

# Homework 4 Solutions

①

(i) + + + - - + -

$$R_x(k) = \frac{1}{7} \sum_{n=1}^7 X(n) X(n+k)$$

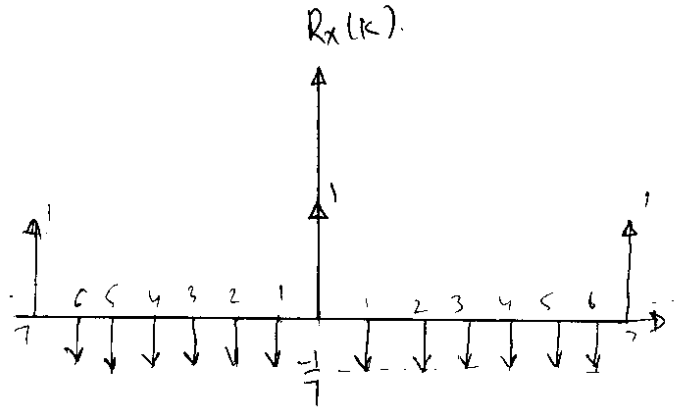
$$R_x(0) = \frac{1}{7} (1+1+1+1+1+1+1) = 1$$

$$R_x(1) = \frac{1}{7} (+1+1-1+1-1-1-1) = -\frac{1}{7}$$

$$R_x(2) = \frac{1}{7} (+1-1-1+1+1-1-1) = -\frac{1}{7}$$

$$R_x(3) = \frac{1}{7} (-1-1+1+1-1+1-1) = -\frac{1}{7}$$

$$\text{Similarly } R_x(4) = R_x(5) = R_x(6) = -\frac{1}{7}$$



(ii) + - + - - + + +

$$R_x(0) = \frac{1}{8} (1+1+1+1+1+1+1+1) = 1$$

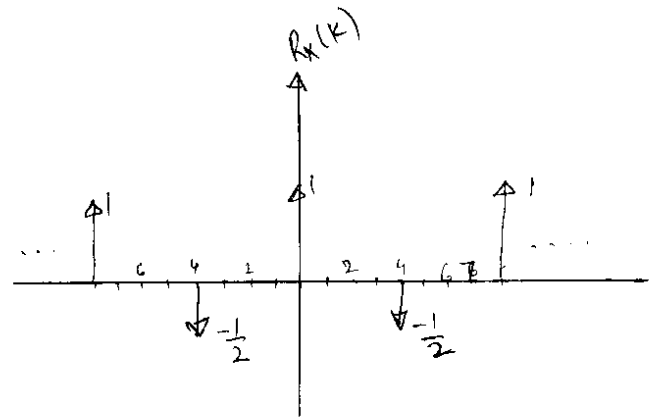
$$R_x(1) = \frac{1}{8} (-1-1-1+1-1+1+1+1) = 0$$

$$R_x(2) = \frac{1}{8} (+1+1-1-1-1+1-1+1) = 0$$

$$R_x(3) = \frac{1}{8} (-1+1+1-1-1+1-1+1) = 0$$

$$R_x(4) = \frac{1}{8} (-1-1+1-1-1+1-1-1) = -\frac{1}{2}$$

$$\text{Similarly } R_x(5) = R_x(6) = R_x(7) = 0$$



(iii) + - + - - +

$$R_x(0) = \frac{1}{6} (1+1+1+1+1+1) = 1$$

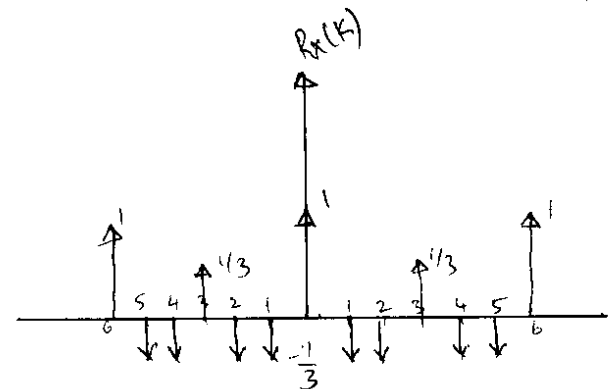
$$R_x(1) = \frac{1}{6} (-1-1-1+1-1+1) = -\frac{1}{3}$$

$$R_x(2) = \frac{1}{6} (+1+1-1-1-1-1) = -\frac{1}{3}$$

$$R_x(3) = \frac{1}{6} (+1+1+1-1+1+1) = \frac{1}{3}$$

$$R_x(4) = \frac{1}{6} (-1-1+1+1-1-1) = -\frac{1}{3}$$

$$R_x(5) = \frac{1}{6} (+1-1-1-1+1-1) = -\frac{1}{3}$$



②  $x(t) = \frac{7}{10} t \quad 0 < t < 10$

(i)  $x(t+\tau) = \frac{7}{10} (t+\tau)$

For  $0 < \tau < 10$

$$R_x(\tau) = \frac{1}{T} \int_{-\infty}^{\infty} x(t)x(t+\tau) dt$$

$$= \frac{1}{10} \int_0^{10-\tau} \frac{7}{10} t \cdot \frac{7}{10} (t+\tau) dt + \frac{1}{10} \int_{10-\tau}^{10} \frac{7}{10} t \cdot \frac{7}{10} (t+\tau-10) dt$$

$$= \frac{49}{1000} \left[ \int_0^{10-\tau} t^2 + t\tau dt + \int_{10-\tau}^{10} t^2 + t\tau - 10t dt \right]$$

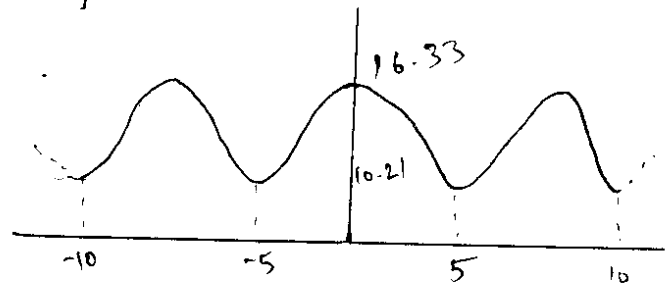
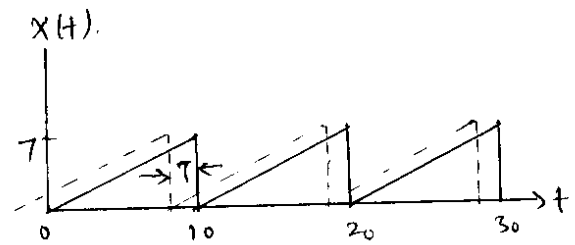
$$= \frac{49}{1000} \left[ \frac{1000}{3} - \frac{100\tau}{2} + \frac{10\tau^2}{2} \right]$$

$$R_x(0) = \frac{49}{1000} \cdot \frac{1000}{3} = 16.33$$

To find minimum value:

$$\frac{dR_x(\tau)}{d\tau} = \frac{49}{1000} \left[ -\frac{100}{2} + 2 \cdot \frac{10\tau}{2} \right] = 0 \Rightarrow \tau = 5$$

$$R_x(5) = \frac{49}{1000} \left[ \frac{1000}{3} - \frac{500}{2} + \frac{250}{2} \right] = 10.21$$

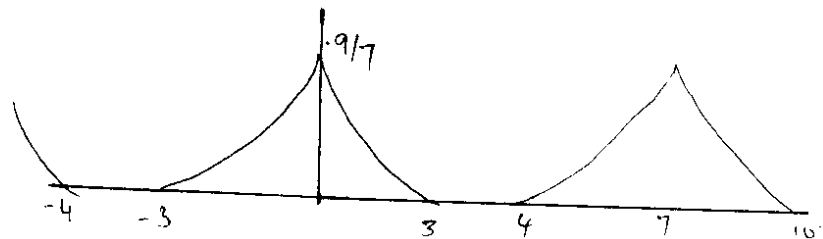
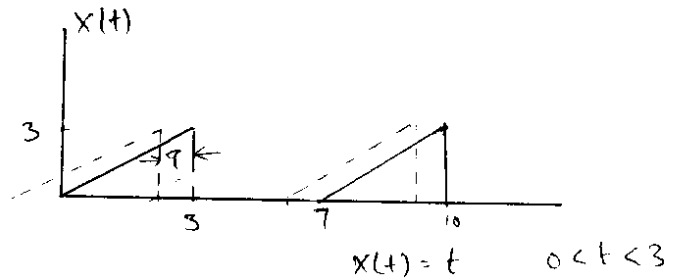


(ii)  $R_x(\tau) = \frac{1}{T} \int_0^{3-\tau} t(t+\tau) dt$

$$= \frac{1}{7} \int_0^{3-\tau} t^2 + t\tau dt$$

$$= \frac{1}{7} \left[ \frac{(3-\tau)^3}{3} + \frac{(3-\tau)^2 \tau}{2} \right]$$

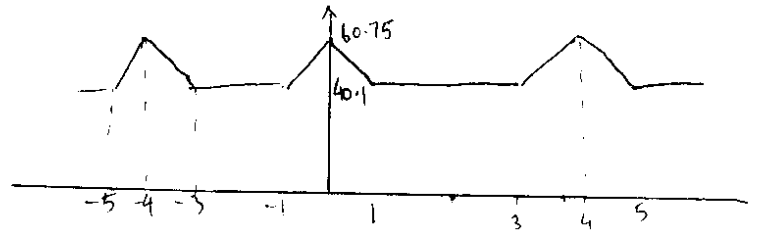
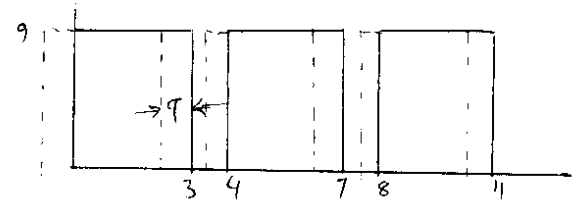
$$R_x(0) = \frac{9}{7} \quad ; \quad R_x(3) = 0$$



$$(iii) R_x(\tau) = \frac{1}{4} \int_0^{3-\tau} 9 \cdot 9 dt$$

$$= \frac{81}{4} (3-\tau) \quad \text{for } \tau \leq 1$$

$$R_x(1) = 40.1 \quad R_x(0) = 60.75$$



$$(3) X(t) = A \cos(3t + \theta)$$

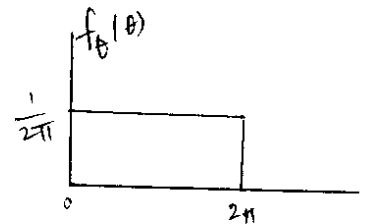
$\theta$  is uniformly distributed in  $(0, 2\pi)$

$$R_x(\tau) = E[A \cos(3t + \theta) \cdot A \cos(3t + \tau + \theta)]$$

$$= \int_0^{2\pi} A \cos(3t + \theta) \cdot A \cos(3t + \tau + \theta) \cdot \frac{1}{2\pi} d\theta$$

$$= \frac{A^2}{4\pi} \left[ \int_0^{2\pi} \cos(6t + 3\tau + 2\theta) d\theta + \cos 3\tau \int_0^{2\pi} d\theta \right]$$

$$= \frac{A^2}{4\pi} \left[ 0 + 2\pi \cos 3\tau \right] = \frac{A^2}{2} \cos 3\tau$$



(4) (i) binary

1	0.4	0.4	0.4	0.4	0.4	0.4	0.4
00	0.3	0.3	0.3	0.3	0.3	0.3	0.3
011	0.12	0.12	0.12	0.12	0.12	0.12	0.12
0100	0.10	0.1	0.1	0.1	0.1	0.1	0.1
010100	0.03	0.03	0.03	0.03	0.03	0.03	0.03
010101	0.02	0.02	0.02	0.02	0.02	0.02	0.02
010110	0.02	0.02	0.02	0.02	0.02	0.02	0.02
010111	0.01	0.01	0.01	0.01	0.01	0.01	0.01

$$L = 0.4 \cdot 1 + 0.3(2) + 0.12(3) + 0.1(4) + 0.03(6) + 0.02(6) + 0.02(6) + 0.01(6)$$

$$= 2.24 \text{ bits/message}$$

④ (i) Ternary

0	0.4	—	0.4	—	0.4	—	0.4	} 1
1	0.3	—	0.3	—	0.3	—	0.3	
20	0.12	—	0.12	—	0.12	} 0.3	} 2	
21	0.10	—	0.10	—	0.1			
220	0.03	—	0.03	} 0.08	} 2			
222	0.02	—	0.03					
2210	0.02	} 0.02	} 2					
2211	0.01							
	0.00	} 2						

$$\begin{aligned} \bar{L} &= 0.4 + 0.3 + 0.12(2) + 0.1(2) + 0.03(3) + 0.02(3) + 0.02(4) \\ &\quad + 0.01(4) \\ &= 1.41 \\ &= 2.235 \text{ bits/message.} \end{aligned}$$

(ii)

2	0.2	—	0.2	} 0.28	} 0.52
00	0.19	—	0.19		
01	0.18	—	0.18	} 0.2	
02	0.15	—	0.15		
10	0.12	—	0.12	} 0.18	
11	0.11	—	0.11		
120	0.03	} 0.05	} 2		
121	0.02				
122	0.00				

$$\begin{aligned} \bar{L} &= 0.2 + 0.38 + 0.36 + 0.3 + 0.24 + 0.22 \\ &\quad + 0.09 + 0.06 \\ &= 1.85 \text{ char/message.} \\ &= 2.92 \text{ bits/message.} \end{aligned}$$

(ii) binary

00	0.2	→ 0.20	→ 0.2	}	0.27	0.34	}	0.39	}	0.61	}
11	0.19	→ 0.19	→ 0.19		}	0.2		0.27		}	
000	0.18	→ 0.18	→ 0.18	}		0.19	0.2	}	0.27		}
010	0.15	→ 0.15	→ 0.16		}	0.18	0.19		}		
011	0.12	→ 0.12	→ 0.15	}		0.16		}			}
0010	0.11	→ 0.11	→ 0.12		}				}		
00110	0.03	→ 0.05		}				}			}
00111	0.02										

$$\begin{aligned} \bar{L} &= 0.4 + 0.38 + 0.54 + 0.45 \\ &\quad + 0.36 + 0.44 + 0.15 + 0.1 \\ &= 2.82 \text{ char/message} \end{aligned}$$

4(iii) binary

0.4	→ 0.4	→ 0.4	→ 0.4	→ 0.4	→ 0.4	→ 0.4	}	0.6	}
0.3	→ 0.3	→ 0.3	→ 0.3	→ 0.3	→ 0.3	→ 0.3		}	
0.2	→ 0.2	→ 0.2	→ 0.2	→ 0.2	→ 0.2	→ 0.2	}		0.3
0.05	→ 0.05	→ 0.05	→ 0.05	→ 0.05	→ 0.05	→ 0.05		}	0.1
0.02	→ 0.02	→ 0.03	→ 0.05				}		
0.01	→ 0.02	→ 0.02						}	
0.01	→ 0.01						}		
0.01									

$$\begin{aligned} \bar{L} &= 0.4 + 0.6 + 0.6 + 0.2 + 0.1 + 0.06 \\ &\quad + 0.07 + 0.07 \\ &= 2.1 \text{ char/message} \\ &= 2.1 \text{ bits/message} \end{aligned}$$

Ternary

0	0.4	→ 0.4	→ 0.4	→ 0.4	}	0.4	}
1	0.3	→ 0.3	→ 0.3	→ 0.3		}	
20	0.2	→ 0.2	→ 0.2	→ 0.2	}		0.3
21	0.05	→ 0.05	→ 0.05	→ 0.05		}	
220	0.02	→ 0.02	→ 0.05		}		
222	0.01	→ 0.02	→ 0.05			}	
2210	0.01	→ 0.01			}		
2211	0.01					}	
2212	0.00						

$$\begin{aligned} \bar{L} &= 0.4 + 0.3 + 0.4 + 0.1 + 0.06 + 0.03 \\ &\quad + 0.04 + 0.04 \\ &= 1.37 \text{ char/message} \\ &= 2.18 \text{ bits/message} \end{aligned}$$

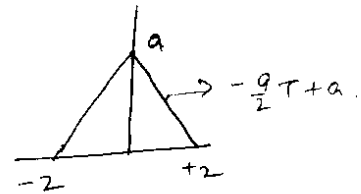
$$\textcircled{5} \quad W = \frac{2 \times 10^6 \log_2 [1 + 10^3]}{6 \times 10^3 \log_2 [1 + 10^6]} \quad \because C = W \log_2 (1 + \text{SNR})$$

$$= 166$$

$$\textcircled{6} \text{ (i)} \quad S_X(\omega) = \mathcal{F} [R_X(\tau)] \\ = \mathcal{F} [e^{-2\alpha|\tau|}] \\ = \frac{1}{j\omega + 2\alpha} + \frac{1}{-j\omega + 2\alpha}$$

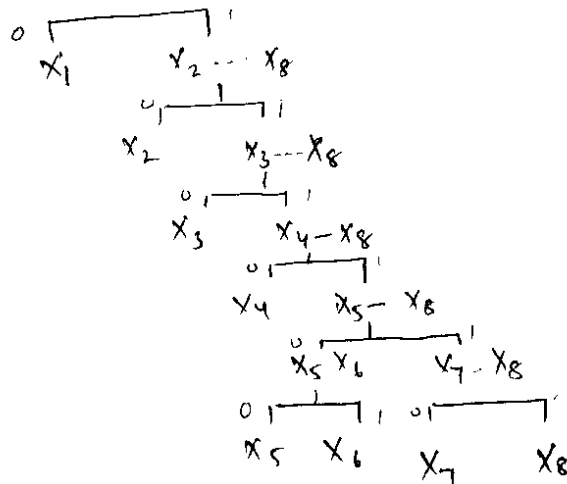
$$\text{(ii)} \quad S_X(\omega) = \int_{-\infty}^{\infty} 2 \cos 2\pi T e^{-j\omega\tau} d\tau \\ = 2 \int_0^{\infty} 2 \cos 2\pi T \cos \omega T d\tau = 2 \int_0^{\infty} [\cos(2\pi + \omega)T + \cos(2\pi - \omega)T] d\tau \\ = 2 \left[ \frac{\sin(2\pi + \omega)}{2\pi + \omega} + \frac{\sin(2\pi - \omega)}{2\pi - \omega} \right]$$

$$\textcircled{7} \quad S_X(\omega) = 2 \int_0^2 \left( -\frac{a}{2} T + a \right) \cos \omega T dT \\ = -a \int_0^2 \cos \omega T dT + 2a \int_0^2 \cos \omega T dT \\ = \frac{a}{\omega^2} [1 - \cos 2\omega]$$



$$\textcircled{8} \quad \begin{matrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\ 0.4 & 0.3 & 0.2 & 0.05 & 0.02 & 0.01 & 0.01 & 0.01 \end{matrix}$$

$$\begin{matrix} X_1 & 0 \\ X_2 & 10 \\ X_3 & 110 \\ X_4 & 1110 \\ X_5 & 11110 \\ X_6 & 111101 \\ X_7 & 111110 \\ X_8 & 111111 \end{matrix}$$



$$\textcircled{9} \quad E(X) = \frac{2}{7} + \frac{2}{7} + \frac{6}{7} + \frac{5}{7} + \frac{6}{7} = 3$$

$$E(X^2) = \frac{2}{7} + \frac{4}{7} + \frac{18}{7} + \frac{25}{7} + \frac{36}{7} = \frac{85}{7}$$

$$E(Y) = \frac{2}{7} + \frac{4}{7} + \frac{4}{7} + \frac{7}{7} = \frac{17}{7}$$

$$E(Y^2) = \frac{2}{7} + \frac{8}{7} + \frac{16}{7} + \frac{49}{7} = \frac{75}{7}$$

$$E(XY) = \frac{1}{7} + \frac{2}{7} + \frac{8}{7} + \frac{3}{7} + \frac{35}{7} + \frac{12}{7} = \frac{61}{7}$$

$$[E(X)]^2 = 9 \quad [E(Y)]^2 = \frac{289}{49}$$

$$\sigma_X = \sqrt{E(X^2) - E(X)^2} = 1.77$$

$$\sigma_Y = \sqrt{E(Y^2) - E(Y)^2} = 2.19$$

$$\rho = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y} = 0.368$$

$\textcircled{10}$  Using the last 3 elements:

$$R_X(0) = \frac{4^2 + 5^2 + 4^2}{3} = \frac{57}{3}$$

$$R_X(1) = \frac{4 \cdot 5 + 5 \cdot 4 + 4 \cdot 3}{3} = \frac{52}{3}$$

$$R_X(2) = \frac{4 \cdot 4 + 5 \cdot 3 + 4 \cdot 3}{3} = \frac{43}{3}$$

$$R_X(3) = \frac{4 \cdot 3 + 5 \cdot 3 + 4 \cdot 3}{3} = \frac{39}{3}$$

$$\begin{bmatrix} C(0) \\ C(1) \\ C(2) \end{bmatrix} = \begin{bmatrix} 52/3 & 43/3 & 39/3 \\ 57/3 & 52/3 & 43/3 \\ 52/3 & 57/3 & 52/3 \end{bmatrix}^{-1} \begin{bmatrix} 57/3 \\ 52/3 \\ 43/3 \end{bmatrix}$$

$$\begin{aligned} X_{(n+1)} &= C(0) \cdot 4 + C(1) \cdot 5 + C(2) \cdot 4 \\ &= 1.94(4) + (-2.22)(5) + (1.32)(4) \\ &= 1.94 \end{aligned}$$

① Weighing once gives  $\log_2 3$  bits of information.

The lighter coin is 1 out of  $n$  coins.

$$I = \log_2 \frac{1}{1/n} = \log_2 n.$$

$$\# \text{ of weighing} = \left\lceil \frac{\log_2 n}{\log_2 3} \right\rceil$$