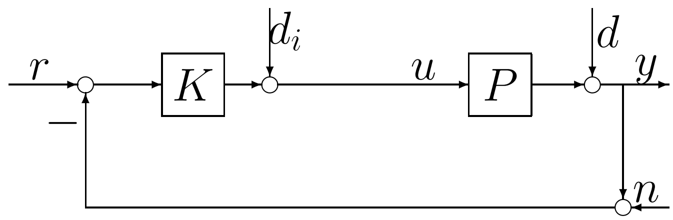
- Let $X \ge 0$ be the stabilizing solution to
$$XA + A^*X - XBB^*X + C^*C = 0$$
then
$$Q = (I + XY)^{-1}X$$
and
$$\gamma_{min} = \frac{1}{\sqrt{1 - \lambda_{max}(YQ)}} = \left(1 - \left\| \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} \right\|_H^2\right)^{-1/2} = \sqrt{1 + \lambda_{max}(XY)}.$$

- Let $P = \tilde{M}^{-1}\tilde{N}$ be a normalized left coprime factorization. Then
$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\|_\infty = \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_\infty .$$

- $\left\| \begin{bmatrix} I \\ K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\| = \left\| \begin{bmatrix} I \\ P \end{bmatrix} (I + KP)^{-1} \begin{bmatrix} I & K \end{bmatrix} \right\|.$

- Let $P = \tilde{M}^{-1}\tilde{N} = NM^{-1}$ be respectively the normalized left and right coprime factorizations. Then
$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\|_\infty = \left\| M^{-1}(I + KP)^{-1} \begin{bmatrix} I & K \end{bmatrix} \right\|_\infty .$$

# $\mathcal{H}_\infty$ Loop Shaping Design

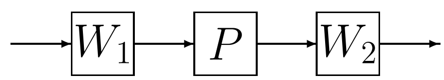


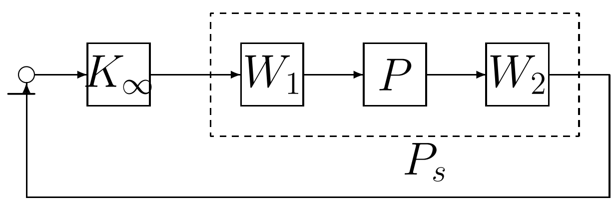

**Loop Shaping Design Procedure**

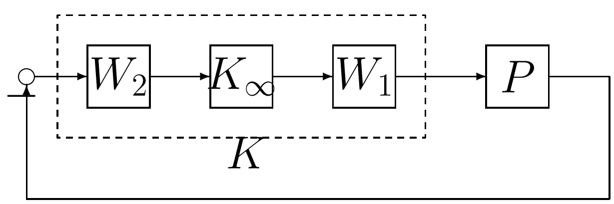
(1) Loop Shaping: Using a precompensator $W_1$ and/or a postcompensator $W_2$, the singular values of the nominal plant are shaped to give a desired open-loop shape.

$$P_s = W_2 P W_1$$

Assume that $W_1$ and $W_2$ are such that $P_s$ contains no hidden modes.




(2) Robust Stabilization: a) Calculate $\epsilon_{max}$, where

$$\begin{aligned}\epsilon_{max} &= \left(\inf_{K\ stabilizing}\left\|\begin{bmatrix} I \\ K \end{bmatrix}(I+P_sK)^{-1}\tilde{M}_s^{-1}\right\|_\infty\right)^{-1} \\ &= \sqrt{1-\left\|\begin{bmatrix}\tilde{N}_s & \tilde{M}_s\end{bmatrix}\right\|_H^2} < 1\end{aligned}$$

$P_s = \tilde{M}_s^{-1}\tilde{N}_s$ and

$$\tilde{M}_s(j\omega)\tilde{M}_s^*(j\omega) + \tilde{N}_s(j\omega)\tilde{N}_s^*(j\omega) = I.$$

If $\epsilon_{max} \ll 1$ return to (1) and adjust $W_1$ and $W_2$.

b) Select $\epsilon \leq \epsilon_{max}$, then synthesize a stabilizing controller $K_\infty$, which satisfies

$$\left\| \begin{bmatrix} I \\ K_\infty \end{bmatrix} (I + P_s K_\infty)^{-1} \tilde{M}_s^{-1} \right\|_\infty \leq \epsilon^{-1}.$$

(3) The final controller $K$

$$K = W_1 K_\infty W_2.$$

A typical design works as follows: the designer inspects the open-loop singular values of the nominal plant, and shapes these by pre- and/or postcompensation until nominal performance (and possibly robust stability) specifications are met. (Recall that the open-loop shape is related to closed-loop objectives.) A feedback controller $K_\infty$ with associated stability margin (for the shaped plant) $\epsilon \leq \epsilon_{max}$, is then synthesized. If $\epsilon_{max}$ is small, then the specified loop shape is incompatible with robust stability requirements, and should be adjusted accordingly, then $K_\infty$ is reevaluated.

## Weighted $\mathcal{H}_\infty$ Control Interpretation

$$\left\| \begin{bmatrix} I \\ K_\infty \end{bmatrix} (I + P_s K_\infty)^{-1} \tilde{M}_s^{-1} \right\|_\infty = \left\| \begin{bmatrix} I \\ K_\infty \end{bmatrix} (I + P_s K_\infty)^{-1} \begin{bmatrix} I & P_s \end{bmatrix} \right\|_\infty$$

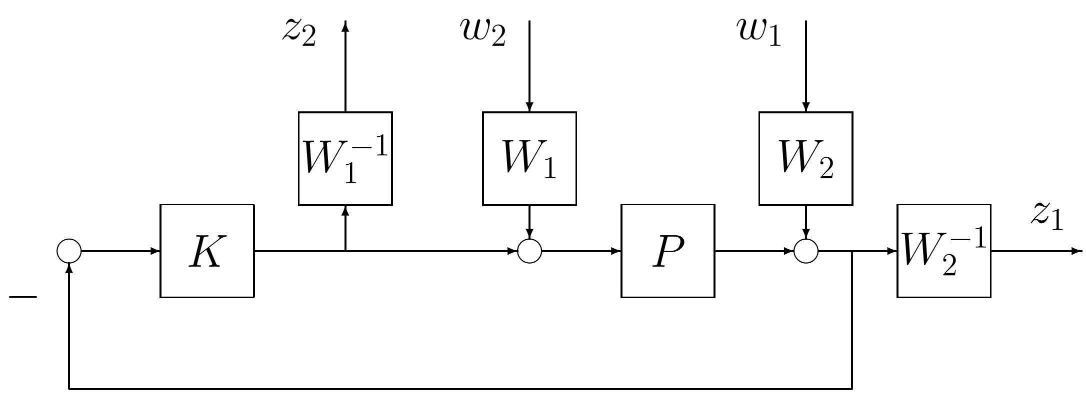
$$= \left\| \begin{bmatrix} W_2 & \\ & W_1^{-1} \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \begin{bmatrix} W_2^{-1} & \\ & W_1 \end{bmatrix} \right\|_\infty$$

$$= \left\| \begin{bmatrix} I \\ P_s \end{bmatrix} (I + K_\infty P_s)^{-1} \begin{bmatrix} I & K_\infty \end{bmatrix} \right\|_\infty$$

$$= \left\| \begin{bmatrix} W_1^{-1} & \\ & W_2 \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} (I + KP)^{-1} \begin{bmatrix} I & P \end{bmatrix} \begin{bmatrix} W_1 & \\ & W_2^{-1} \end{bmatrix} \right\|_\infty$$

This shows how all the closed-loop objective are incorporated.

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} W_2 & \\ & W_1^{-1} \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \begin{bmatrix} W_2^{-1} & \\ & W_1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$



Define

$$b_{P,K} := \begin{cases} \left( \left\| \begin{bmatrix} I \\ K \end{bmatrix} (I+PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_\infty \right)^{-1} & \text{if } K \text{ stabilizes } P \\ \\ 0 & \text{otherwise} \end{cases}$$

and

$$b_{\text{opt}} := \sup_K b_{P,K}.$$

Then $b_{P,K} = b_{K,P}$ and

$$b_{\text{opt}} = \sqrt{1 - \lambda_{\max}(YQ)} = \sqrt{1 - \left\| \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} \right\|_H^2}.$$

SISO $P$:

$$\text{gain margin} \geq \frac{1 + b_{P,K}}{1 - b_{P,K}}$$

and

$$\text{phase margin} \geq 2 \arcsin(b_{P,K}).$$

**Proof.** Note that for SISO system

$$b_{P,K} \leq \frac{|1 + P(j\omega)K(j\omega)|}{\sqrt{1 + |P(j\omega)|^2}\sqrt{1 + |K(j\omega)|^2}}, \quad \forall \omega.$$

So, at frequencies where $k := -P(j\omega)K(j\omega) \in \mathbb{R}^+$,

$$b_{P,K} \leq \frac{|1-k|}{\sqrt{(1 + |P|^2)(1 + \frac{k^2}{|P|^2})}} \leq \frac{|1-k|}{\sqrt{\min_P \left\{ (1 + |P|^2)(1 + \frac{k^2}{|P|^2}) \right\}}} = \left| \frac{1-k}{1+k} \right|,$$

which implies that

$$k \leq \frac{1 - b_{P,K}}{1 + b_{P,K}}, \quad \text{or} \quad k \geq \frac{1 + b_{P,K}}{1 - b_{P,K}}$$

from which the gain margin result follows. Similarly, at frequencies where $P(j\omega)K(j\omega) = -e^{j\theta}$,

$$b_{P,K} \le \frac{|1 - e^{j\theta}|}{\sqrt{(1+|P|^2)(1+\frac{1}{|P|^2})}} \le \frac{|2\sin\frac{\theta}{2}|}{\sqrt{\min_P \left\{(1+|P|^2)(1+\frac{1}{|P|^2})\right\}}} = \frac{|2\sin\frac{\theta}{2}|}{2},$$

which implies $\theta \ge 2\arcsin b_{P,K}$. $\square$

For example, $b_{P,K} = 1/2$ guarantees a gain margin of 3 and a phase margin of $60^{\text{o}}$.

$\gg$ $\mathbf{b_{p,k}} = \mathbf{emargin}(\mathbf{P}, \mathbf{K})$; % given $P$ and $K$, compute $b_{P,K}$.

$\gg$ $[\mathbf{K}_{\text{opt}}, \mathbf{b_{p,k}}] = \mathbf{ncfsyn}(\mathbf{P}, \mathbf{1})$; % find the optimal controller $K_{\text{opt}}$.

$\gg$ $[\mathbf{K}_{\text{sub}}, \mathbf{b_{p,k}}] = \mathbf{ncfsyn}(\mathbf{P}, \mathbf{2})$; % find a suboptimal controller $K_{sub}$.

## Further Guidelines for Loop Shaping

$P = NM^{-1}$: normalized right coprime factorization.

$$b_{\rm opt}(P) \le \lambda(P) := \inf_{\Re s > 0} \underline{\sigma}\left(\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}\right).$$

small $\lambda(P) \Longrightarrow$ small $b_{\rm opt}(P)$.
open right-half plane zeros and poles of $P$:

$$z_1, z_2, \ldots, z_m, \qquad p_1, p_2, \ldots, p_k$$

Define

$$N_z(s) = \frac{z_1 - s}{z_1 + s}\frac{z_2 - s}{z_2 + s}\cdots\frac{z_m - s}{z_m + s}, \quad N_p(s) = \frac{p_1 - s}{p_1 + s}\frac{p_2 - s}{p_2 + s}\cdots\frac{p_k - s}{p_k + s}.$$

Then

$$P(s) = P_0(s)N_z(s)/N_p(s)$$

where $P_0(s)$ has no open right-half plane poles or zeros.

Let $N_0(s)$ and $M_0(s)$ be stable and minimum phase spectral factors:

$$N_0(s)N_0^\sim(s) = \left(1 + \frac{1}{P(s)P^\sim(s)}\right)^{-1}, \;\; M_0(s)M_0^\sim(s) = (1+P(s)P^\sim(s))^{-1}.$$

Then $P_0 = N_0/M_0$ is a normalized coprime factorization and $(N_0 N_z)$ and $(M_0 N_p)$ form a pair of normalized coprime factorizations of $P$. Thus

$$b_{\rm opt}(P) \le \sqrt{|N_0(s)N_z(s)|^2 + |M_0(s)N_p(s)|^2}, \quad \forall \Re(s) > 0.$$

$$\begin{aligned} \ln|N_0(re^{j\theta})| &= \int_{-\infty}^{\infty} \ln\left(\frac{1}{\sqrt{1 + 1/|P(j\omega)|^2}}\right) K_\theta(\omega/r)\; d(\ln\omega) \\ \ln|M_0(re^{j\theta})| &= \int_{-\infty}^{\infty} \ln\left(\frac{1}{\sqrt{1 + |P(j\omega)|^2}}\right) K_\theta(\omega/r)\; d(\ln\omega) \end{aligned}$$

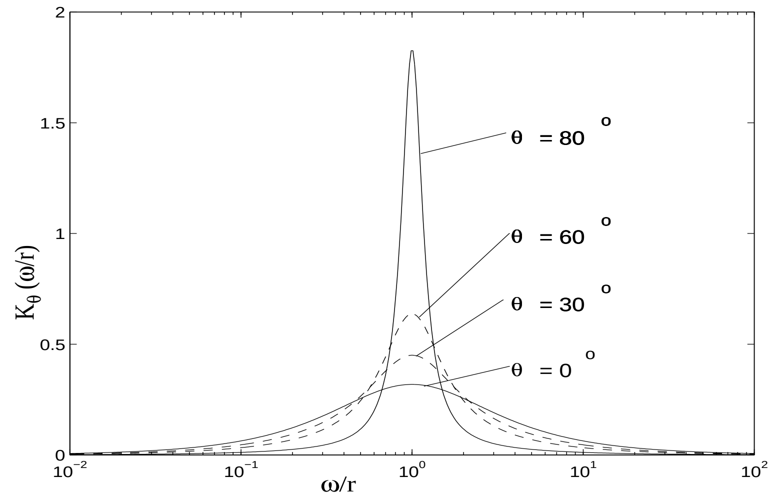



Figure 0.29: $K_\theta(\omega/r)$ vs. normalized frequency $\omega/r$

where $r > 0$, $-\pi/2 < \theta < \pi/2$, and

$$K_\theta(\omega/r) \ = \ \frac{1}{\pi}\frac{2(\omega/r)[1+(\omega/r)^2]\cos\theta}{[1-(\omega/r)^2]^2+4(\omega/r)^2\cos^2\theta}$$

$K_\theta(\omega/r)$ large near $\omega = r$: $|N_0(re^{j\theta})|$ will be small if $|P(j\omega)|$ is small near $\omega = r$ and $|M_0(re^{j\theta})|$ will be small if $|P(j\omega)|$ is large near $\omega = r$.

Large $\theta$: $K_\theta(\omega/r)$ very near $\omega = r$ and small otherwise. Hence $|N_0(re^{j\theta})|$ and $|M_0(re^{j\theta})|$ will essentially be determined by $|P(j\omega)|$ in a very narrow frequency range near $\omega = r$ when $\theta$ is large. On the other hand, when $\theta$ is small, a larger range of frequency response $|P(j\omega)|$ around $\omega = r$ will have affect on the value $|N_0(re^{j\theta})|$ and $|M_0(re^{j\theta})|$. (This, in fact, will imply that a right-plane zero (pole) with a much larger real part than the imaginary part will have much worse effect on the performance than a right-plane zero (pole) with a much larger imaginary part than the real part.)

# When can $b_{\text{opt}}(P)$ be small

Let $s = re^{j\theta}$ and note that $N_z(z_i) = 0$ and $N_p(p_j) = 0$. Then the bound can be small if

- ▷ $|N_z(s)|$ and $|N_p(s)|$ are both small for some $s$. That is, $|N_z(s)| \approx 0$ (i.e., $s$ is close to a right-half plane zero of $P$) and $|N_p(s)| \approx 0$ (i.e., $s$ is close to a right-half plane pole of $P$). This is only possible if $P(s)$ has a right-half plane zero near a right-half plane pole. (See Example 0.1.)
- ▷ $|N_z(s)|$ and $|M_0(s)|$ are both small for some $s$. That is, $|N_z(s)| \approx 0$ (i.e., $s$ is close to a right-half plane zero of $P$) and $|M_0(s)| \approx 0$ (i.e., $|P(j\omega)|$ is large around $\omega = |s| = r$). This is only possible if $|P(j\omega)|$ is large around $\omega = r$, where $r$ is the modulus of a right-half plane zero of $P$. (See Example 0.2.)
- ▷ $|N_p(s)|$ and $|N_0(s)|$ are both small for some $s$. That is, $|N_p(s)| \approx 0$ (i.e., $s$ is close to a right-half plane pole of $P$) and $|N_0(s)| \approx 0$ (i.e., $|P(j\omega)|$ is small around $\omega = |s| = r$). This is only possible if $|P(j\omega)|$ is small around $\omega = r$, where $r$ is the modulus of a right-half plane pole of $P$. (See Example 0.3.)
- ▷ $|N_0(s)|$ and $|M_0(s)|$ are both small for some $s$. That is, $|N_0(s)| \approx 0$ (i.e., $|P(j\omega)|$ is small around $\omega = |s| = r$) and $|M_0(s)| \approx 0$ (i.e., $|P(j\omega)|$ is large around $\omega = |s| = r$). The only way in which $|P(j\omega)|$ can be both small and large at frequencies near $\omega = r$ is that $|P(j\omega)|$ is approximately equal to 1 and the absolute value of the slope of $|P(j\omega)|$ is large. (See Example 0.4.)

# RHP Poles/Zeros are close

## Example 0.1

$$P_1(s) = \frac{K(s-r)}{(s+1)(s-1)}.$$

$b_{\text{opt}}(P_1)$ will be very small for all $K$ whenever $r$ is close to 1 (i.e., whenever there is an unstable pole close to an unstable zero).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $K = 0.1$ | $r$ | 0.5 | 0.7 | 0.9 | 1.1 | 1.3 | 1.5 |
| | $b_{\text{opt}}(P_1)$ | 0.0125 | 0.0075 | 0.0025 | 0.0025 | 0.0074 | 0.0124 |
| $K = 1$ | $r$ | 0.5 | 0.7 | 0.9 | 1.1 | 1.3 | 1.5 |
| | $b_{\text{opt}}(P_1)$ | 0.1036 | 0.0579 | 0.0179 | 0.0165 | 0.0457 | 0.0706 |
| $K = 10$ | $r$ | 0.5 | 0.7 | 0.9 | 1.1 | 1.3 | 1.5 |
| | $b_{\text{opt}}(P_1)$ | 0.0658 | 0.0312 | 0.0088 | 0.0077 | 0.0208 | 0.0318 |

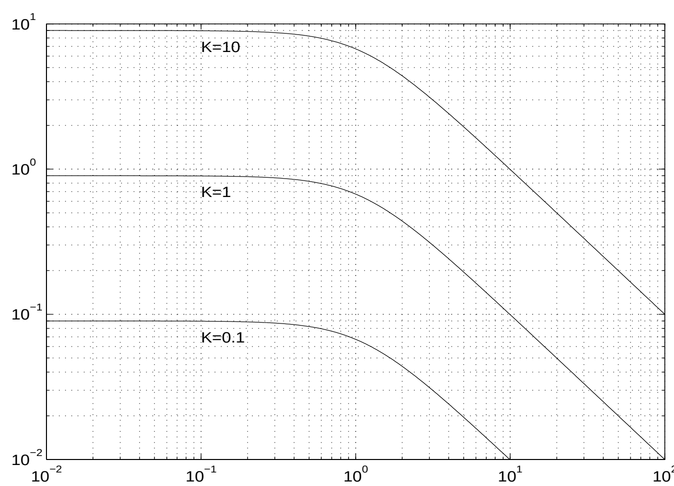



Figure 0.30: Frequency responses of $P_1$ for $r = 0.9$ and $K = 0.1, 1,$ and $10$

## Example 0.2

$$P_2(s) = \frac{K(s-1)}{s(s+1)}.$$

$b_{\rm opt}(P_2)$ will be small if $|P_2(j\omega)|$ is large around $\omega = 1$, the modulus of the right-half plane zero.

| $K$ | 0.01 | 0.1 | 1 | 10 | 100 |
|---|---|---|---|---|---|
| $b_{\rm opt}(P_2)$ | 0.7001 | 0.6451 | 0.3827 | 0.0841 | 0.0098 |

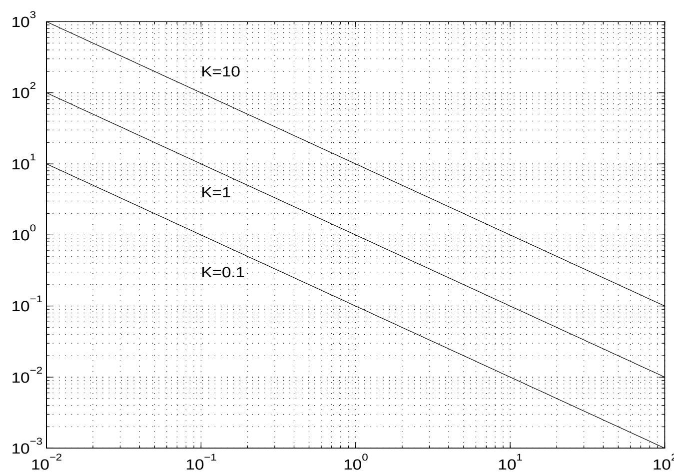



Figure 0.31: Frequency responses of $P_2$ and $P_3$ for $K = 0.1, 1$, and 10

Note that $b_{\rm opt}(L/s) = 0.707$ for any $L$ and $b_{\rm opt}(P_2) \longrightarrow 0.707$ as $K \longrightarrow 0$. This is because $|P_2(j\omega)|$ around the frequency of the right-half plane zero is very small as $K \longrightarrow 0$.

## Complex Nonminimum Phase Zeros

$$P_3(s) = \frac{K[(s - \cos\theta)^2 + \sin^2\theta]}{s[(s + \cos\theta)^2 + \sin^2\theta]}.$$

| | | | | | | |
|---|---|---|---|---|---|---|
| | $\theta$ (degree) | 0 | 45 | 60 | 80 | 85 |
| $K = 0.1$ | $b_{\rm opt}(P_3)$ | 0.5952 | 0.6230 | 0.6447 | 0.6835 | 0.6950 |
| | $\theta$ (degree) | 0 | 45 | 60 | 80 | 85 |
| $K = 1$ | $b_{\rm opt}(P_3)$ | 0.2588 | 0.3078 | 0.3568 | 0.4881 | 0.5512 |
| | $\theta$ (degree) | 0 | 45 | 60 | 80 | 85 |
| $K = 10$ | $b_{\rm opt}(P_3)$ | 0.0391 | 0.0488 | 0.0584 | 0.0813 | 0.0897 |

- $b_{\rm opt}(P_3)$ will be small if $|P_3(j\omega)|$ is large around the frequency of $\omega = 1$ (the modulus of the right-half plane zero).
- for zeros with the same modulus, $b_{\rm opt}(P_3)$ will be smaller for a plant with relatively larger real part zeros than for a plant with relatively larger imaginary part zeros (i.e., a pair of real right-half plane zeros has a much worse effect on the performance than a pair of almost pure imaginary axis right-half plane zeros of the same modulus).

## Example 0.3

$$P_4(s) = \frac{K(s+1)}{s(s-1)}.$$

$b_{\mathrm{opt}}(P_4)$ will be small if $|P_4(j\omega)|$ is small around $\omega = 1$ (the modulus of the right-half plane pole).

| $K$ | 0.01 | 0.1 | 1 | 10 | 100 |
|---|---|---|---|---|---|
| $b_{\mathrm{opt}}(P_4)$ | 0.0098 | 0.0841 | 0.3827 | 0.6451 | 0.7001 |

Note that $b_{\mathrm{opt}}(P_4) \longrightarrow 0.707$ as $K \longrightarrow \infty$. This is because $|P_4(j\omega)|$ is very large around the frequency of the modulus of the right-half plane pole as $K \longrightarrow \infty$.

$$P_5(s) = \frac{K[(s+\cos\theta)^2 + \sin^2\theta]}{s[(s-\cos\theta)^2 + \sin^2\theta]}.$$

The optimal $b_{\mathrm{opt}}(P_5)$ for various $\theta$'s are listed in the following table:

| | | | | | | |
|---|---|---|---|---|---|---|
| $K = 0.1$ | $\theta$ (degree) | 0 | 45 | 60 | 80 | 85 |
| | $b_{\mathrm{opt}}(P_5)$ | 0.0391 | 0.0488 | 0.0584 | 0.0813 | 0.0897 |
| $K = 1$ | $\theta$ (degree) | 0 | 45 | 60 | 80 | 85 |
| | $b_{\mathrm{opt}}(P_5)$ | 0.2588 | 0.3078 | 0.3568 | 0.4881 | 0.5512 |
| $K = 10$ | $\theta$ (degree) | 0 | 45 | 60 | 80 | 85 |
| | $b_{\mathrm{opt}}(P_5)$ | 0.5952 | 0.6230 | 0.6447 | 0.6835 | 0.6950 |

- $b_{\mathrm{opt}}(P_5)$ will be small if $|P_5(j\omega)|$ is small around the frequency of the modulus of the right-half plane pole.

- for poles with the same modulus, $b_{\mathrm{opt}}(P_5)$ will be smaller for a plant with relatively larger real part poles than for a plant with relatively larger imaginary part poles (i.e., a pair of real right-half plane poles has a much worse effect on the performance than a pair of almost pure imaginary axis right-half plane poles of the same modulus).

# Large Slope near Crossover

## Example 0.4

$$P_6(s) = \frac{K(0.2s+1)^4}{s(s+1)^4}.$$

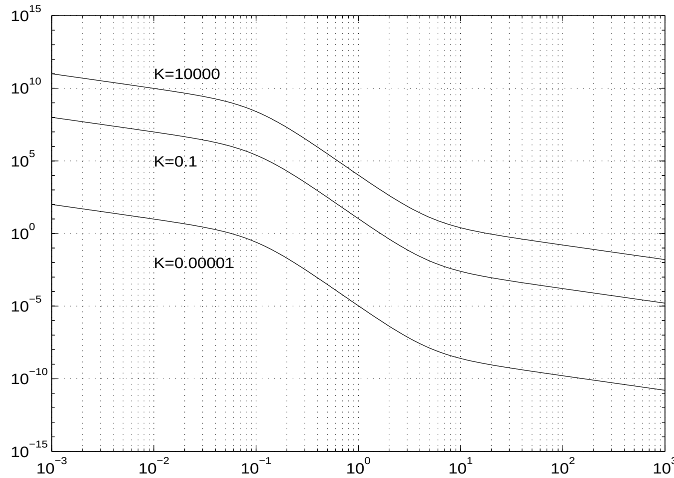



Figure 0.32: Frequency response of $P_6$ for $K = 10^{-5}, 10^{-1}$ and $10^4$

- $K = 10^{-5}$: slope near crossover is not too large $\Longrightarrow$ $b_{\text{opt}}(P_6)$ not too small.
- $K = 10^4$: Similar.
- $K = 0.1$: slope near crossover is quite large $\Longrightarrow$ $b_{\text{opt}}(P_6)$ quite small.

| $K$ | $10^{-5}$ | $10^{-3}$ | 0.1 | 1 | 10 | $10^2$ | $10^4$ |
|---|---|---|---|---|---|---|---|
| $b_{\text{opt}}(P_6)$ | 0.3566 | 0.0938 | 0.0569 | 0.0597 | 0.0765 | 0.1226 | 0.4933 |

## Guidelines

Based on the preceding discussion, we can give some guidelines for the loop-shaping design.

♡ The loop transfer function should be shaped in such a way that it has low gain around the frequency of the modulus of any right-half plane zero $z$. Typically, it requires that the crossover frequency be much smaller than the modulus of the right-half plane zero; say, $\omega_c < |z|/2$ for any real zero and $\omega_c < |z|$ for any complex zero with a much larger imaginary part than the real part (see Figure 0.29).

♡ The loop transfer function should have a large gain around the frequency of the modulus of any right-half plane pole.

♡ The loop transfer function should not have a large slope near the crossover frequencies.

These guidelines are consistent with the rules used in classical control theory (see Bode [1945] and Horowitz [1963]).

# Chapter 17: Gap metric and $\nu$-Gap Metric

- Gap metric
- $\nu$-Gap metric
- Geometric interpretation of $\nu$-gap metric
- Extended loop-shaping design
- controller order reduction

## Example

Measure of Distance:

$$P_1(s) = \frac{1}{s}, \quad P_2(s) = \frac{1}{s+0.1}.$$

Closed-loop:

$$\left\| P_1(I+P_1)^{-1} - P_2(I+P_2)^{-1} \right\|_\infty = 0.0909,$$

Open-loop:

$$\|P_1 - P_2\|_\infty = \infty.$$

Need new measure.

normalized right and left stable coprime factorizations:

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N}.$$

$$M^{\sim}M + N^{\sim}N = I, \quad \tilde{M}\tilde{M}^{\sim} + \tilde{N}\tilde{N}^{\sim} = I.$$

The graph of the operator $P$ is the subspace of $\mathcal{H}_2$ consisting of all pairs $(u, y)$ such that $y = Pu$. This is given by

$$\begin{bmatrix} M \\ N \end{bmatrix} \mathcal{H}_2$$

and is a closed subspace of $\mathcal{H}_2$. The gap between two systems $P_1$ and $P_2$ is defined by

$$\delta_g(P_1, P_2) = \left\| \Pi_{\begin{bmatrix} M_1 \\ N_1 \end{bmatrix}\mathcal{H}_2} - \Pi_{\begin{bmatrix} M_2 \\ N_2 \end{bmatrix}\mathcal{H}_2} \right\|$$

where $\Pi_K$ denotes the orthogonal projection onto $K$ and $P_1 = N_1 M_1^{-1}$ and $P_2 = N_2 M_2^{-1}$ are normalized right coprime factorizations.

**Theorem 0.1** *Let $P_1 = N_1 M_1^{-1}$ and $P_2 = N_2 M_2^{-1}$ be normalized right coprime factorizations. Then*

$$\delta_g(P_1, P_2) = \max\left\{ \vec{\delta}(P_1, P_2), \vec{\delta}(P_2, P_1) \right\}$$

*where $\vec{\delta}_g(P_1, P_2)$ is the directed gap and can be computed by*

$$\vec{\delta}_g(P_1, P_2) = \inf_{Q \in \mathcal{H}_\infty} \left\| \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} Q \right\|_\infty .$$

$$\gg \ \delta_{\mathbf{g}}(\mathbf{P_1}, \mathbf{P_2}) = \mathbf{gap}(\mathbf{P_1}, \mathbf{P_2}, \mathbf{tol})$$

$$\delta_g\left(\frac{1}{s}, \frac{1}{s+0.1}\right) = 0.0995,$$

## Lower Bound of Gap

Let

$$\Phi = \begin{bmatrix} M_2^{\sim} & N_2^{\sim} \\ -\tilde{N}_2 & \tilde{M}_2 \end{bmatrix}.$$

Then $\Phi^{\sim}\Phi = \Phi\Phi^{\sim} = I$ and

$$\begin{aligned}
\vec{\delta}_g(P_1, P_2) &= \inf_{Q \in \mathcal{H}_\infty} \left\| \begin{bmatrix} M_2^{\sim} & N_2^{\sim} \\ -\tilde{N}_2 & \tilde{M}_2 \end{bmatrix} \left\{ \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} Q \right\} \right\|_\infty \\
&= \inf_{Q \in \mathcal{H}_\infty} \left\| \begin{bmatrix} M_2^{\sim} M_1 + N_2^{\sim} N_1 - Q \\ -\tilde{N}_2 M_1 + \tilde{M}_2 N_1 \end{bmatrix} \right\|_\infty \\
&\geq \|\Psi(P_1, P_2)\|_\infty
\end{aligned}$$

where

$$\Psi(P_1, P_2) := -\tilde{N}_2 M_1 + \tilde{M}_2 N_1 = \begin{bmatrix} \tilde{M}_2 & \tilde{N}_2 \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} M_1 \\ N_1 \end{bmatrix}.$$

$\|\Psi(P_1, P_2)\|_\infty$ is related to the $\nu$-gap metric.

$$P_1 = \frac{k_1}{s+1}, \quad P_2 = \frac{k_2}{s+1}.$$

Then it is easy to verify that $P_i = N_i/M_i$, $i = 1, 2$, with

$$N_i = \frac{k_i}{s + \sqrt{1 + k_i^2}}, \quad M_i = \frac{s+1}{s + \sqrt{1 + k_i^2}},$$

are normalized coprime factorizations and it can be further shown, as in Georgiou and Smith [1990], that

$$\delta_g(P_1, P_2) = \|\Psi(P_1, P_2)\|_\infty = \begin{cases} \dfrac{|k_1 - k_2|}{|k_1| + |k_2|}, & \text{if } |k_1 k_2| > 1; \\[2ex] \dfrac{|k_1 - k_2|}{\sqrt{(1 + k_1^2)(1 + k_2^2)}}, & \text{if } |k_1 k_2| \leq 1. \end{cases}$$

## Connection with Coprime Factor Uncertainty

**Corollary 0.2** *Let $P$ have a normalized coprime factorization $P = NM^{-1}$. Then for all* $0 < b \leq 1$,

$$\left\{P_1 : \ \vec{\delta}_g(P, P_1) < b\right\}$$

$$= \left\{P_1 : \quad P_1 = (N + \Delta_N)(M + \Delta_M)^{-1}, \ \Delta_N, \ \Delta_M \in \mathcal{H}_\infty, \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_\infty < b \right\}.$$

**Proof**. Suppose $\vec{\delta}_g(P, P_1) < b$ and let $P_1 = N_1 M_1^{-1}$ be a normalized right coprime factorization. Then there exists a $Q \in \mathcal{H}_\infty$ such that

$$\left\| \begin{bmatrix} M \\ N \end{bmatrix} - \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} Q \right\|_\infty < b.$$

Define

$$\begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix} := \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} Q - \begin{bmatrix} M \\ N \end{bmatrix} \in \mathcal{H}_\infty.$$

Then $\left\| \begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix} \right\|_\infty < b$ and $P_1 = (N_1 Q)(M_1 Q)^{-1} = (N + \Delta_N)(M + \Delta_M)^{-1}$.

To show the converse, note that $P_1 = (N + \Delta_N)(M + \Delta_M)^{-1}$ and there exists a $\tilde{Q}^{-1} \in \mathcal{H}_\infty$ such that $P_1 = \left\{(N + \Delta_N)\tilde{Q}\right\}\left\{(M + \Delta_M)\tilde{Q}\right\}^{-1}$ is a normalized right coprime factorization. Hence by definition, $\vec{\delta}_g(P, P_1)$ can be computed as

$$\vec{\delta}_g(P, P_1) = \inf_Q \left\| \begin{bmatrix} M \\ N \end{bmatrix} - \begin{bmatrix} M + \Delta_M \\ N + \Delta_N \end{bmatrix} \tilde{Q} Q \right\|_\infty \leq \left\| \begin{bmatrix} M \\ N \end{bmatrix} - \begin{bmatrix} M + \Delta_M \\ N + \Delta_N \end{bmatrix} \right\|_\infty < b$$

where the first inequality follows by taking $Q = \tilde{Q}^{-1} \in \mathcal{H}_\infty$. □

- If $\delta_g(P_1, P_2) < 1$, then $\delta_g(P_1, P_2) = \vec{\delta}_g(P_1, P_2) = \vec{\delta}_g(P_2, P_1)$.
- If $b \le \lambda(P) := \inf_{\Re s > 0} \underline{\sigma}\left(\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}\right)$, then
$$\left\{P_1 :\ \vec{\delta}(P, P_1) < b\right\} = \left\{P_1 :\ \delta(P, P_1) < b\right\}.$$

Recall that

$$\begin{aligned} b_{\mathrm{obt}}(P) &:= \left\{ \inf_{K \text{ stabilizing}} \left\| \begin{bmatrix} I \\ K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_\infty \right\}^{-1} \\ &= \sqrt{1 - \lambda_{\max}(YQ)} = \sqrt{1 - \left\| \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} \right\|_H^2} \end{aligned}$$

and

$$b_{P,K} := \left\| \begin{bmatrix} I \\ K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_\infty^{-1} = \left\| \begin{bmatrix} I \\ P \end{bmatrix} (I + KP)^{-1} \begin{bmatrix} I & K \end{bmatrix} \right\|_\infty^{-1}.$$

**Theorem 0.3** *Suppose the feedback system with the pair $(P_0, K_0)$ is stable. Let $\mathcal{P} := \{P :\ \delta_g(P, P_0) < r_1\}$ and $\mathcal{K} := \{K :\ \delta_g(K, K_0) < r_2\}$. Then*

*(a) The feedback system with the pair $(P, K)$ is also stable for all $P \in \mathcal{P}$ and $K \in \mathcal{K}$ if and only if*
$$\arcsin b_{P_0,K_0} \ge \arcsin r_1 + \arcsin r_2.$$

*(b) The worst possible performance resulting from these sets of plants and controllers is given by*
$$\inf_{P \in \mathcal{P},\ K \in \mathcal{K}} \arcsin b_{P,K} = \arcsin b_{P_0,K_0} - \arcsin r_1 - \arcsin r_2.$$

one can take either $r_1 = 0$ or $r_2 = 0$.

## Example

Consider

$$P_1 = \frac{s-1}{s+1} = N_1/M_1, \quad P_2 = \frac{2s-1}{s+1} = N_2/M_2.$$

$$N_1 = \frac{1}{\sqrt{2}}\frac{s-1}{s+1}, \quad M_1 = \frac{1}{\sqrt{2}}, \quad N_2 = \frac{2s-1}{\sqrt{5}s+\sqrt{2}}, \quad M_2 = \frac{s+1}{\sqrt{5}s+\sqrt{2}}$$

$$\delta_g(P_1, P_2) = 1/3 > \|\Psi(P_1, P_2)\|_\infty = \sup_\omega \frac{|\omega|}{\sqrt{10\omega^2+4}} = \frac{1}{\sqrt{10}},$$

$$\gg \boldsymbol{\delta_g(P_1, P_2) = \mathbf{gap}(P_1, P_2, tol)}$$

Next, note that $b_{\text{obt}}(P_1) = 1/\sqrt{2}$ and the optimal controller achieving $b_{\text{obt}}(P_1)$ is $K_{\text{obt}} = 0$. There must be a plant $P$ with $\delta_\nu(P_1, P) = b_{\text{obt}}(P_1) = 1/\sqrt{2}$ that can not be stabilized by $K_{\text{obt}} = 0$; that is, there must be an unstable plant $P$ such that $\delta_\nu(P_1, P) = b_{\text{obt}}(P_1) = 1/\sqrt{2}$. A such $P$ can be found using Corollary 0.2:

$$\{P : \ \delta_g(P_1, P) \le b_{\text{obt}}(P_1)\}$$

$$= \left\{ P : \quad P = \frac{N_1 + \Delta_N}{M_1 + \Delta_M}, \ \Delta_N, \ \Delta_M \in \mathcal{H}_\infty, \ \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_\infty \le b_{\text{obt}}(P_1) \right\}.$$

that is, there must be $\Delta_N, \ \Delta_M \in \mathcal{H}_\infty$, $\left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_\infty = b_{\text{obt}}(P_1)$ such that

$$P = \frac{N_1 + \Delta_N}{M_1 + \Delta_M}$$

is unstable. Let

$$\Delta_N = 0, \quad \Delta_M = \frac{1}{\sqrt{2}}\frac{s-1}{s+1}.$$

Then

$$P = \frac{N_1 + \Delta_N}{M_1 + \Delta_M} = \frac{s-1}{2s}, \quad \delta_\nu(P_1, P) = b_{\text{obt}}(P_1) = 1/\sqrt{2}.$$

## Example

Question: Given an uncertain plant

$$P(s) = \frac{k}{s-1}, \quad k \in [k_1, k_2],$$

(a) Find the best nominal design model $P_0 = \dfrac{k_0}{s-1}$ in the sense

$$\inf_{k_0 \in [k_1, k_2]} \sup_{k \in [k_1, k_2]} \delta_g(P, P_0).$$

(b) Let $k_1$ be fixed and $k_2$ be variable. Find the $k_0$ so that the largest family of the plant $P$ can be guaranteed to be stabilized a priori by any controller satisfying $b_{P_0,K} = b_{\text{obt}}(P_0)$.

For simplicity, suppose $k_1 \geq 1$. It can be shown that $\delta_g(P, P_0) = \frac{|k_0 - k|}{k_0 + k}$. Then the optimal $k_0$ for question (a) satisfies

$$\frac{k_0 - k_1}{k_0 + k_1} = \frac{k_2 - k_0}{k_2 + k_0};$$

that is, $k_0 = \sqrt{k_1 k_2}$ and

$$\inf_{k_0 \in [k_1, k_2]} \sup_{k \in [k_1, k_2]} \delta_g(P, P_0) = \frac{\sqrt{k_2} - \sqrt{k_1}}{\sqrt{k_2} + \sqrt{k_1}}.$$

To answer question (b), we note that by Theorem 0.3, a family of plants satisfying $\delta_g(P, P_0) \leq r$ with $P_0 = k_0/(s+1)$ is stabilizable a priori by any controller satisfying $b_{P_0,K} = b_{\text{obt}}(P_0)$ if, and only if, $r < b_{P_0,K}$. Since $P_0 = N_0/M_0$ with

$$N_0 = \frac{k_0}{s + \sqrt{1 + k_0^2}}, \quad M_0 = \frac{s-1}{s + \sqrt{1 + k_0^2}}$$

is a normalized coprime factorization, it is easy to show that

$$\left\| \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} \right\|_H = \frac{\sqrt{k_0^2 + (1 - \sqrt{1 + k_0^2})^2}}{2\sqrt{1 + k_0^2}}$$

and

$$b_{\text{obt}}(P_0) = \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{1+k_0^2}}\right)}.$$

Hence we need to find a $k_0$ such that

$$b_{\text{obt}}(P_0) \geq \max\left\{\frac{k_0 - k_1}{k_0 + k_1}, \frac{k_2 - k_0}{k_2 + k_0}\right\};$$

that is,

$$\sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{1+k_0^2}}\right)} \geq \max\left\{\frac{k_0 - k_1}{k_0 + k_1}, \frac{k_2 - k_0}{k_2 + k_0}\right\}$$

for a largest possible $k_2$. The optimal $k_0$ is given by the solution of the equation:

$$\sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{1+k_0^2}}\right)} = \frac{k_0 - k_1}{k_0 + k_1}$$

and the largest $k_2 = k_0^2/k_1$. For example, if $k_1 = 1$, then $k_0 = 7.147$ and $k_2 = 51.0793$.

In general, given a family of plant $P$, it is not easy to see how to choose a best nominal model $P_0$ such that (a) or (b) is true. This is still a very important open question.

**Definition 0.2** *The the winding number of $g(s)$ with respect to this contour, denoted by* $\text{wno}(g)$*, is the number of counterclockwise encirclements around the origin by $g(s)$ evaluated on the Nyquist contour $\Gamma$. (A clockwise encirclement counts as a negative encirclement.)*

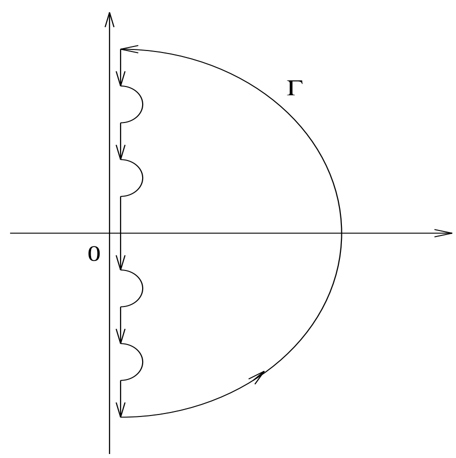



Figure 0.33: The Nyquist contour

**Lemma 0.4 (The Argument Principle)** *Let $\Gamma$ be a closed contour in the complex plane. Let $f(s)$ be a function analytic along the contour; that is, $f(s)$ has no poles on $\Gamma$. Assume $f(s)$ has $Z$ zeros and $P$ poles inside $\Gamma$. Then $f(s)$ evaluated along the contour $\Gamma$ once in an anti-clockwise direction will make $Z - P$ anti-clockwise encirclements of the origin.*

## Properties of wno

Denote $\eta(G)$ and $\eta_0(G)$, respectively, the number of open right-half plane and imaginary axis poles of $G(s)$.

**Lemma 0.5** *Let $g$ and $h$ be biproper rational scalar transfer functions and let $F$ be a square transfer matrix. Then*

*(a)* $\text{wno}(gh) = \text{wno}(g) + \text{wno}(h)$*;*

*(b)* $\text{wno}(g) = \eta(g^{-1}) - \eta(g)$*;*

*(c)* $\text{wno}(g^{\sim}) = -\text{wno}(g) - \eta_0(g^{-1}) + \eta_0(g)$*;*

*(d)* $\text{wno}(1+g) = 0$ *if* $g \in \mathcal{RL}_\infty$ *and* $\|g\|_\infty < 1$*;*

*(e)* $\text{wno}\det(I+F) = 0$ *if* $F \in \mathcal{RL}_\infty$ *and* $\|F\|_\infty < 1$*.*

**Proof**.

(a) obvious.

(b) the number of right-half plane zeros of $g$ is the number of right-half plane poles of $g^{-1}$.

(c) Suppose the order of $g$ is $n$. Then $\eta(g^{\sim}) = n - \eta(g) - \eta_0(g)$ and $\eta\left[(g^{\sim})^{-1}\right] = n - \eta(g^{-1}) - \eta_0(g^{-1})$, which gives $\text{wno}(g^{\sim}) = \eta\left[(g^{\sim})^{-1}\right] - \eta(g^{\sim}) = \eta(g) - \eta(g^{-1}) - \eta_0(g^{-1}) + \eta_0(g) = -\text{wno}(g) - \eta_0(g^{-1}) + \eta_0(g)$.

(d) follows from the fact that $1 + \Re g(j\omega) > 0, \ \forall\omega$ since $\|g\|_\infty < 1$.

(e) follows from part (d) and $\det(I+F) = \Pi_{i=1}^{m}(1+\lambda_i(F))$ with $|\lambda_i(F)| < 1$.

$\square$

## Example

Let

$$g_1 = \frac{1.2(s+3)}{s-5}, \quad g_2 = \frac{s-1}{s-2}, \quad g_3 = \frac{2(s-1)(s-2)}{(s+3)(s+4)}, \quad g_4 = \frac{(s-1)(s+3)}{(s-2)(s-4)}.$$

Figure 0.34 shows the functions, $g_1, g_2, g_3$, and $g_4$, evaluated on the Nyquist contour $\Gamma$. Clearly, we have

$$\text{wno}(g_1) = -1, \quad \text{wno}(g_2) = 0, \quad \text{wno}(g_3) = 2, \quad \text{wno}(g_4) = -1$$

and they are consistent with the results computed from using Lemma 0.5.

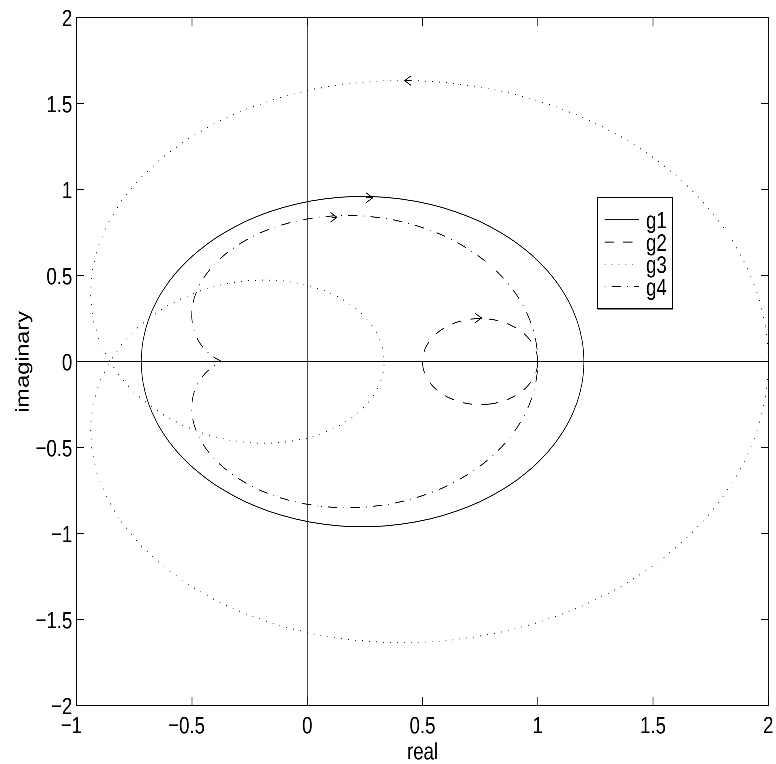



Figure 0.34: $g_1, g_2, g_3$, and $g_4$ evaluated on $\Gamma$

**Definition 0.3** *The $\nu$-gap metric is defined as*

$$\delta_\nu(P_1, P_2) = \begin{cases} \|\Psi(P_1, P_2)\|_\infty, & \text{if } \det\Theta(j\omega) \neq 0 \ \forall\omega \\ & \text{and } \operatorname{wno}\det\Theta(s) = 0, \\ \\ 1, & \text{otherwise} \end{cases}$$

*where* $\Theta(s) := N_2^{\sim} N_1 + M_2^{\sim} M_1$ *and* $\Psi(P_1, P_2) := -\tilde{N}_2 M_1 + \tilde{M}_2 N_1$.

$$\delta_\nu(P_1, P_2) = \delta_\nu(P_2, P_1) = \delta_\nu(P_1^T, P_2^T)$$

$$\gg \ \delta_\nu(\mathbf{P_1}, \mathbf{P_2}) = \mathbf{nugap}(\mathbf{P_1}, \mathbf{P_2}, \mathbf{tol})$$

where tol is the computational tolerance.

Consider, for example, $P_1 = 1$ and $P_2 = \dfrac{1}{s}$. Then

$$M_1 = N_1 = \frac{1}{\sqrt{2}}, \quad M_2 = \frac{s}{s+1}, \quad N_2 = \frac{1}{s+1}.$$

Hence

$$\Theta(s) = \frac{1}{\sqrt{2}}\frac{1-s}{1-s} = \frac{1}{\sqrt{2}}, \quad \Psi(P_1, P_2) = \frac{1}{\sqrt{2}}\frac{s-1}{s+1},$$

and $\delta_\nu(P_1, P_2) = \frac{1}{\sqrt{2}}$. (Note that $\Theta$ has no poles or zeros!)

**Theorem 0.6** *The $\nu$-gap metric can be defined as*

$$\delta_\nu(P_1,P_2) = \begin{cases} \|\Psi(P_1,P_2)\|_\infty, & \text{if } \det(I+P_2^\sim P_1) \neq 0 \ \forall\omega \ \text{ and} \\ & \text{wno}\det(I+P_2^\sim P_1)+\eta(P_1) \\ & -\eta(P_2)-\eta_0(P_2)=0, \\ 1, & \text{otherwise} \end{cases}$$

*where $\Psi(P_1,P_2)$ can be written as*

$$\Psi(P_1,P_2) = (I+P_2P_2^\sim)^{-1/2}(P_1-P_2)(I+P_1^\sim P_1)^{-1/2}.$$

**Proof.** Since the number of unstable zeros of $M_1$ ($M_2$) is equal to the number of unstable poles of $P_1$ ($P_2$), and

$$N_2^\sim N_1 + M_2^\sim M_1 = M_2^\sim (I+P_2^\sim P_1)M_1,$$

we have

$$\begin{aligned} \text{wno}\det(N_2^\sim N_1 + M_2^\sim M_1) &= \text{wno}\det\left\{M_2^\sim (I+P_2^\sim P_1)M_1\right\} \\ &= \text{wno}\det M_2^\sim + \text{wno}\det(I+P_2^\sim P_1) + \text{wno}\det M_1. \end{aligned}$$

Note that $\text{wno}\det M_1 = \eta(P_1)$, $\text{wno}\det M_2^\sim = -\text{wno}\det M_2 - \eta_0(M_2^{-1}) = -\eta(P_2)-\eta_0(P_2)$, and

$$\text{wno}\det(N_2^\sim N_1+M_2^\sim M_1) = -\eta(P_2)-\eta_0(P_2)+\text{wno}\det(I+P_2^\sim P_1)+\eta(P_1).$$

Furthermore,

$$\det(N_2^\sim N_1 + M_2^\sim M_1) \neq 0,\ \forall\omega \iff \det(I+P_2^\sim P_1)\neq 0,\ \forall\omega.$$

The theorem follows by noting that

$$\Psi(P_1,P_2) = (I+P_2P_2^\sim)^{-1/2}(P_1-P_2)(I+P_1^\sim P_1)^{-1/2}$$

since $\Psi(P_1,P_2) = -\tilde{N}_2 M_1 + \tilde{M}_2 N_1 = \tilde{M}_2(P_1-P_2)M_1$ and

$$\tilde{M}_2^\sim \tilde{M}_2 = (I+P_2P_2^\sim)^{-1}, \quad M_1 M_1^\sim = (I+P_1^\sim P_1)^{-1}.$$

$\square$

**Theorem 0.7** *Let $P_1 = N_1 M_1^{-1}$ and $P_2 = N_2 M_2^{-1}$ be normalized right coprime factorizations. Then*

$$\delta_\nu(P_1, P_2) = \inf_{\substack{Q,\, Q^{-1} \in \mathcal{L}_\infty \\ \text{wno } \det(Q) = 0}} \left\| \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} Q \right\|_\infty .$$

*Moreover, $\delta_g(P_1, P_2) b_{\text{obt}}(P_1) \le \delta_\nu(P_1, P_2) \le \delta_g(P_1, P_2)$.*

It is now easy to see that

$$\{P : \; \delta_\nu(P_0, P) < r\}$$

$$\supset \left\{ P = (N_0 + \Delta_N)(M_0 + \Delta_M)^{-1} : \; \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \in \mathcal{H}_\infty, \; \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_\infty < r \right\}.$$

Define

$$\frac{1}{b_{P,K}(\omega)} := \overline{\sigma} \left( \begin{bmatrix} I \\ K(j\omega) \end{bmatrix} (I + P(j\omega)K(j\omega))^{-1} \begin{bmatrix} I & P(j\omega) \end{bmatrix} \right)$$

and

$$\psi(P_1(j\omega), P_2(j\omega)) = \overline{\sigma}\left(\Psi(P_1(j\omega), P_2(j\omega))\right).$$

The following theorem states that robust stability can be checked using the frequency-by-frequency test.

**Theorem 0.8** *Suppose $(P_0, K)$ is stable and $\delta_\nu(P_0, P_1) < 1$. Then $(P_1, K)$ is stable if*

$$b_{P_0,K}(\omega) > \psi(P_0(j\omega), P_1(j\omega)), \quad \forall \omega.$$

*Moreover,*

$$\arcsin b_{P_1,K}(\omega) \ge \arcsin b_{P_0,K}(\omega) - \arcsin \psi(P_0(j\omega), P_1(j\omega)), \quad \forall \omega$$

*and*

$$\arcsin b_{P_1,K} \ge \arcsin b_{P_0,K} - \arcsin \delta_\nu(P_0, P_1).$$

**Proof.** Let $P_1 = \tilde{M}_1^{-1}\tilde{N}_1$, $P_0 = N_0 M_0^{-1} = \tilde{M}_0^{-1}\tilde{N}_0$ and $K = UV^{-1}$ be normalized coprime factorizations, respectively. Then

$$\frac{1}{b_{P_1,K}(\omega)} = \overline{\sigma}\left(\begin{bmatrix} V \\ U \end{bmatrix} (\tilde{M}_1 V + \tilde{N}_1 U)^{-1} \begin{bmatrix} \tilde{M}_1 & \tilde{N}_1 \end{bmatrix}\right) = \overline{\sigma}\left((\tilde{M}_1 V + \tilde{N}_1 U)^{-1}\right).$$

That is,

$$b_{P_1,K}(\omega) = \underline{\sigma}(\tilde{M}_1 V + \tilde{N}_1 U) = \underline{\sigma}\left(\begin{bmatrix} \tilde{M}_1 & \tilde{N}_1 \end{bmatrix}\begin{bmatrix} V \\ U \end{bmatrix}\right).$$

Similarly,

$$b_{P_0,K}(\omega) = \underline{\sigma}(\tilde{M}_0 V + \tilde{N}_0 U) = \underline{\sigma}\left(\begin{bmatrix} \tilde{M}_0 & \tilde{N}_0 \end{bmatrix}\begin{bmatrix} V \\ U \end{bmatrix}\right).$$

Note that

$$\psi(P_0(j\omega), P_1(j\omega)) = \overline{\sigma}\left(\begin{bmatrix} \tilde{M}_1 & \tilde{N}_1 \end{bmatrix}\begin{bmatrix} N_0 \\ -M_0 \end{bmatrix}\right)$$

$$\begin{bmatrix} N_0 & \tilde{M}_0^{\sim} \\ -M_0 & \tilde{N}_0^{\sim} \end{bmatrix}^{\sim}\begin{bmatrix} N_0 & \tilde{M}_0^{\sim} \\ -M_0 & \tilde{N}_0^{\sim} \end{bmatrix} = I.$$

To simplify the derivation, define

$$G_0 = \begin{bmatrix} N_0 \\ -M_0 \end{bmatrix}, \quad \tilde{G}_0 = \begin{bmatrix} \tilde{M}_0 & \tilde{N}_0 \end{bmatrix}, \quad \tilde{G}_1 = \begin{bmatrix} \tilde{M}_1 & \tilde{N}_1 \end{bmatrix}, \quad F = \begin{bmatrix} V \\ U \end{bmatrix}.$$

Then

$$\psi(P_0, P_1) = \overline{\sigma}(\tilde{G}_1 G_0), \quad b_{P_0,K}(\omega) = \underline{\sigma}(\tilde{G}_0 F), \quad b_{P_1,K}(\omega) = \underline{\sigma}(\tilde{G}_1 F)$$

and

$$\begin{bmatrix} G_0 & \tilde{G}_0^{\sim} \end{bmatrix}^{\sim}\begin{bmatrix} G_0 & \tilde{G}_0^{\sim} \end{bmatrix} = I \Longrightarrow \begin{bmatrix} G_0 & \tilde{G}_0^{\sim} \end{bmatrix}\begin{bmatrix} G_0 & \tilde{G}_0^{\sim} \end{bmatrix}^{\sim} = I.$$

That is,

$$G_0 G_0^\sim + \tilde{G}_0^\sim \tilde{G}_0 = I.$$

Note that

$$I = \tilde{G}_1 \tilde{G}_1^\sim = \tilde{G}_1 (G_0 G_0^\sim + \tilde{G}_0^\sim \tilde{G}_0) \tilde{G}_1^\sim = (\tilde{G}_1 G_0)(\tilde{G}_1 G_0)^\sim + (\tilde{G}_1 \tilde{G}_0^\sim)(\tilde{G}_1 \tilde{G}_0^\sim)^\sim.$$

Hence

$$\underline{\sigma}^2(\tilde{G}_1 \tilde{G}_0^\sim) = 1 - \overline{\sigma}^2(\tilde{G}_1 G_0).$$

Similarly,

$$I = F^\sim F = F^\sim (G_0 G_0^\sim + \tilde{G}_0^\sim \tilde{G}_0) F = (G_0^\sim F)^\sim (G_0^\sim F) + (\tilde{G}_0 F)^\sim (\tilde{G}_0 F)$$

$$\Longrightarrow \overline{\sigma}^2(G_0^\sim F) = 1 - \underline{\sigma}^2(\tilde{G}_0 F).$$

By the assumption, $\psi(P_0, P_1) < b_{P_0,K}(\omega)$; that is,

$$\overline{\sigma}(\tilde{G}_1 G_0) < \underline{\sigma}(\tilde{G}_0 F), \quad \forall \omega$$

and

$$\overline{\sigma}(G_0^\sim F) = \sqrt{1 - \underline{\sigma}^2(\tilde{G}_0 F)} < \sqrt{1 - \overline{\sigma}^2(\tilde{G}_1 G_0)} = \underline{\sigma}(\tilde{G}_1 \tilde{G}_0^\sim).$$

Hence

$$\overline{\sigma}(\tilde{G}_1 G_0) \overline{\sigma}(G_0^\sim F) < \underline{\sigma}(\tilde{G}_1 \tilde{G}_0^\sim) \underline{\sigma}(\tilde{G}_0 F);$$

that is,

$$\overline{\sigma}(\tilde{G}_1 G_0 G_0^\sim F) < \underline{\sigma}(\tilde{G}_1 \tilde{G}_0^\sim \tilde{G}_0 F), \quad \forall\, \omega$$

$$\Longrightarrow \left\| (\tilde{G}_1 \tilde{G}_0^\sim G_0 F)^{-1} (\tilde{G}_1 G_0 G_0^\sim F) \right\|_\infty < 1.$$

Now

$$\tilde{G}_1 F = \tilde{G}_1 (\tilde{G}_0^\sim \tilde{G}_0 + G_0 G_0^\sim) F = (\tilde{G}_1 \tilde{G}_0^\sim \tilde{G}_0 F) + (\tilde{G}_1 G_0 G_0^\sim F)$$

$$= (\tilde{G}_1 \tilde{G}_0^\sim \tilde{G}_0 F) \left( I + (\tilde{G}_1 \tilde{G}_0^\sim \tilde{G}_0 F)^{-1} (\tilde{G}_1 G_0 G_0^\sim F) \right).$$

By Lemma 0.5,

$$\text{wno}\ \det(\tilde{G}_1 F) = \text{wno}\ \det(\tilde{G}_1 \tilde{G}_0^\sim \tilde{G}_0 F) = \text{wno}\ \det(\tilde{G}_1 \tilde{G}_0^\sim) + \text{wno}\ \det(\tilde{G}_0 F).$$

Since $(P_0, K)$ is stable $\Longrightarrow (\tilde{G}_0 F)^{-1} \in \mathcal{H}_\infty \Longrightarrow \eta((\tilde{G}_0 F)^{-1}) = 0$

$$\Longrightarrow \text{wno } \det(\tilde{G}_0 F) := \eta((\tilde{G}_0 F)^{-1}) - \eta(\tilde{G}_0 F) = 0.$$

Next, note that

$$P_0^T = (\tilde{N}_0^T)(\tilde{M}_0^T)^{-1}, \quad P_1^T = (\tilde{N}_1^T)(\tilde{M}_1^T)^{-1}$$

and $\delta_\nu(P_0^T, P_1^T) = \delta_\nu(P_0, P_1) < 1$; then, by definition of $\delta_\nu(P_0^T, P_1^T)$,

$$\text{wno } \det((\tilde{N}_0^T)^\sim(\tilde{N}_1^T)+(\tilde{M}_0^T)^\sim(\tilde{M}_1^T)) = \text{wno } \det(\tilde{G}_1\tilde{G}_0^\sim)^T = \text{wno } \det(\tilde{G}_1\tilde{G}_0^\sim) = 0$$

Hence wno $\det(\tilde{G}_1 F) = 0$, but wno $\det(\tilde{G}_1 F) := \eta((\tilde{G}_1 F)^{-1}) - \eta(\tilde{G}_1 F) = \eta((\tilde{G}_1 F)^{-1})$ since $\eta(\tilde{G}_1 F) = 0$, so $\eta((\tilde{G}_1 F)^{-1}) = 0$; that is, $(P_1, K)$ is stable.

Finally, note that

$$\tilde{G}_1 F = \tilde{G}_1(\tilde{G}_0^\sim \tilde{G}_0 + G_0 G_0^\sim)F = (\tilde{G}_1\tilde{G}_0^\sim)(\tilde{G}_0 F) + (\tilde{G}_1 G_0)(G_0^\sim F)$$

and

$$\begin{aligned}
\underline{\sigma}(\tilde{G}_1 F) &\geq \underline{\sigma}(\tilde{G}_1\tilde{G}_0^\sim)\underline{\sigma}(\tilde{G}_0 F) - \overline{\sigma}(\tilde{G}_1 G_0)\overline{\sigma}(G_0^\sim F) \\
&= \sqrt{1 - \overline{\sigma}^2(\tilde{G}_1 G_0)}\,\underline{\sigma}(\tilde{G}_0 F) - \overline{\sigma}(\tilde{G}_1 G_0)\sqrt{1 - \underline{\sigma}^2(\tilde{G}_0 F)} \\
&= \sin(\arcsin \underline{\sigma}(\tilde{G}_0 F) - \arcsin \overline{\sigma}(\tilde{G}_1 G_0)) \\
&= \sin(\arcsin b_{P_0,K}(\omega) - \arcsin \psi(P_0(j\omega), P_1(j\omega)))
\end{aligned}$$

and, consequently,

$$\arcsin b_{P_1,K}(\omega) \geq \arcsin b_{P_0,K}(\omega) - \arcsin \psi(P_0(j\omega), P_1(j\omega))$$

and

$$\inf_\omega \arcsin b_{P_1,K}(\omega) \geq \inf_\omega \arcsin b_{P_0,K}(\omega) - \sup_\omega \arcsin \psi(P_0(j\omega), P_1(j\omega)).$$

That is, $\arcsin b_{P_1,K} \geq \arcsin b_{P_0,K} - \arcsin \delta_\nu(P_0, P_1)$. $\square$

The significance of the preceding theorem can be illustrated using Figure 0.35. It is clear from the figure that $\delta_\nu(P_0, P_1) > b_{P_0,K}$. Thus a frequency-independent stability test cannot conclude that a stabilizing controller $K$ for $P_0$ will stabilize $P_1$. However, the frequency-dependent test in the preceding theorem shows that $K$ stabilizes both $P_0$ and $P_1$ since $b_{P_0,K}(\omega) > \psi(P_0(j\omega), P_1(j\omega))$ for all $\omega$. Furthermore,

$$b_{P_1,K} \geq \inf_{\omega} \sin\left(\arcsin b_{P_0,K}(\omega) - \arcsin \psi(P_0, P_1)\right) > 0.$$

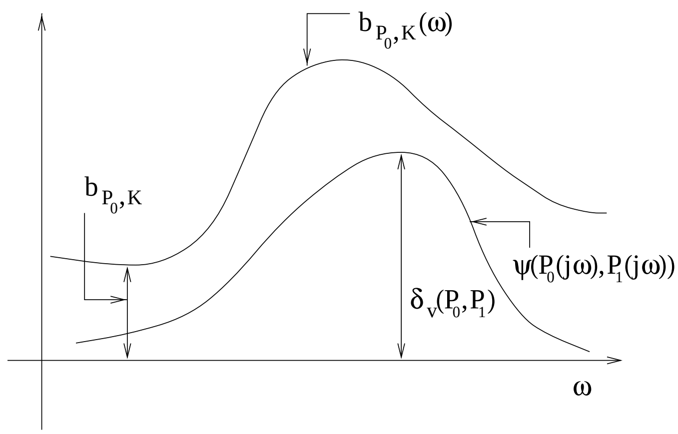



Figure 0.35: $K$ stabilizes both $P_0$ and $P_1$ since $b_{P_0,K}(\omega) > \psi(P_0, P_1)$ for all $\omega$

**Theorem 0.9** *Let $P_0$ be a nominal plant and $\beta \leq \alpha < b_{\text{obt}}(P_0)$.*

*(i) For a given controller $K$,*

$$\arcsin b_{P,K} > \arcsin \alpha - \arcsin \beta$$

*for all $P$ satisfying $\delta_\nu(P_0, P) \leq \beta$ if and only if $b_{P_0,K} > \alpha$.*

*(ii) For a given plant $P$,*

$$\arcsin b_{P,K} > \arcsin \alpha - \arcsin \beta$$

*for all $K$ satisfying $b_{P_0,K} > \alpha$ if and only if $\delta_\nu(P_0, P) \leq \beta$.*

**Theorem 0.10** *Suppose the feedback system with the pair $(P_0, K_0)$ is stable. Then*

$$\arcsin b_{P,K} \geq \arcsin b_{P_0,K_0} - \arcsin \delta_\nu(P_0, P) - \arcsin \delta_\nu(K_0, K)$$

*for any $P$ and $K$.*

**Proof.** Use the fact that $b_{P,K} = b_{K,P}$ and apply Theorem 0.8 to get

$$\arcsin b_{P,K} \geq \arcsin b_{P_0,K} - \arcsin \delta_\nu(P_0, P).$$

Dually, we have

$$\arcsin b_{P_0,K} \geq \arcsin b_{P_0,K_0} - \arcsin \delta_\nu(K_0, K).$$

Hence the result follows. $\square$

## Example

Consider again the following example, studied in Vinnicombe [1993b], with

$$P_1 = \frac{s-1}{s+1}, \quad P_2 = \frac{2s-1}{s+1}$$

and note that

$$1 + P_2^{\sim} P_1 = 1 + \frac{-2s-1}{-s+1}\frac{s-1}{s+1} = \frac{3s+2}{s+1}.$$

Then

$$1 + P_2^{\sim}(j\omega)P_1(j\omega) \neq 0, \quad \forall \omega, \quad \text{wno } \det(I + P_2^{\sim}P_1) + \eta(P_1) - \eta(P_2) = 0$$

and

$$\delta_\nu(P_1, P_2) = \|\Psi(P_1, P_2)\|_\infty = \sup_\omega \frac{|P_1 - P_2|}{\sqrt{1+|P_1|^2}\sqrt{1+|P_2|^2}}$$
$$= \sup_\omega \frac{|\omega|}{\sqrt{10\omega^2+4}} = \frac{1}{\sqrt{10}}.$$

This implies that any controller $K$ that stabilizes $P_1$ and achieves only $b_{P_1,K} > 1/\sqrt{10}$ will actually stabilize $P_2$. This result is clearly less conservative than that of using the gap metric. Furthermore, there exists a controller such that $b_{P_1,K} = 1/\sqrt{10}$ that *destabilizes* $P_2$. Such a controller is $K = -1/2$, which results in a closed-loop system with $P_2$ ill-posed.

## Example

$$P_1 = \frac{100}{2s+1}, \quad P_2 = \frac{100}{2s-1}, \quad P_3 = \frac{100}{(s+1)^2}.$$

$$\delta_\nu(P_1, P_2) = \delta_g(P_1, P_2) = 0.02, \quad \delta_\nu(P_1, P_3) = \delta_g(P_1, P_3) = 0.8988,$$
$$\delta_\nu(P_2, P_3) = \delta_g(P_2, P_3) = 0.8941,$$

which show that $P_1$ and $P_2$ are very close while $P_1$ and $P_3$ (or $P_2$ and $P_3$) are quite far away. It is not surprising that any reasonable controller for $P_1$ will do well for $P_2$ but not necessarily for $P_3$.

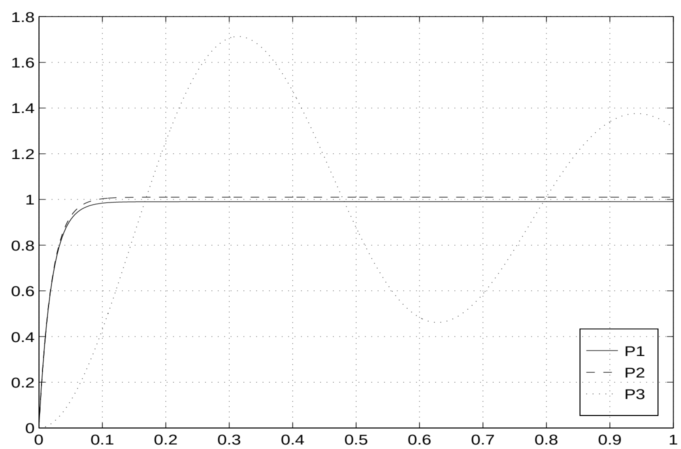



Figure 0.36: Closed-loop step responses with $K_1 = 1$

The corresponding stability margins for the closed-loop systems with $P_1$ and $P_2$ are

$$b_{P_1,K_1} = 0.7071, \quad \text{and} \quad b_{P_2,K_1} = 0.7,$$

respectively, which are very close to their maximally possible margins,

$$b_{\text{obt}}(P_1) = 0.7106, \quad \text{and} \quad b_{\text{obt}}(P_2) = 0.7036$$

(in fact, the optimal controllers for $P_1$ and $P_2$ are $K = 0.99$ and $K = 1.01$, respectively). While the stability margin for the closed-loop system with $P_3$ is

$$b_{P_3,K_1} = 0.0995,$$

which is far away from its optimal value, $b_{\mathrm{obt}}(P_3) = 0.4307$, and results in poor performance of the closed loop. In fact, it is not hard to find a controller that will perform well for both $P_1$ and $P_2$ but will destabilize $P_3$.

Of course, this does not necessarily mean that all controllers performing reasonably well with $P_1$ and $P_2$ will do badly with $P_3$, merely that some do — the unit feedback being an example. It may be harder to find a controller that will perform reasonably well with all three plants; the maximally stabilizing controller of $P_3$,

$$K_3 = \frac{2.0954s + 10.8184}{s + 23.2649},$$

is a such controller, which gives

$$b_{P_1,K_3} = 0.4307, \quad b_{P_2,K_3} = 0.4126, \quad \text{and} \quad b_{P_3,K_3} = 0.4307.$$

The step responses under this control law are shown in Figure 0.37.

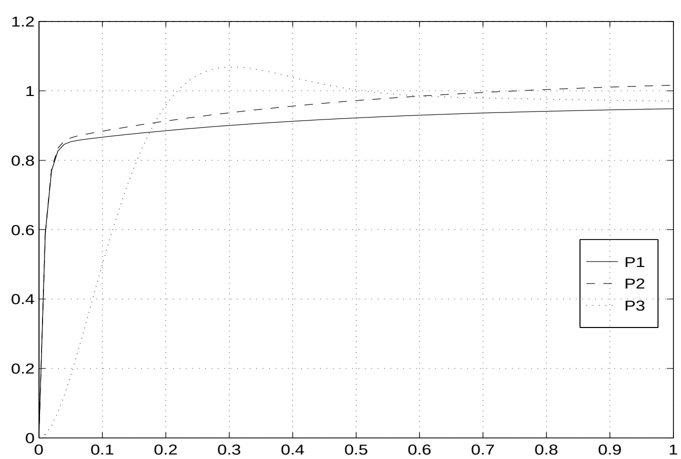



Figure 0.37: Closed-loop step responses with $K_3 = \dfrac{2.0954s + 10.8184}{s + 23.2649}$

# Geometric Interpretation of $\nu$-Gap Metric

$$\delta_\nu(P_1, P_2) = \sup_\omega \psi(P_1(j\omega),\ P_2(j\omega))$$

In particular, for a single-input single-output system,

$$\psi(P_1(j\omega),\ P_2(j\omega)) = \frac{|P_1(j\omega) - P_2(j\omega)|}{\sqrt{1 + |P_1(j\omega)|^2}\,\sqrt{1 + |P_2(j\omega)|^2}}. \tag{0.15}$$

This function has the interpretation of being the chordal distance between $P_1(j\omega)$ and $P_2(j\omega)$.

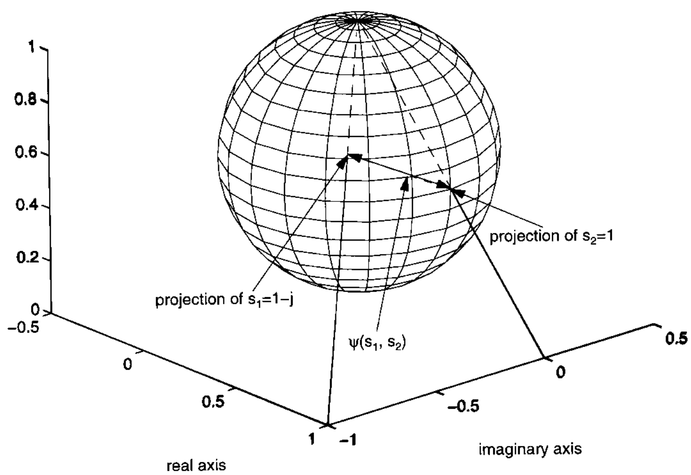



Figure 0.38: Projection onto the Riemann sphere

Now consider a circle of chordal radius $r$ centered at $P_0(j\omega_0)$ on the Riemann sphere for some frequency $\omega_0$; that is,

$$\frac{|P(j\omega_0) - P_0(j\omega_0)|}{\sqrt{1 + |P(j\omega_0)|^2}\,\sqrt{1 + |P_0(j\omega_0)|^2}} = r.$$

Let $P(j\omega_0) = R + jI$ and $P_0(j\omega_0) = R_0 + jI_0$. Then it is easy to show that

$$\left(R - \frac{R_0}{1-\alpha}\right)^2 + \left(I - \frac{I_0}{1-\alpha}\right)^2 = \frac{\alpha(1 + |P_0|^2 - \alpha)}{(1-\alpha)^2}, \quad \text{if } \alpha \neq 1$$

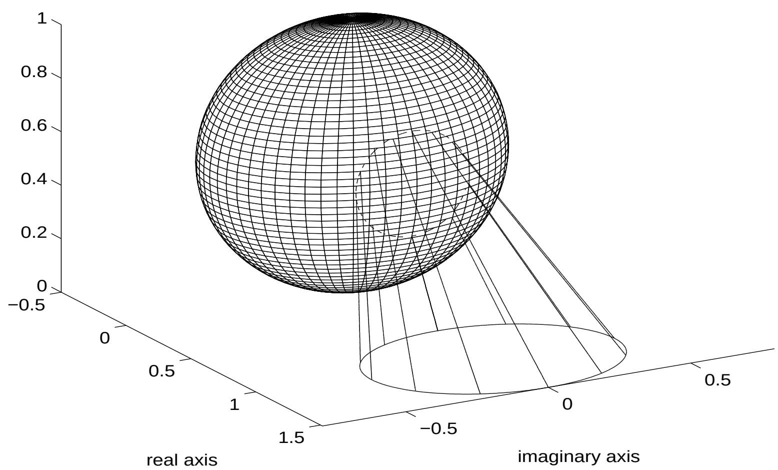



Figure 0.39: Projection of a disk on the Nyquist diagram onto the Riemann sphere

where $\alpha = r^2(1 + |P_0|^2)$.

For example, an uncertainty of 0.2 at $|p_0(j\omega_0)| = 1$ for some $\omega_0$ (i.e., $\delta_\nu(p_0, p) \leq 0.2$) implies that $0.661 \leq |p(j\omega_0)| \leq 1.513$ and the phase difference between $p_0$ and $p$ is no more than $23.0739^{\rm o}$ at $\omega_0$.

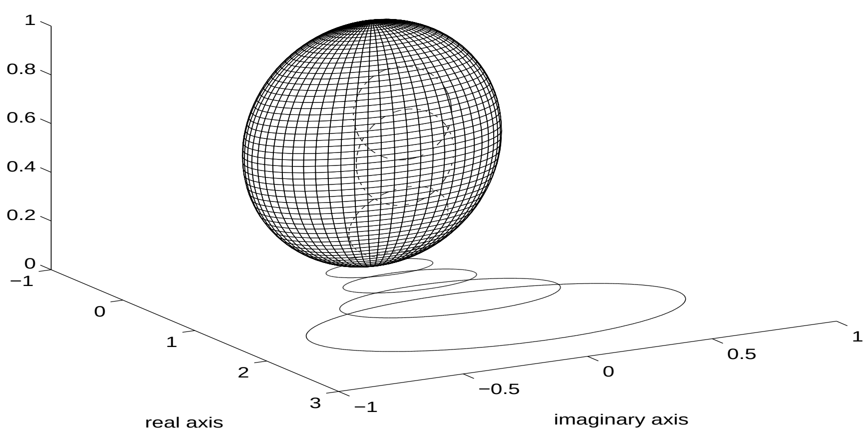



Figure 0.40: Uncertainty on the Riemann sphere and the corresponding uncertainty on the Nyquist diagram

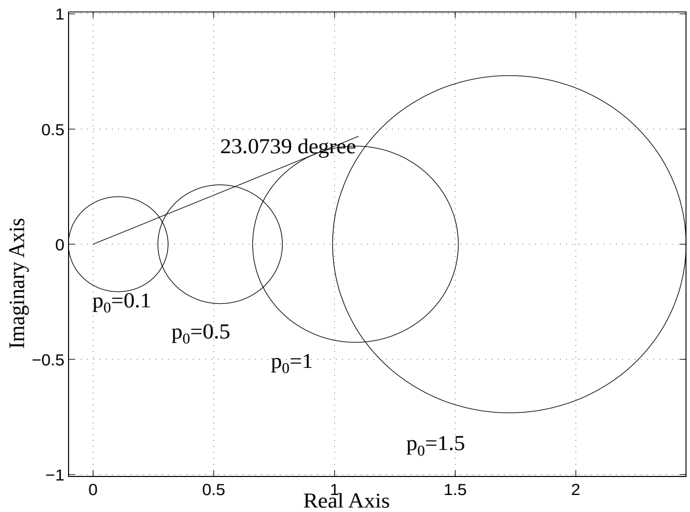



Figure 0.41: Uncertainty on the Nyquist diagram corresponding to the balls of uncertainty on the Riemann sphere centered at $p_0$ with chordal radius 0.2

## The Necessity of WNO

$\|\Psi(P_1, P_2)\|_\infty$ on its own without the winding number condition is useless for the study of feedback systems.

Consider

$$P_1 = 1, \quad P_2 = \frac{s-1-\epsilon}{s-1}.$$

It is clear that $P_2$ becomes increasingly difficult to stabilize as $\epsilon \longrightarrow 0$ due to the near unstable pole/zero cancellation. In fact, any stabilizing controller for $P_1$ will destabilize all $P_2$ for $\epsilon$ sufficiently small. This is confirmed by noting that $b_{\text{obt}}(P_1) = 1,\ b_{\text{obt}}(P_2) \approx \epsilon/2$, and

$$\delta_g(P_1, P_2) = \delta_\nu(P_1, P_2) = 1, \quad \epsilon \geq -2.$$

However, $\|\Psi(P_1, P_2)\|_\infty = \frac{|\epsilon|}{\sqrt{4+4\epsilon+2\epsilon^2}} \approx \frac{\epsilon}{2}$ in itself fails to indicate the difficulty of the problem.

Let $\mathcal{P}$ be a family of parametric uncertainty systems and let $P_0 \in \mathcal{P}$ be a nominal design model. We are interested in finding a controller so that we have the largest possible robust stability margin; that is,

$$\sup_K \inf_{P\in\mathcal{P}} b_{P,K}.$$

Note that by Theorem 0.8, for any $P_1 \in \mathcal{P}$, we have

$$\arcsin b_{P_1,K}(\omega) \geq \arcsin b_{P_0,K}(\omega) - \arcsin \psi(P_0(j\omega), P_1(j\omega)), \quad \forall\omega.$$

Now suppose we need $\inf_{P\in\mathcal{P}} b_{P,K} > \alpha$. Then it is sufficient to have

$$\arcsin b_{P_0,K}(\omega) - \arcsin \psi(P_0(j\omega), P_1(j\omega)) > \arcsin \alpha, \quad \forall\omega, \quad P_1 \in \mathcal{P};$$

that is,

$$b_{P_0,K}(\omega) > \sin\left(\arcsin \psi(P_0(j\omega), P_1(j\omega)) + \arcsin \alpha\right), \quad \forall\omega, \quad P_1 \in \mathcal{P}.$$

Let $W(s) \in \mathcal{H}_\infty$ be such that

$$|W(j\omega)| \geq \sin\left(\arcsin \psi(P_0(j\omega), P_1(j\omega)) + \arcsin \alpha\right), \quad \forall\omega, \quad P_1 \in \mathcal{P}.$$

Then it is sufficient to guarantee

$$\frac{|W(j\omega)|}{b_{P_0,K}(\omega)} < 1.$$

Let $P_0 = \tilde{M}_0^{-1}\tilde{N}_0$ be a normalized left coprime factorization and note that

$$\frac{1}{b_{P_0,K}(\omega)} := \overline{\sigma}\left(\begin{bmatrix} I \\ K(j\omega) \end{bmatrix} (I + P_0(j\omega)K(j\omega))^{-1}\, \tilde{M}_0^{-1}(j\omega)\right).$$

Then it is sufficient to find a controller so that

$$\left\| \begin{bmatrix} I \\ K \end{bmatrix} (I + P_0K)^{-1}\, \tilde{M}_0^{-1} W \right\|_\infty < 1.$$

The process can be iterated to find the largest possible $\alpha$.

**Design Procedure:**

Let $\mathcal{P}$ be a family of parametric uncertain systems and let $P_0$ be a nominal model.

(a) Loop-Shaping: The singular values of the nominal plant are shaped, using a precompensator $W_1$ and/or a postcompensator $W_2$, to give a desired open-loop shape. The nominal plant $P_0$ and the shaping functions $W_1, W_2$ are combined to form the shaped plant, $P_s$, where $P_s = W_2 P_0 W_1$. We assume that $W_1$ and $W_2$ are such that $P_s$ contains no hidden modes.

(b) Compute *frequency-by-frequency*:

$$f(\omega) = \sup_{P \in \mathcal{P}} \psi(P_s(j\omega), W_2(j\omega)P(j\omega)W_1(j\omega)).$$

Set $\alpha = 0$.

(b) Fit a stable and minimum phase rational transfer function $W(s)$ so that

$$|W(j\omega)| \geq \sin(\arcsin f(\omega) + \arcsin \alpha) \ \ \forall \omega.$$

(c) Find a $K_\infty$ such that

$$\beta := \inf_{K_\infty} \left\| \begin{bmatrix} I \\ K_\infty \end{bmatrix} (I + P_0 K_\infty)^{-1} \tilde{M}_0^{-1} W \right\|_\infty .$$

(d) If $\beta \approx 1$, stop and the final controller is $K = W_1 K_\infty W_2$. If $\beta \ll 1$, increase $\alpha$ and go back to (b). If $\beta \gg 1$, decrease $\alpha$ and go back to (b).

## Controller Order Reduction

**Theorem 0.11** *Let $P_0$ be a nominal plant and $K_0$ be a stabilizing controller such that $b_{P_0,K_0} \le b_{\rm obt}(P_0)$. Let $K_0 = UV^{-1}$ be a normalized coprime factorization and let $\hat{U}, \hat{V} \in \mathcal{RH}_\infty$ be such that*

$$\left\| \begin{bmatrix} U \\ V \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right\|_\infty \le \varepsilon.$$

*Then $K := \hat{U}\hat{V}^{-1}$ stabilizes $P_0$ if $\varepsilon < b_{P_0,K_0}$. Furthermore,*

$$\arcsin b_{P,K} \ge \arcsin b_{P_0,K_0} - \arcsin \varepsilon - \arcsin \beta$$

*for all $\{P : \delta_\nu(P, P_0) \le \beta\}$.*

Hence to reduce the controller order one only needs to approximate the normalized coprime factors of the controller.

## Chapter 18: Miscellaneous Topics

- Model Validation
- Mixed $\mu$

# Model Validation

Question: how one can decide if a model description is appropriate (i.e., how to validate a model).

Consider a set of uncertain discrete-time dynamical systems:

$$\mathbf{\Delta} := \{\Delta : \ \Delta \in \mathcal{H}_\infty, \quad \|\Delta\|_\infty \leq 1\}$$

where $\|\Delta(z)\|_\infty = \sup_{|z|>1} \overline{\sigma}\left(\Delta(z)\right)$.

experimental data:

$$u = (u_0, u_1, \ldots, u_{l-1}), \quad y = (y_0, y_1, \ldots, y_{l-1})$$

Question: are these data consistent with our modeling assumption?

Does there exist a model $\Delta \in \mathbf{\Delta}$ such that $y = (y_0, y_1, \ldots, y_{l-1})$ with the input $u = (u_0, u_1, \ldots, u_{l-1})$?

- No, the model is invalidated.
- Yes, the model is not invalidated.

Let $\Delta$ be a stable, causal, LTI system with

$$\Delta(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \cdots$$

where $h_i, i = 0, 1, \ldots$ are the matrix Markov parameters.

Suppose input sequence $u = (u_0, u_1, \ldots, u_{l-1})$ generates the output $y = (y_0, y_1, \ldots, y_{l-1})$ for the period $t = 0, 1, \ldots, \ell - 1$,

Then

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{l-1} \end{bmatrix} = \begin{bmatrix} h_0 & 0 & \cdots & 0 \\ h_1 & h_0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ h_{l-1} & h_{l-2} & \cdots & h_0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{l-1} \end{bmatrix}.$$

- if $u_0 \neq 0$ and $\Delta$ is SISO, $h_0, \ldots, h_{\ell-1}$ are uniquely determined by $u_i$ and $y_i$.
- The model is not invalidated if the remaining Markov parameters can be chosen so that $\Delta(z) \in \mathbf{\Delta}$.
- Answer: classical tangential Carathéodory-Fejér interpolation problem

Let $\pi_\ell$ denote the truncation operator such that

$$\pi_\ell(v_0, v_1, \ldots, v_{\ell-1}, v_\ell, v_{\ell+1}, \ldots) = (v_0, v_1, \ldots, v_{\ell-1}) =: v.$$

Denote

$$T_v := \begin{bmatrix} v_0 & 0 & \cdots & 0 \\ v_1 & v_0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ v_{l-1} & v_{l-2} & \cdots & v_0 \end{bmatrix}.$$

**Theorem 0.12** *Given $u = (u_0, u_1, \ldots, u_{l-1})$ and $y = (y_0, y_1, \ldots, y_{l-1})$, there exists a $\Delta \in \mathcal{H}_\infty$, $\|\Delta\|_\infty \leq 1$ such that*

$$y = \pi_\ell \Delta u$$

*if and only if $T_y^* T_y \leq T_u^* T_u$ or $\overline{\sigma}\left(T_y (T_u^* T_u)^{-\frac{1}{2}}\right) \leq 1$ if $u_0 \neq 0$.*

Note that the output of $\Delta$ after time $t = \ell - 1$ is irrelevant to the test. The condition $T_y^* T_y \leq T_u^* T_u$ is equivalent to

$$\sum_{j=1}^{i} \|y_j\|^2 \leq \sum_{j=1}^{i} \|u_j\|^2, i = 0, 1, \ldots, \ell - 1$$

or

$$\|\pi_i y\|_2 \leq \|\pi_i u\|_2, i = 0, 1, \ldots, \ell - 1,$$

which is obviously necessary.

# Model Validation: Additive Example

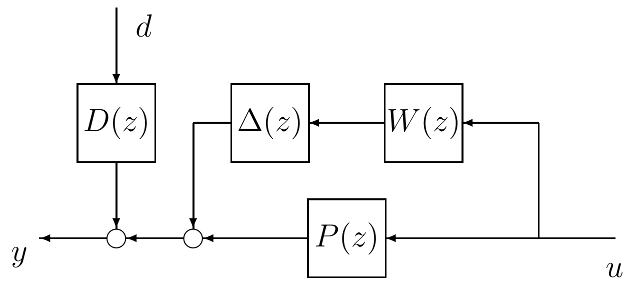



Figure 0.42: Model validation for additive uncertainty

$$y = (P + \Delta W)u + Dd, \quad \|\Delta\|_\infty \leq 1$$

$d \in \mathcal{D}_{\text{convex}}$

Assume $W(\infty)$ is of full column rank. Let

$$D(z) = D_0 + D_1 z^{-1} + D_2 z^{-2} + \cdots.$$

**Theorem 0.13** *Given data* $u_{\text{expt}} = (u_0, u_1, \ldots, u_{\ell-1})$ *with* $u_0 \neq 0$, $y_{\text{expt}} = (y_0, y_1, \ldots, y_{\ell-1})$ *with* $d \in \mathcal{D}_{\text{convex}}$, *let*

$$\hat{u} = (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{\ell-1}) = \pi_\ell(W u_{\text{expt}})$$

$$\hat{y} = (\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_{\ell-1}) = y_{\text{expt}} - \pi_\ell P u_{\text{expt}}.$$

*Then there exists a* $\Delta \in \mathcal{H}_\infty$, $\|\Delta\|_\infty \leq 1$ *such that*

$$y_{\text{expt}} = \pi_\ell\left((P + \Delta W)u_{\text{expt}} + Dd\right)$$

*for some* $d \in \mathcal{D}_{\text{convex}}$ *iff there exists a* $d = (d_0, d_1, \ldots, d_{l-1}) \in \pi_\ell \mathcal{D}_{\text{convex}}$ *such that*

$$\overline{\sigma}\left[(T_{\hat{y}} - T_D T_d)(T_{\hat{u}}^* T_{\hat{u}})^{-1/2}\right] \leq 1$$

*where*

$$T_D := \begin{bmatrix} D_0 & 0 & \cdots & 0 \\ D_1 & D_0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ D_{l-1} & D_{l-2} & \cdots & D_0 \end{bmatrix}.$$

**Proof**.

$$(y - Pu) - Dd = \Delta(Wu).$$

Since $P, W, D$, and $\Delta$ are causal, linear, and time invariant, we have $\pi_\ell Dd = \pi_\ell D \pi_\ell d$, $\pi_\ell(y - Pu) = y_{\text{expt}} - \pi_\ell P \pi_\ell u = y_{\text{expt}} - \pi_\ell P u_{\text{expt}}$ and $\pi_\ell W u = \pi_\ell W \pi_\ell u = \pi_\ell W u_{\text{expt}}$. Denote

$$\hat{d} = (\hat{d}_0, \hat{d}_1, \ldots, \hat{d}_{\ell-1}) = \pi_\ell(Dd).$$

Then it is easy to show that

$$\begin{bmatrix} \hat{d}_0 \\ \hat{d}_1 \\ \vdots \\ \hat{d}_{\ell-1} \end{bmatrix} = T_D \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{\ell-1} \end{bmatrix}$$

and $T_{\hat{d}} = T_D T_d$. Now note that

$$T_{\pi_\ell(y-Pu-Dd)} = T_{\pi_\ell(y-Pu)} - T_{\pi_\ell(Dd)} = T_{\hat{y}} - T_D T_d, \quad T_{\pi_\ell W u} = T_{\hat{u}}$$

and $\pi_\ell \Delta W u = \pi_\ell \Delta \pi_\ell (Wu)$ since $\Delta$ is causal. Applying Theorem 0.12, there exists a $\Delta \in \mathcal{H}_\infty$, $\|\Delta\|_\infty \leq 1$ such that

$$\pi_\ell \left[(y - Pu) - Dd\right] = \pi_\ell \Delta(Wu) = \pi_\ell \Delta \pi_\ell (Wu)$$

if and only if

$$\begin{aligned} & (T_{\hat{y}} - T_D T_d)^*(T_{\hat{y}} - T_D T_d) \leq T_{\hat{u}}^* T_{\hat{u}} \\ \iff \quad & \overline{\sigma}\left[(T_{\hat{y}} - T_D T_d)(T_{\hat{u}}^* T_{\hat{u}})^{-\frac{1}{2}}\right] \leq 1. \end{aligned}$$

Note that $T_{\hat{u}}$ is of full column rank since $W(\infty)$ is of full column rank and $u_0 \neq 0$, which implies $\hat{u}_0 \neq 0$. $\square$

Note that

$$\inf_{d \in \mathcal{D}_{\text{convex}}} \overline{\sigma}\left[(T_{\hat{y}} - T_D T_d)(T_{\hat{u}}^* T_{\hat{u}})^{-\frac{1}{2}}\right] \leq 1$$

is a convex problem and can be checked numerically.

**Mixed $\mu$ Analysis and Synthesis**

uncertainties $\mathbf{\Delta} \subset \mathbb{C}^{n \times n}$ is defined as

$$\begin{aligned}\mathbf{\Delta} &= \Big\{\text{diag}\ \Big[\phi_1 I_{k_1}, \ldots, \phi_{s_r} I_{k_{s_r}}, \delta_1 I_{r_1}, \ldots, \delta_{s_c} I_{r_{s_c}}, \\ &\quad \Delta_1, \ldots, \Delta_F\Big] : \phi_i \in \mathbb{R},\ \delta_j \in \mathbb{C},\ \Delta_\ell \in \mathbb{C}^{m_\ell \times m_\ell}\Big\}.\end{aligned}$$

Then

$$\mu_{\mathbf{\Delta}}(M) := \left(\min\left\{\overline{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \det\left(I - M\Delta\right) = 0\right\}\right)^{-1}$$

unless no $\Delta \in \mathbf{\Delta}$ makes $I - M\Delta$ singular, in which case $\mu_{\mathbf{\Delta}}(M) := 0$. Or, equivalently,

$$\frac{1}{\mu_{\mathbf{\Delta}}(M)} := \inf\left\{\alpha :\ \det(I - \alpha M\Delta) = 0,\ \overline{\sigma}(\Delta) \le 1,\ \Delta \in \mathbf{\Delta}\right\}.$$

Let $\rho_R(M)$ be the real spectral radius (i.e., the largest magnitude of the real eigenvalues of $M$). Then

$$\mu_{\mathbf{\Delta}}(M) = \max_{\Delta \in \mathbf{B\Delta}} \rho_R(M\Delta)$$

where $\mathbf{B\Delta} := \{\Delta :\ \Delta \in \mathbf{\Delta},\ \overline{\sigma}(\Delta) \le 1\}$.

Define

$$\mathcal{Q} = \{\Delta \in \mathbf{\Delta} :\ \phi_i \in [-1, 1],\ |\delta_i| = 1,\ \Delta_i \Delta_i^* = I_{m_i}\}$$

$$\mathcal{D} = \left\{\begin{array}{l}\text{diag}\left[\tilde{D}_1, \ldots, \tilde{D}_{s_r}, D_1, \ldots, D_{s_c}, d_1 I_{m_1}, \ldots, d_{F-1} I_{m_{F-1}}, I_{m_F}\right] : \\ \ \tilde{D}_i \in \mathbb{C}^{k_i \times k_i},\ \tilde{D}_i = \tilde{D}_i^* > 0,\ D_i \in \mathbb{C}^{r_i \times r_i},\ D_i = D_i^* > 0,\ d_j \in \mathbb{R}, d_j > 0\end{array}\right\}$$

$$\mathcal{G} = \left\{\text{diag}\left[G_1, \ldots, G_{s_r}, 0, \ldots, 0\right] :\ \ G_i = G_i^* \in \mathbb{C}^{k_i \times k_i}\right\}.$$

Then

$$\mu_{\mathbf{\Delta}}(M) = \max_{Q \in \mathcal{Q}} \rho_R(QM)$$

- not necessarily achieved on the vertices for the real parameters
- may not be a continuous function of the data
- NP hard problem

## Upper Bound of Mixed $\mu$

$$\mu_{\mathbf{\Delta}}(M) \leq \inf_{D \in \mathcal{D}} \overline{\sigma}(DMD^{-1}).$$

LMI form:

$$\overline{\sigma}(DMD^{-1}) \leq \beta \iff (DMD^{-1})^* DMD^{-1} \leq \beta^2 I$$

$$\iff M^* D^* DM - \beta^2 D^* D \leq 0.$$

Since $D^* D = D^2 \in \mathcal{D}$, we have

$$\mu_{\mathbf{\Delta}}(M) \leq \inf_{D \in \mathcal{D}} \min_{\beta} \left\{ \beta : \ M^* DM - \beta^2 D \leq 0 \right\}.$$

**Theorem 0.14** *Let $M \in \mathbb{C}^{n \times n}$ and $\Delta \in \mathbf{\Delta}$. Then*

$$\mu_{\mathbf{\Delta}}(M) \leq \inf_{D \in \mathcal{D}, G \in \mathcal{G}} \min_{\beta} \left\{ \beta : \ M^* DM + j(GM - M^* G) - \beta^2 D \leq 0 \right\}.$$

**Proof**. Suppose we have a $Q \in \mathcal{Q}$ such that $QM$ has a real eigenvalue $\lambda \in \mathbb{R}$. Then there is a vector $x \in \mathbb{C}^n$ such that

$$QMx = \lambda x.$$

Let $D \in \mathcal{D}$. Then $D^{\frac{1}{2}} \in \mathcal{D}$, $D^{\frac{1}{2}} Q = Q D^{\frac{1}{2}}$ and

$$D^{\frac{1}{2}} QMx = QD^{\frac{1}{2}} Mx = \lambda D^{\frac{1}{2}} x.$$

Since $\overline{\sigma}(Q) \leq 1$, it follows that

$$\lambda^2 \left\| D^{\frac{1}{2}} x \right\|^2 = \left\| QD^{\frac{1}{2}} Mx \right\|^2 \leq \left\| D^{\frac{1}{2}} Mx \right\|^2.$$

Hence

$$x^* (M^* DM - \lambda^2 D) x \geq 0.$$

Next, let $G \in \mathcal{G}$ and note that $Q = Q^*$ and $Q^* G = QG = GQ$; then

$$x^* GMx = \left( \frac{1}{\lambda} QMx \right)^* GMx = \frac{1}{\lambda} x^* M^* Q^* GMx = \frac{1}{\lambda} x^* M^* QGMx$$

$$= \frac{1}{\lambda} x^* M^* G Q M x = \frac{1}{\lambda} x^* M^* G (QMx) = x^* M^* G x.$$

That is,

$$x^*(GM - M^*G)x = 0.$$

Note that $j(GM - M^*G)$ is a Hermitian matrix, so it follows that for such $x$

$$x^*(M^*DM + j(GM - M^*G) - \lambda^2 D)x \geq 0.$$

It is now easy to see that if we have $D \in \mathcal{D}$, $G \in \mathcal{G}$ and $0 \leq \beta \in \mathbb{R}$ such that

$$M^*DM + j(GM - M^*G) - \beta^2 D \leq 0$$

then $|\lambda| \leq \beta$, and hence $\mu_{\mathbf{\Delta}}(M) \leq \beta$. $\square$

## Interpretation of the Bound

Interpretation: covering the uncertainties on the real axis using possibly off-axis disks.

Example: $M \in \mathbb{C}$ and $\Delta \in [-1, 1]$. The off-axis disk is

$$j\frac{G}{\beta} + \sqrt{1 + \left(\frac{G}{\beta}\right)^2}\ \tilde{\Delta}, \quad \tilde{\Delta} \in \mathbb{C},\ \left|\tilde{\Delta}\right| \le 1.$$

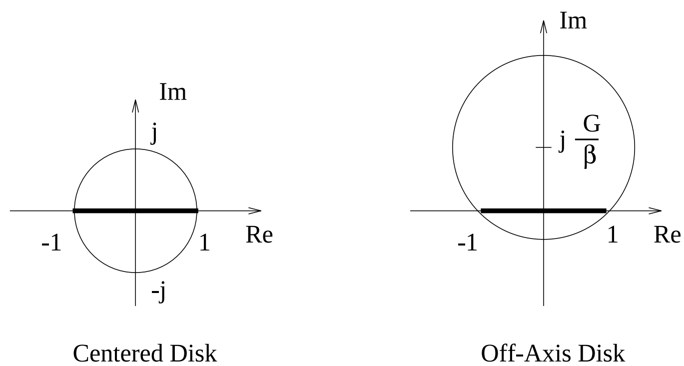



Figure 0.43: Covering real parameters with disks

Hence $1 - \Delta\frac{M}{\beta} \neq 0$ for all $\Delta \in [-1, 1]$ is guaranteed if

$$1 - \left(j\frac{G}{\beta} + \sqrt{1 + \left(\frac{G}{\beta}\right)^2}\ \tilde{\Delta}\right)\frac{M}{\beta} \neq 0, \quad \tilde{\Delta} \in \mathbb{C},\ \left|\tilde{\Delta}\right| \le 1$$

$$\Longleftrightarrow \quad 1 - \frac{\sqrt{1 + \left(\frac{G}{\beta}\right)^2}\ \frac{M}{\beta}}{1 - j\frac{G}{\beta}\frac{M}{\beta}}\ \tilde{\Delta} \neq 0, \quad \tilde{\Delta} \in \mathbb{C},\ \left|\tilde{\Delta}\right| \le 1$$

$$\Longleftrightarrow \quad \left(\frac{\sqrt{1 + \left(\frac{G}{\beta}\right)^2}\ \frac{M}{\beta}}{1 - j\frac{G}{\beta}\frac{M}{\beta}}\right)^* \left(\frac{\sqrt{1 + \left(\frac{G}{\beta}\right)^2}\ \frac{M}{\beta}}{1 - j\frac{G}{\beta}\frac{M}{\beta}}\right) \le 1$$

$$\Longleftrightarrow \quad \frac{M^*}{\beta}\frac{M}{\beta} + j\left(\frac{G}{\beta}\frac{M}{\beta} - \frac{M^*}{\beta}\frac{G}{\beta}\right) - 1 \le 0$$

$$\Longleftrightarrow \quad M^*M + j(GM - M^*G) - \beta^2 \le 0.$$

The scaling $G$ allows one to exploit the phase information about the real parameters so that a better upper bound can be obtained. We shall demonstrate this further using a simple example.

# Example

$$G(s) = \frac{s^2 + 2s + 1}{s^3 + s^2 + 2s + 1}.$$

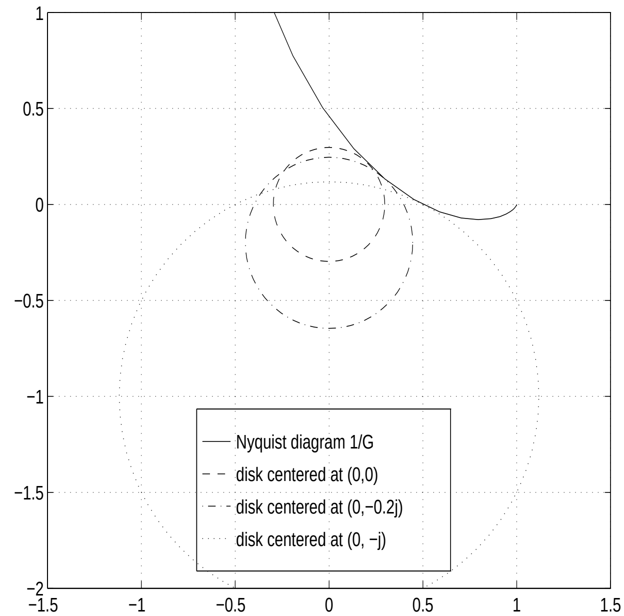



Figure 0.44: Computing the real stability margin by covering with disks

Find the largest $k$ such that $1 + \Delta G(s)$ has no zero in the right-half plane for all $\Delta \in [-k, k]$.

$$k_{\max} = \left(\sup_{\omega} \mu_{\mathbf{\Delta}}(G(j\omega))\right)^{-1} = \inf_{\omega} \left\{ \frac{1}{|G(j\omega)|} : \ \Im G(j\omega) = 0 \right\} = 0.5.$$

Now we use the complex covering idea to find the best possible $k$: find the smallest $|\Delta|$ so that $1 + \Delta G(j\omega_0) = 0$ for some $\omega_0 \leftrightarrow \Delta + 1/G(j\omega_0) = 0$.

disks covering an interval $[-k, k]$:

a centered disk: $k = 1/\left\|G\right\|_{\infty} = 0.2970$

an off-axis disk centered at $(0, -0.2j)$: $k = 0.3984$

an off-axis disk centered at $(0, -j)$: $k = 0.5$.

# Alternative characterization of the upper bound

**Theorem 0.15** *Given $\beta > 0$, there exist $D \in \mathcal{D}$ and $G \in \mathcal{G}$ such that*

$$M^* D M + j(GM - M^* G) - \beta^2 D \leq 0$$

*if and only if there are $D_1 \in \mathcal{D}$ and $G_1 \in \mathcal{G}$ such that*

$$\overline{\sigma}\left(\left(\frac{D_1 M D_1^{-1}}{\beta} - jG_1\right)(I + G_1^2)^{-\frac{1}{2}}\right) \leq 1.$$

**Proof.** Let $D = D_1^2$ and $G = \beta D_1 G_1 D_1$. Then

$$M^* D M + j(GM - M^* G) - \beta^2 D \leq 0$$

$$\iff M^* D_1^2 M + j(\beta D_1 G_1 D_1 M - \beta M^* D_1 G_1 D_1) - \beta^2 D_1^2 \leq 0$$

$$\iff (D_1 M D_1^{-1})^* (D_1 M D_1^{-1}) + j(\beta G_1 D_1 M D_1^{-1} - \beta (D_1 M D_1^{-1})^* G_1) - \beta^2 I \leq 0$$

$$\iff \left(\frac{D_1 M D_1^{-1}}{\beta} - jG_1\right)^* \left(\frac{D_1 M D_1^{-1}}{\beta} - jG_1\right) - (I + G_1^2) \leq 0$$

$$\iff \overline{\sigma}\left[\left(\frac{D_1 M D_1^{-1}}{\beta} - jG_1\right)(I + G_1^2)^{-\frac{1}{2}}\right] \leq 1.$$

$\square$

**Corollary 0.16** *$\mu_{\mathbf{\Delta}}(M) \le r\beta$ if there are $D_1 \in \mathcal{D}$ and $G_1 \in \mathcal{G}$ such that*

$$\overline{\sigma}\left(\left(\frac{D_1 M D_1^{-1}}{\beta} - jG_1\right)(I + G_1^2)^{-\frac{1}{2}}\right) \le r \le 1.$$

**Proof.** This follows by noting that

$$\overline{\sigma}\left(\left(\frac{D_1 M D_1^{-1}}{\beta} - jG_1\right)(I + G_1^2)^{-\frac{1}{2}}\right) \le r \le 1$$

$$\Longrightarrow \left(\frac{D_1 M D_1^{-1}}{r\beta} - j\frac{G_1}{r}\right)^* \left(\frac{D_1 M D_1^{-1}}{r\beta} - j\frac{G_1}{r}\right) \le I + G_1^2 \le I + \left(\frac{G_1}{r}\right)^2.$$

Let $G_2 = \dfrac{G_1}{r} \in \mathcal{G}$. Then

$$\left(\frac{D_1 M D_1^{-1}}{r\beta} - jG_2\right)^* \left(\frac{D_1 M D_1^{-1}}{r\beta} - jG_2\right) \le I + G_2^2$$

$$\Longrightarrow \overline{\sigma}\left(\left(\frac{D_1 M D_1^{-1}}{r\beta} - jG_2\right)(I + G_2^2)^{-\frac{1}{2}}\right) \le 1$$

$$\Longrightarrow \mu_{\mathbf{\Delta}}(M) \le r\beta.$$

□

# D, G-K Iteration

Find $K$ so that

$$\min_K \sup_\omega \mu_{\mathbf{\Delta}}\left(\mathcal{F}_\ell(P, K)\right) \le \beta.$$

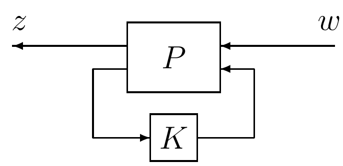



Figure 0.45: Synthesis framework

Note that $\exists\ D_\omega \in \mathcal{D}$ and $G_\omega \in \mathcal{G}$ such that

$$\sup_\omega \overline{\sigma}\left[\left(\frac{D_\omega\left(\mathcal{F}_\ell\left(P(j\omega), K(j\omega)\right)\right) D_\omega^{-1}}{\beta} - jG_\omega\right)(I + G_\omega^2)^{-\frac{1}{2}}\right] \le 1, \quad \forall \omega.$$

$$\Downarrow$$

$$\mu_{\mathbf{\Delta}}\left(\mathcal{F}_\ell\left(P(j\omega), K(j\omega)\right)\right) \le \beta,\ \forall \omega$$

## $D, G - K$ Iteration:

(1) Let $K$ be a stabilizing controller. Find initial estimates of the scaling matrices $D_\omega \in \mathcal{D}$, $G_\omega \in \mathcal{G}$ and a scalar $\beta_1 > 0$ such that

$$\sup_\omega \overline{\sigma}\left[\left(\frac{D_\omega\left(\mathcal{F}_\ell\left(P(j\omega), K(j\omega)\right)\right) D_\omega^{-1}}{\beta_1} - jG_\omega\right)(I + G_\omega^2)^{-\frac{1}{2}}\right] \le 1, \quad \forall \omega.$$

Obviously, one may start with $D_\omega = I$, $G_\omega = 0$, and a large $\beta_1 > 0$.

(2) Fit the frequency response matrices $D_\omega$ and $jG_\omega$ with $D(s)$ and $G(s)$ so that

$$D(j\omega) \approx D_\omega, \quad G(j\omega) \approx jG_\omega, \quad \forall\ \omega.$$

Then for $s = j\omega$

$$\sup_\omega \overline{\sigma}\left(\left(\frac{D_\omega\left(\mathcal{F}_\ell\left(P(j\omega), K(j\omega)\right)\right) D_\omega^{-1}}{\beta_1} - jG_\omega\right)(I + G_\omega^2)^{-\frac{1}{2}}\right)$$

$$\approx \sup_{\omega} \overline{\sigma} \left[ \left( \frac{D(s) \left( \mathcal{F}_\ell \left( P(s), K(s) \right) \right) D^{-1}(s)}{\beta_1} - G(s) \right) \left( I + G^{\sim}(s) G(s) \right)^{-\frac{1}{2}} \right].$$

(3) Let $D(s)$ be factorized as

$$D(s) = D_{ap}(s) D_{\min}(s), \quad D_{ap}^{\sim}(s) D_{ap}(s) = I, \quad D_{\min}(s),\ D_{\min}^{-1}(s) \in \mathcal{H}_\infty.$$

That is, $D_{ap}$ is an all-pass and $D_{\min}$ is a stable and minimum phase transfer matrix. Find a normalized right coprime factorization

$$D_{ap}^{\sim}(s) G(s) D_{ap}(s) = G_N G_M^{-1}, \quad G_N, \quad G_M \in \mathcal{H}_\infty$$

such that

$$G_M^{\sim} G_M + G_N^{\sim} G_N = I.$$

Then

$$G_M^{-1} D_{ap}^{\sim} (I + G^{\sim} G)^{-1} D_{ap} (G_M^{-1})^{\sim} = I$$

and, for each frequency $s = j\omega$, we have

$$\begin{aligned}
&\overline{\sigma} \left[ \left( \frac{D(s) \left( \mathcal{F}_\ell \left( P(s), K(s) \right) \right) D^{-1}(s)}{\beta_1} - G(s) \right) \left( I + G^{\sim}(s) G(s) \right)^{-\frac{1}{2}} \right] \\
&\quad = \overline{\sigma} \left[ \left( \frac{D_{\min} \left( \mathcal{F}_\ell \left( P, K \right) \right) D_{\min}^{-1}}{\beta_1} - D_{ap}^{\sim} G D_{ap} \right) D_{ap}^{\sim} (I + G^{\sim} G)^{-\frac{1}{2}} \right] \\
&\quad = \overline{\sigma} \left[ \left( \frac{D_{\min} \left( \mathcal{F}_\ell \left( P, K \right) \right) D_{\min}^{-1}}{\beta_1} - G_N G_M^{-1} \right) D_{ap}^{\sim} (I + G^{\sim} G)^{-\frac{1}{2}} \right] \\
&\quad = \overline{\sigma} \left[ \left( \frac{D_{\min} \left( \mathcal{F}_\ell \left( P, K \right) \right) D_{\min}^{-1} G_M}{\beta_1} - G_N \right) G_M^{-1} D_{ap}^{\sim} (I + G^{\sim} G)^{-\frac{1}{2}} \right] \\
&\quad = \overline{\sigma} \left[ \frac{D_{\min} \left( \mathcal{F}_\ell \left( P, K \right) \right) D_{\min}^{-1} G_M}{\beta_1} - G_N \right].
\end{aligned}$$

(4) Define

$$P_a = \begin{bmatrix} D_{\min}(s) & \\ & I \end{bmatrix} P(s) \begin{bmatrix} D_{\min}^{-1}(s) G_M(s) & \\ & I \end{bmatrix} - \beta_1 \begin{bmatrix} G_N & \\ & 0 \end{bmatrix}$$

and find a controller $K_{\text{new}}$ minimizing $\|\mathcal{F}_\ell(P_a, K)\|_\infty$.

(5) Compute a new $\beta_1$ as

$$\beta_1 = \sup_{\omega} \inf_{\tilde{D}_\omega \in \mathcal{D}, \tilde{G}_\omega \in \mathcal{G}} \{\beta(\omega) : \quad \Gamma \leq 1\}$$

where

$$\Gamma := \overline{\sigma} \left[ \left( \frac{\tilde{D}_\omega \mathcal{F}_\ell(P, K_{\mathrm{new}}) \tilde{D}_\omega^{-1}}{\beta(\omega)} - j\tilde{G}_\omega \right) (I + \tilde{G}_\omega^2)^{-\frac{1}{2}} \right].$$

(6) Find $\hat{D}_\omega$ and $\hat{G}_\omega$ such that

$$\inf_{\hat{D}_\omega \in \mathcal{D}, \hat{G}_\omega \in \mathcal{G}} \overline{\sigma} \left[ \left( \frac{\hat{D}_\omega \mathcal{F}_\ell(P, K_{\mathrm{new}}) \hat{D}_\omega^{-1}}{\beta_1} - j\hat{G}_\omega \right) (I + \hat{G}_\omega^2)^{-\frac{1}{2}} \right].$$

(7) Compare the new scaling matrices $\hat{D}_\omega$ and $\hat{G}_\omega$ with the previous estimates $D_\omega$ and $G_\omega$. Stop if they are close, else replace $D_\omega$, $G_\omega$ and $K$ with $\hat{D}_\omega$, $\hat{G}_\omega$ and $K_{\mathrm{new}}$, respectively, and go back to step (2).