Homework 4 Solutions

①

(i) $+ + + - - + -$

$$R_x(k) = \frac{1}{T}\sum_{n=1}^{T} X(n)\,X(n+k)$$

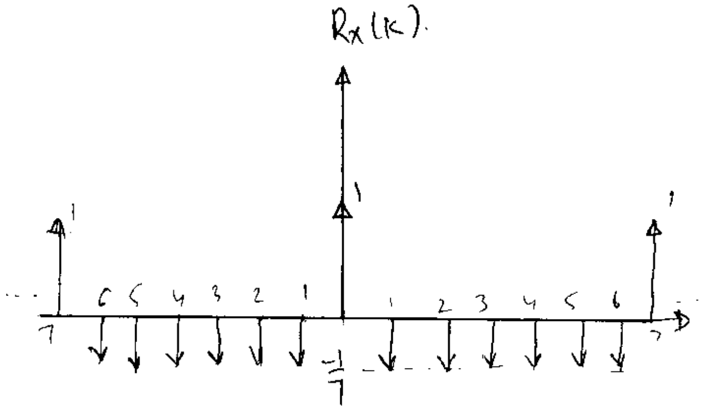
$$R_x(0) = \frac{1}{7}(1+1+1+1+1+1+1) = 1$$

$$R_x(1) = \frac{1}{7}(+1+1-1+1-1-1-1) = -\frac{1}{7}$$

$$R_x(2) = \frac{1}{7}(+1-1-1-1+1+1-1) = \frac{-1}{7}$$

$$R_x(3) = \frac{1}{7}(-1-1+1+1-1+1-1) = \frac{-1}{7}$$

Similarly $R_x(4) = R_x(5) = R_x(6) = \frac{-1}{7}$



(ii) $+ - + - - + + +$

$$R_x(0) = \frac{1}{8}(1+1+1+1+1+1+1+1) = 1$$

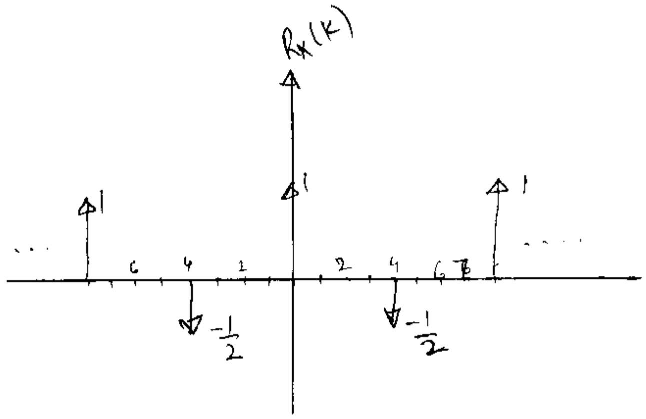
$$R_x(1) = \frac{1}{8}(-1-1-1+1-1+1+1+1) = 0$$

$$R_x(2) = \frac{1}{8}(+1+1-1-1-1+1-1+1) = 0$$

$$R_x(3) = \frac{1}{8}(-1+1+1-1-1+1-1+1) = 0$$

$$R_x(4) = \frac{1}{8}(-1-1+1-1-1-1+1-1) = -\frac{1}{2}$$

Similarly $R_x(5) = R_x(6) = R_x(7) = 0$



(iii) $+ - + - - +$

$$R_x(0) = \frac{1}{6}(1+1+1+1+1+1) = 1$$

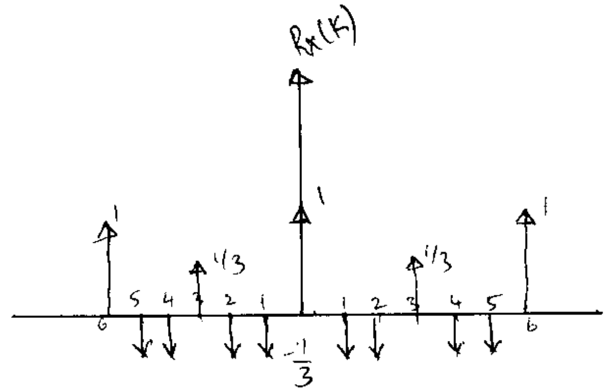
$$R_x(1) = \frac{1}{6}(-1-1-1+1-1+1) = -\frac{1}{3}$$

$$R_x(2) = \frac{1}{6}(+1+1-1-1-1-1) = \frac{-1}{3}$$

$$R_x(3) = \frac{1}{6}(-1+1+1-1+1+1) = \frac{1}{3}$$

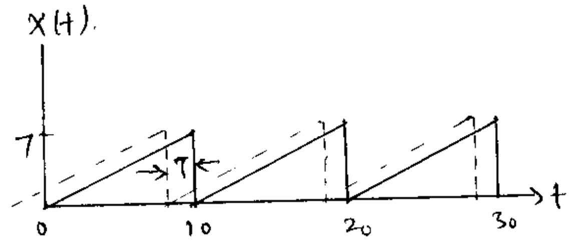
$$R_x(4) = \frac{1}{6}(-1-1+1+1-1-1) = \frac{-1}{3}$$

$$R_x(5) = \frac{1}{6}(+1-1-1-1+1-1) = \frac{-1}{3}$$



② $x(t) = \frac{7}{10} t \qquad 0 < t < 10$

(i) $x(t+\tau) = \frac{7}{10}(t+\tau)$



For $0 < \tau < 10$

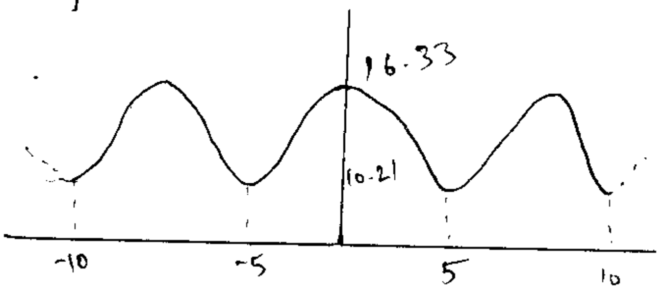
$$R_x(\tau) = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\,x(t+\tau)\,dt$$

$$= \frac{1}{10}\int_{0}^{10-\tau} \frac{7}{10}t \cdot \frac{7}{10}(t+\tau)\,dt + \frac{1}{10}\int_{10-\tau}^{10} \frac{7}{10}t \cdot \frac{7}{10}(t+\tau-10)\,dt$$

$$= \frac{49}{1000}\left[\int_{0}^{10-\tau} t^2 + t\tau\,dt + \int_{10-\tau}^{10} t^2 + t\tau - 10t\,dt\right]$$

$$= \frac{49}{1000}\left[\frac{1000}{3} - \frac{100\tau}{2} + \frac{10\tau^2}{2}\right]$$

$$R_x(0) = \frac{49}{1000} \cdot \frac{1000}{3} = 16.33$$



To find minimum value:

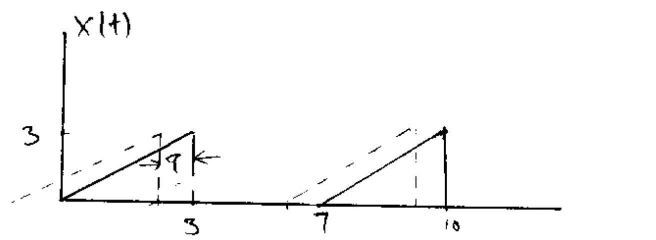
$$\frac{dR_x(\tau)}{d\tau} = \frac{49}{1000}\left[-\frac{100}{2} + 2\cdot\frac{10\tau}{2}\right] = 0 \qquad \Rightarrow \tau = 5$$

$$R_x(5) = \frac{49}{1000}\left[\frac{1000}{3} - \frac{500}{2} + \frac{250}{2}\right] = 10.21$$

(ii) $$R_x(\tau) = \frac{1}{7}\int_{0}^{3-\tau} t\,(t+\tau)\,dt$$

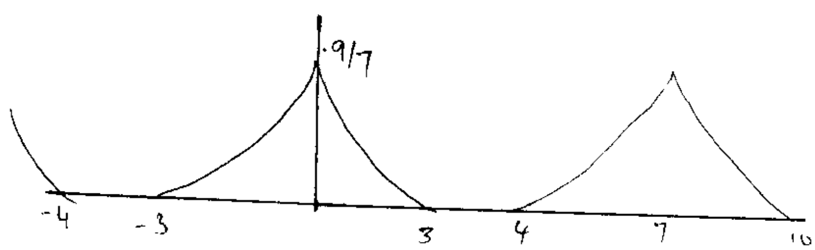
$$= \frac{1}{7}\int_{0}^{3-\tau} t^2 + t\tau\,dt$$

$$= \frac{1}{7}\left[\frac{(3-\tau)^3}{3} + \frac{(3-\tau)^2\tau}{2}\right]$$



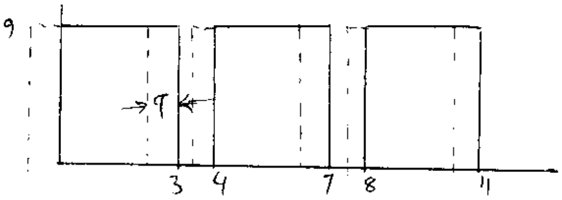
$x(t) = t \qquad 0 < t < 3$

$R_x(0) = \frac{9}{7}$; $R_x(3) = 0$

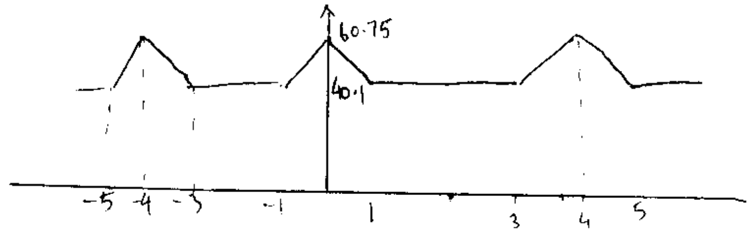


(iii) $R_X(\tau) = \frac{1}{4}\int_{0}^{3-\tau} 9.9\, dt$

$= \frac{81}{4}(3-\tau)$ for $\tau \leq 1$

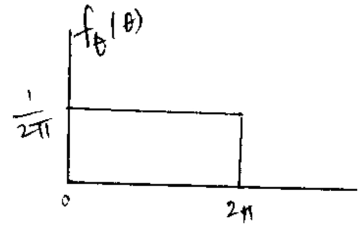


$R_X(1) = 40.1 \qquad R_X(0) = 60.75$



(3) $X(t) = A\cos(3t+\theta)$

θ is uniformly distributed in $(0, 2\pi)$



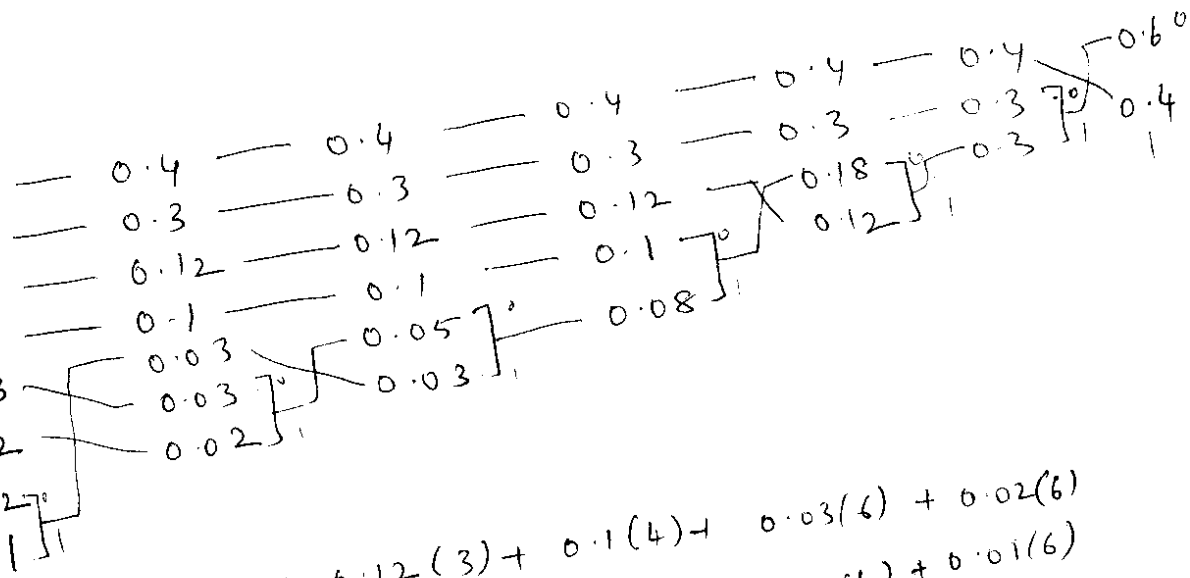
$R_X(\tau) = E\left[A\cos(3t+\theta)\cdot A\cos(3(t+\tau)+\theta)\right]$

$= \int_0^{2\pi} A\cos(3t+\theta)\cdot A\cos(3t+3\tau+\theta)\cdot\frac{1}{2\pi}\, d\theta$

$= \frac{A^2}{4\pi}\left[\int_0^{2\pi}\cos(6t+3\tau+2\theta)\,d\theta + \cos 3\tau\int_0^{2\pi} d\theta\right]$

$= \frac{A^2}{4\pi}\left[0 + 2\pi\cos 3\tau\right] = \underline{\frac{A^2}{2}\cos 3\tau}$

(4) (i) binary



1	0.4
00	0.3
011	0.12
0100	0.10
010100	0.03
010101	0.02
010110	0.02
010111	0.01

$\bar{L} = 0.4 + 0.3(2) + 0.12(3) + 0.1(4) + 0.03(6) + 0.02(6) + 0.02(6) + 0.01(6)$

$= 2.24$ bits/message

4

④ (i) Ternary

0	0.4	0.4	0.4	0.4
1	0.3	0.3	0.3	0.3
20	0.12	0.12	0.12	0.3
21	0.10	0.10	0.1	
220	0.03	0.03	0.08	
222	0.02	0.03		
2210	0.02	0.02		
2211	0.01			
	0.00			

$$L = 0.4 + 0.3 + 0.12(2) + 0.1(2) + 0.03(3) + 0.02(3) + 0.02(4) + 0.01(4)$$

$$= 1.41$$

$$= 2.235 \text{ bits/message}.$$

(ii)

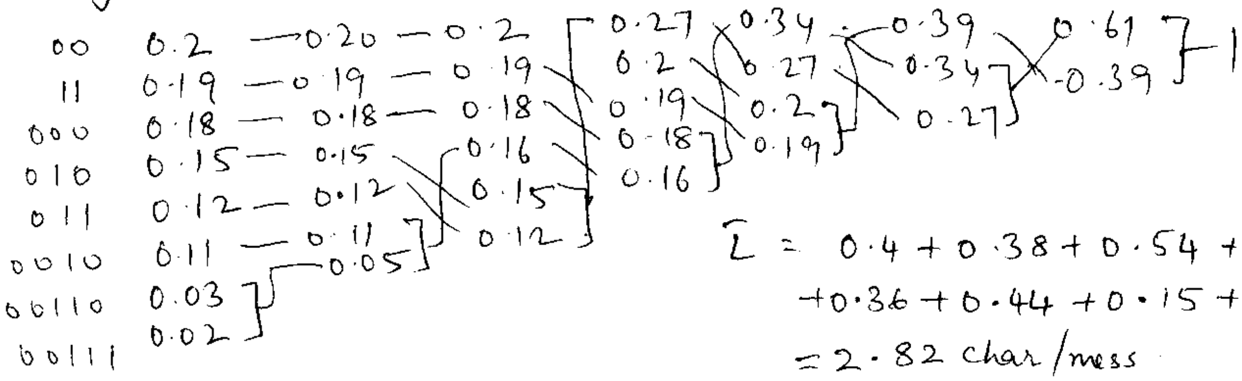
2	0.2	0.2	0.28	0.52
00	0.19	0.19	0.2	0.28
01	0.18	0.18	0.19	0.2
02	0.15	0.15	0.18	
10	0.12	0.12	0.15	
11	0.11	0.11		
120	0.03	0.05		
121	0.02			
122	0.00			

$$L = 0.2 + 0.38 + 0.36 + 0.3 + 0.24 + 0.22 + 0.09 + 0.06$$

$$= 1.85 \text{ char/message}.$$

$$= 2.92 \text{ bits/message}.$$

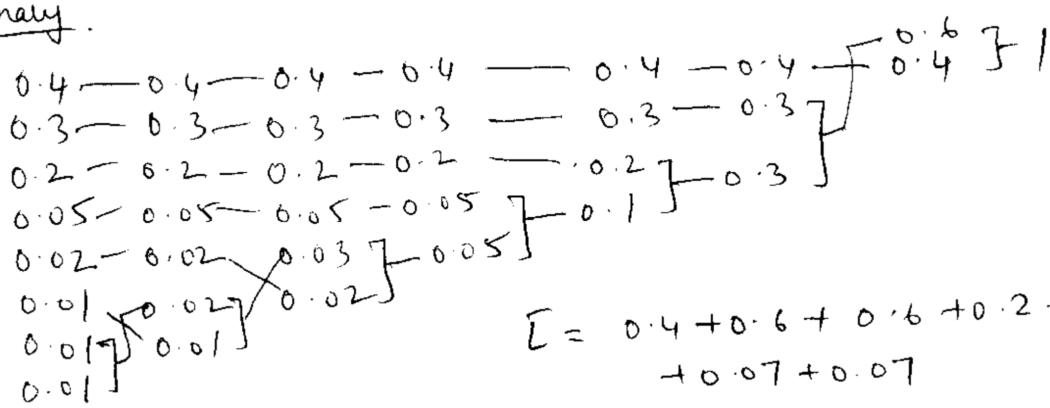
(ii) binary



$$\bar{L} = 0.4 + 0.38 + 0.54 + 0.45 + 0.36 + 0.44 + 0.15 + 0.1$$

$= 2.82$ char/mess

4(iii) binary.

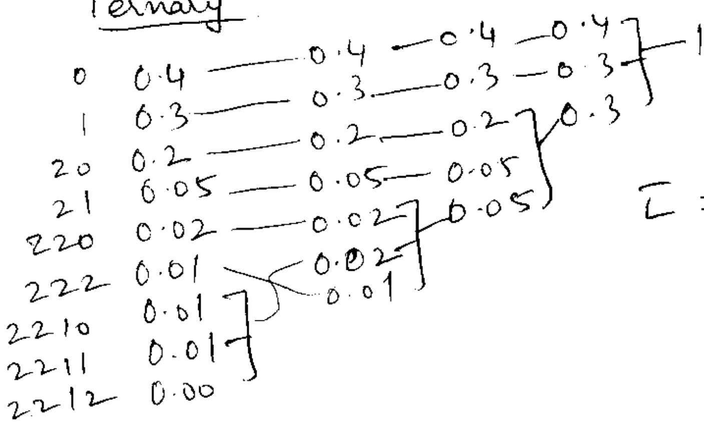


$$\bar{L} = 0.4 + 0.6 + 0.6 + 0.2 + 0.1 + 0.06 + 0.07 + 0.07$$

$= 2.1$ char/mess

$= 2.1$ bits/message.

Ternary



$$\bar{L} = 0.4 + 0.3 + 0.4 + 0.1 + 0.06 + 0.03 + 0.04 + 0.04$$

$= 1.37$ char/message.

$= 2.18$ bits/message.

6

(5) $N = \dfrac{2\times10^6 \log_2[1+10^3]}{6\times10^3 \log_2[1+10^6]}$ $\quad \because \quad C = W\log_2(1+SNR)$

$= 166.$

(6) (i) $S_x(\omega) = \mathcal{F}[R_x(\tau)]$

$= \mathcal{F}[e^{-2\alpha|\tau|}]$

$= \dfrac{1}{j\omega+2\alpha} + \dfrac{1}{-j\omega+2\alpha}$

(ii) $S_x(\omega) = \int_{-\infty}^{\infty} 2\cos 2\pi\tau \, e^{-j\omega\tau} d\tau.$

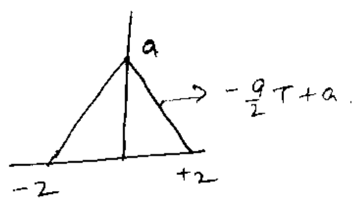
$= 2\int_0^1 2\cos 2\pi\tau \cos\omega\tau \, d\tau \quad = 2\int_0^1 \cos(2\pi+\omega)\tau d\tau + \cos(2\pi-\omega)\tau \, d\tau.$

$= 2\left[\dfrac{\sin(2\pi+\omega)}{2\pi+\omega} + \dfrac{\sin(2\pi-\omega)}{2\pi-\omega}\right]$

(7) $S_x(\omega) = 2\int_0^2 (-\frac{a}{2}\tau + a)\cos\omega\tau \, d\tau$

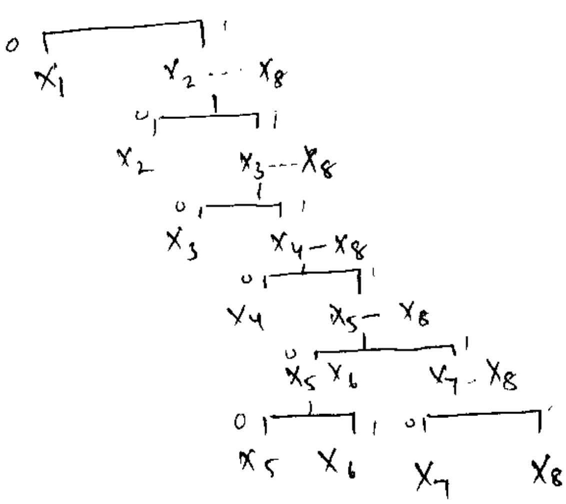
$= -a\int_0^2 \cos\omega\tau \, d\tau + 2a\int_0^2 \cos\omega\tau \, d\tau.$

$= \dfrac{a}{\omega^2}[1-\cos 2\omega]$



(8)

X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
0.4	0.3	0.2	0.05	0.02	0.01	0.01	0.01



X_1 0
X_2 10
X_3 110
X_4 1110
X_5 11110
X_6 111101
X_7 111110
X_8 111111

(9) $E(X) = \frac{2}{7} + \frac{2}{7} + \frac{6}{7} + \frac{5}{7} + \frac{6}{7} = 3$

$E(X^2) = \frac{2}{7} + \frac{4}{7} + \frac{18}{7} + \frac{25}{7} + \frac{36}{7} = \frac{85}{7}$

$E(Y) = \frac{2}{7} + \frac{4}{7} + \frac{4}{7} + \frac{7}{7} = \frac{17}{7}$

$E(Y^2) = \frac{2}{7} + \frac{8}{7} + \frac{16}{7} + \frac{49}{7} = \frac{75}{7}$

$E(XY) = \frac{1}{7} + \frac{2}{7} + \frac{8}{7} + \frac{3}{7} + \frac{35}{7} + \frac{12}{7} = \frac{61}{7}$

$[E(x)]^2 = 9 \quad [E(Y)]^2 = \frac{289}{49}$

$\sigma_X = \sqrt{E(X^2) - E(X)^2} = 1.77$

$\sigma_Y = \sqrt{E(Y^2) - E(Y)^2} = 2.19$

$\rho = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y} = 0.368$

(10) Using the last 3 element:

$R_X(0) = \frac{4^2+5^2+4^2}{3} = \frac{57}{3}$

$R_X(1) = \frac{4.5 + 5.4 + 4.3}{3} = \frac{52}{3}$

$R_X(2) = \frac{4.4 + 5.3 + 4.3}{3} = \frac{43}{3}$

$R_X(3) = \frac{4.3 + 5.3 + 4.3}{3} = \frac{39}{3}$

$$\begin{bmatrix} C(0) \\ C(1) \\ C(2) \end{bmatrix} = \begin{bmatrix} 52/3 & 43/3 & 39/3 \\ 57/3 & 52/3 & 43/3 \\ 52/3 & 57/3 & 52/3 \end{bmatrix}^{-1} \begin{bmatrix} 57/3 \\ 52/3 \\ 43/3 \end{bmatrix}$$

$X(n+1) = C(0) \cdot 4 + C(1) \cdot 5 + C(2) \cdot 4$

$= 1.94(4) + (-2.22)(5) + (1.32)(4)$

$= 1.94$

(11) Weighing once gives $\log_2 3$ bits of information.

The lighter coin is 1 out of n coins.

$$I = \log_2 \frac{1}{1/n} = \log_2 n.$$

$$\# \text{ of weighing} = \left\lceil \frac{\log_2 n}{\log_2 3} \right\rceil$$